Dirk Bollaerts

Exercise notes on the continuity of real functions of two variables

With Illustrations

February 2023

1 introduction

1.1 frequently asked questions

It is our experience that questions about continuity of real functions in two real variables arise very frequently. So we have made an effort to collect exercises that we have discussed in the past. I hope they are interesting and of some help for students studying this theory.

1.2 prerequisites

We will assume that the reader is acquainted with the theory of continuity in one real variable. We assume for example that the reader knows about standard continuity theorems like $\lim_{\alpha\to 0}\frac{\sin(\alpha)}{\alpha} = 1$ or $\lim_{\alpha\to 0}\frac{\tan(\alpha)}{\alpha} = 1$. We assume that the reader is able to write down simple ϵ-δ proofs in one variable. Knowledge of exponential functions and basic behaviour of e.g. the arctan function is also necessary. We will try to avoid building upon notions of differentiability in one variable because we are basically working with continuity and prefer for that reason topological continuity arguments. This is a deliberate choice and if the instructor of readers of this text thinks otherwise, please follow his instructions.

We give in the appendix clues and hints for proving basic essential inequalities about logarithms and exponential functions without making use of differentiability arguments.

We assume also that the reader is familiar with level curves plots of functions of two variables. But this is only important for a good understanding of the graphical interpretations of the exercises.

1.3 types of exercises

All exercises are of the kind that a reader encounters in standard courses of general mathematics. Some exercises can be found in almost all text books and can be considered as standard examples. We have written down the solutions of all exercises. The solutions are meant to be in a style and a technical level that is commonly used in general calculus undergraduate texts.

Very important functions as e.g. functions defined by the limit of a sequence of functions are generally not known when a student meets for the first time functions of two variables in a standard course.

Important topological topics as the Cantor set and the Cantor function are in this context also not considered.

Functions defined by series are not treated here.

1.4 notation

It is understood that all functions of two variables are real functions. So we understand that they are of the type $f : \mathbf{R}^2 \to \mathbf{R} : (x, y) \mapsto f(x, y)$. All functions of one variable are of the type $g : \mathbf{R} \to \mathbf{R} : x \mapsto g(x)$. Because no confusing can arise, we are not going to repeat that throughout.

1.5 list of all exercises

We list all exercises in the following pages. This can help the readers who are interested in searching for special functions that capture their current interest. In order to keep the list short and concise, we give only the main parts of the definitions of the functions.

Table 1: We give a list of the main part of the definitions of the functions in the exercises in this table. The exercises that are marked with “see below" are based on definitions that are too long to put them in the table and they are placed immediately below the table.

1.	$\dfrac{x\,y}{x^2+y^2}$	**2.**	$\dfrac{(2\,x+y)^3}{x^2+y^2}$
3.	$\dfrac{1}{x^2+y^2}$	**4.**	$\dfrac{1}{x-y}$
5.	$\dfrac{x^2\,y}{x^2+y^2}$	**6.**	$\dfrac{x\,y^2}{x^2+y^4}$
7.	$\dfrac{x\,y^3}{x^2+y^4}$	**8.**	$\dfrac{x\,y}{3\,x^2+y^2}$
9.	$\dfrac{x\,y\,\cos(y)}{x^2+y^2}$	**10.**	$\dfrac{e^{x^2+y^2}-1}{x+y}$
11.	$\dfrac{5\,x^2\,y-y^3}{x^2+y^2}$	**12.**	$\dfrac{x^3\,y^2}{x^4+y^4}$
13.	$\dfrac{\sin(x\,y)}{x+y}$	**14.**	$\dfrac{\sin(x\,y)}{x^2+y^2}$
15.	$\dfrac{5\,x^2\,y-y^3}{x^2+y^2}$	**16.**	$\dfrac{x^2\,y^2}{x^2+y^2}$
17.	$\dfrac{x^3+y^3}{x^2+y^2}$	**18.**	$\dfrac{x^3\,y}{x^4+y^4}$
19.	$\dfrac{x^3-y^3}{x^2+y^2}$	**20.**	$\dfrac{2\,x-y}{x^2+y^2}$
21.	$\dfrac{(x-1)^2\,\ln(x)}{(x-1)^2+y^2}$	**22.**	$\dfrac{\sin((x-y)^2)}{\lvert x\rvert+\lvert y\rvert}$
23.	$\dfrac{1}{x^2-y}$	**24.**	$\arctan\left(\dfrac{x\,y^2}{x+y}\right)$

25.	$\dfrac{x^2 - y^2}{x^2 + y^2}$	**26.**	$\dfrac{x}{\sqrt{x^2 + y^2}}$
27.	$\dfrac{x - y}{x + y}$	**28.**	$\dfrac{x^2 + y}{y}$
29.	$\dfrac{x^3 - x\,y^3}{x^2 + y^2}$	**30.**	$\dfrac{x^2}{x^2 + y^2}$
31.	$\dfrac{x^2 + y^2 - x^3\,y^3}{x^2 + y^2}$	**32.**	$\dfrac{x^2\,y}{x^4 + y^4}$
33.	$\dfrac{x^2\,y^2}{x^2 + y^2}$	**34.**	$\dfrac{\lvert x\rvert^a\,\lvert y\rvert^b}{(x^2 + y^2)^c}$
35.	$(x + y)\,\sin(1/x)\,\sin(1/y)$	**36.**	$\dfrac{x\,y}{x^2 + y^2} + y\,\sin(1/x)$
37.	$x\,\sin(1/y) + y\,\sin(1/x)$	**38.**	$\dfrac{x^2\,y^2}{x^3 + y^3}$
39.	$\dfrac{x\,y\,(x^2 - y^2)}{x^2 + y^2}$	**40.**	$\dfrac{x^2 + \sin^2(y)}{2\,x^2 + y^2}$
41.	$\dfrac{x\,y}{\sqrt{x^2 + y^2}}$	**42.**	$\dfrac{\sin(x\,y)}{\sqrt{x^2 + y^2}}$
43.	$\dfrac{\arcsin(x/y)}{1 + x\,y}$	**44.**	$\dfrac{x + y}{x^2 + y^2 + 1}$
45.	$\dfrac{(x - 1)^2\,\ln(x)}{(x - 1)^2 + y^2}$	**46.**	$\dfrac{x + y}{x^2 + y^2}$
47.	$\dfrac{x^3 - y^3}{x^3 + y^3}$	**48.**	$\arctan\left(\dfrac{\lvert x\rvert + \lvert y\rvert}{x^2 + y^2}\right)$
49.	$\dfrac{e^{-x^2-y^2} - 1}{x^2 + y^2} + 1$	**50.**	See below.
51.	$\dfrac{x\,y}{\lvert x\rvert + \lvert y\rvert}$	**52.**	$\dfrac{x^2 - 2\,y}{y^2 + 2\,x}$

53.	$\dfrac{x^2\,y^2}{(x^4+y^2)}$	**54.**	$\dfrac{\sin\,(x^2+4\,y^2)}{x^2+y^2}$
55.	$\lvert x\rvert^{y}-1$	**56.**	$\dfrac{e^{-\frac{x}{y}}}{y}$
57.	$\dfrac{\sin\,(x\,y^3)}{x^2}$	**58.**	$\dfrac{x^3+y^5}{x^2+2\,y^2}$
59.	$\dfrac{x^2+\sin^2(y)}{x^2+2\,y^2}$	**60.**	$\dfrac{\tan\left(\frac{x\,y}{x\,y+1}\right)}{x\,y}$
61.	$(x^2+y^2)^{x^2\,y^2}$	**62.**	$\dfrac{(x+1)\,(y-x)}{(y+1)\,(x+y)}$
63.	$\dfrac{\lvert y\rvert^a\,\lvert x\rvert^b}{\lvert x\rvert^a+\lvert y\rvert^b}$	**64.**	$\arctan\left(\dfrac{y}{x}\right)$
65.	$\dfrac{x^3\,y^3}{x^6+y^6}$	**66.**	$\dfrac{x^3\,y}{x^6+y^2}$
67.	$\dfrac{(x-1)\,y^2}{x^2+y^2}$	**68.**	$\dfrac{x^2-y^6}{x\,y^3}$
69.	$\arctan\left(\dfrac{2\,y}{x^2+y^2}\right)$	**70.**	$\lvert y\rvert\,\sin\left(\dfrac{x}{\sqrt{\lvert y\rvert}}\right)$
71.	$x^2\left(1-\cos\left(\dfrac{y}{x}\right)\right)$	**72.**	$\sin\left(\dfrac{1}{x\,y}\right)$
73.	$\sin\left(\dfrac{1}{x^2+y^2}\right)$	**74.**	$(x^2+y^2)^{\mu}\,\sin\left(\dfrac{1}{x^2+y^2}\right)$
75.	$\dfrac{\cos(x\,y)\,\sin\,(4\,x\,\sqrt{\lvert y\rvert})}{\sqrt{\lvert x\,y\rvert}}$	**76.**	$x^n\,y^m\,\sin\left(\dfrac{1}{x^2+y^2}\right)$
77.	$\dfrac{1-\cos\,(x^2\,y)}{x^2+2\,y^2}$	**78.**	$\dfrac{x^2\,y}{x^2-y^2}$
79.	$\dfrac{x-y}{x+y}$		

The following exercise is not in the table.
50. Is the following function continuous in $(0,0)$?

$$f(x,y) = \begin{cases} 0 & \text{if } x\,y \neq 0; \\ \\ 1 & \text{if } x\,y = 0. \end{cases}$$

1.6 graphical illustrations of the exercises

We give a graphical interpretations of almost all exercises. These are not a part of the solution of the exercises but only serve to have a better understanding of what is going on. The reader can safely skip these illustrations if he is only interested in the mathematical solutions.

Illustrations are a good motivational and educational help tool in a general mathematics course.

The drawback of visualisation is that one can wrongly assume that drawing is a tool that is sufficient for deciding whether or not a function is continuous. This can be the case for the type of functions used in this text. But one should realise that calculus texts do not in general use more refined functions for which a visual representation is of little or no help at all. Continuity is as a matter of fact an extremely difficult and also very subtle concept when considered in a more general context.

It is the opinion of this author that no visual tool can replace an alphabetical argument in mathematics.

We should also mention the fact that we stretched the possibilities of the visualisation tool to the absolute maximum. There are exercises in which the visualisation is not an exact description of the mathematical facts but it can even in those cases be a help to understand what is exactly going on. We encounter examples in which the oscillations are too wild, or that discontinuities cause gaps in the graphical presentation of the function.

1.7 solution strategies

At least as far as this author knows, there is no general method with which to proceed. One can however learn by experience how to handle a problem. The basic difficulty is deciding if a function is continuous or not. If one is sure that the function is continuous, then a classical ϵ-δ proof is the way to go. If one feels sure that the function is discontinuous, then a typical attack strategy is showing that the function behaves differently if one restricts it to continuous curves going through the point in which the continuity has to be investigated. Another way to go is to search for numerical information. We can calculate some numerical approximations of function values in the neighbourhood of the point in which we are interested. But basically one has to develop a gut feeling for deciding whether to go for a discontinuity proof or a continuity proof. This intuitive feeling can be developed by making plenty of exercises.

1.8 style of solutions

The solutions are written in a style of a technical level that is suited for general calculus students in their first year. Powerful tools like homeomorphisms are thus out of the question. It is probably very unlikely that students have heard about it when following a general calculus course.

Important and fundamental inequalities like the inequality of the geometrical mean and the arithmetic mean are also not used because many students for which this text document was originally written do not know this. That inequality could make some inequalities in the text much easier and more elegant.

But I would advise any student who is an advanced student or an adventurous student to develop his own style and change the arguments used in the text suiting this style and his own mathematical background. Any mathematical creativity is encouraged. The reader should never be entirely pleased by mathematics created by another person. Create your own solutions!

1.9 use with care

The reader should realize that these notes are exercise notes. This text is never peer reviewed as it is in the case of theoretical publications.

Use the text with care and ask whenever necessary advise from your instructor. He is best placed to guide you. And remember that Murphy's law is merciless in all circumstances. Remarks are welcome! The reader can find my contact address at the end of this text.

1.10 create your own examples

This section is not suited for a first reading. It can also considerably lessen the fun in solving this type of exercises. The attention of the reader will then invariably undergo an inevitable change of attention from the solution to how the function is created and why it works in the first place.

When we create a function that behaves critically only around $(0,0)$, we can use a denominator like x^2+y^2 or other even powers like x^4+y^2. We can also use absolute values as in $|x|+|y|$. Then we have to take care that the function behaves in some directions very differently as in other directions. In order to ensure that, we can e.g. use x^2y in the numerator or $x+y$. This ensures in the first case that in the X-direction and the Y-direction the function is identically zero. In the second case the function is identically zero in the direction $y=-x$. If the function behaves now differently in another direction, say $y=\lambda x$ for a suitably λ, then we are done and have our first exercise. If one wants the function to be continuous, make the degrees of polynomials in the numerator large enough to ensure that. In the other case, lessen the degrees of polynomials in the numerator.

Another way to go is making the exercise technically a little bit more difficult by using asymptotically equivalent functions. Replace for e.g. $x\,y$ with $\sin(x\,y)$. They have both an equivalent converging behaviour in a neighbourhood of $(0,0)$.

It is also interesting to start from the behaviour of functions like $x\,\ln(x)$ in a neighbourhood of 0. Another example is e^x-1 in a neighbourhood of 0. Because these functions behave nicely in a neighbourhood of 0, we can for example use $(x+y)\ln(x+y)$ in the build up of our function.

If we want diverging behaviour, divide somewhere by a suitable $x^n y^m$ or multiply with something like $e^{(1/(x^2+y^2))}$.

We can also start from something that is essentially bounded but behaves wildly around $(0,0)$. This is then e.g. $\sin(1/(x^2+y^2))$. Then multiply it with e.g. a power of x or y in order to force continuity. We can end up with something like $(x+y)\sin(1/x)$. We could also start with a bounded function of one variable and substitute something like $e^{-x^2-y^2}$ for that variable. Then multiply it again with e.g. a power of x or y in order to force continuity.

We can let our phantasy run rampant with this type of exercises.

1.11 mathematical notation

We are going a function $f(x,y)$ restricted to the subset $\{(x,y) \mid y = h(x)\}$ where $h(x)$ is a function in one variable by

$$f\big|_{y=h(x)}(x,y).$$

We are always going to restrict to a continuous curve $y = h(x)$ so that $f\big|_{y=h(x)}$ is continuous if f is continuous.

1.12 limits

Almost all exercises can be asked in two styles. Either one asks if the function $f(x)$ with $f(a) = b$ is continuous in $x = a$ or one asks this question in the following way: is the following limit $\lim_{x\to a} f(x) = b$ valid? So one can translate almost every continuity exercise in an exercise about limits and vice versa.

2 exercises and solutions

Exercise 1.

Is the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

continuous in $(0, 0)$?

Solution.

Let us calculate the function restricted to a line $y = \lambda x$. We have in that case:

$$f|_{y=\lambda x}(x, y) = \begin{cases} f(x, \lambda x) = \dfrac{\lambda x^2}{x^2 + \lambda^2 x^2} = \dfrac{\lambda}{1 + \lambda^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So the limit for $x \to 0$ is then

$$\lim_{x \to 0} f(x, \lambda x) = \frac{\lambda}{1 + \lambda^2}.$$

This gives us different values for at least two different lines. If the function is continuous, then all these values should give us the function value in $(0, 0)$, that is $f(0, 0) = 0$. So this function is not continuous.

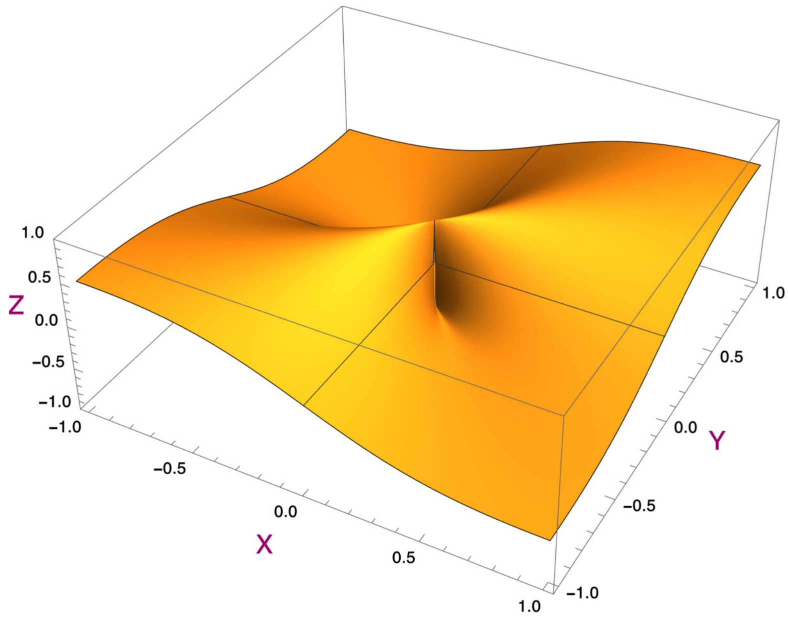


Figure 1. We see here a three dimensional figure of the graph of the function. We see that the function above the point $(0,0)$ shows a vertical line. The reason for this is that the limit points of the level curves unavoidably are visible above $(0,0)$. The graph does not show a function any more, because it looks like $(0,0)$ is mapped onto many points. But it seems to be unavoidable. Otherwise we could show nothing at all.

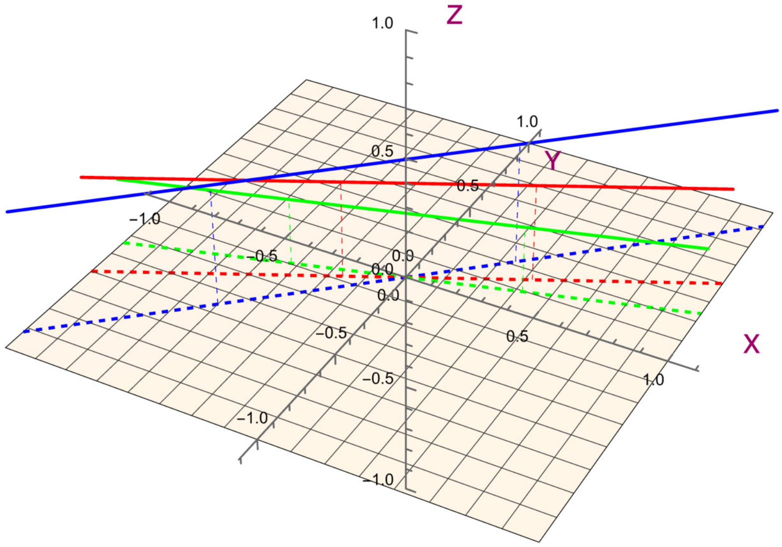


Figure 2. We have restricted the function here to $y = 1/2\,x$ and $y = 3/10\,x$ and $y = 9/10\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

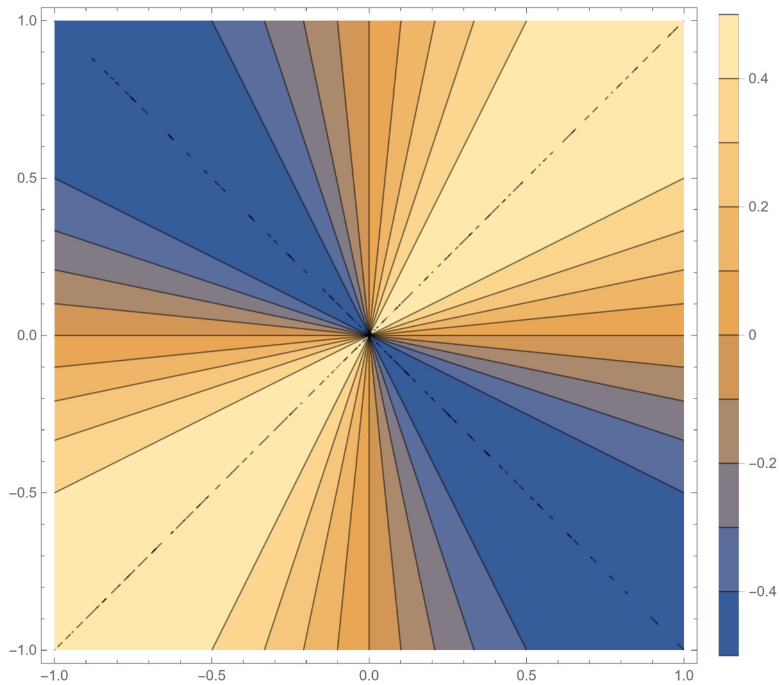


Figure 3. We see here a figure of the contour plot of the function. Remark that level curves of very different levels approach $(0,0)$ infinitesimally. This is always very suspicious and it is almost certain that the function is not continuous in a neighbourhood of $(0,0)$. The level curves plot shows a very peculiar picture. It consists of straight lines through the origin which are level lines with many different levels. This gives us a picture of a spiral staircase in which the lines represent the steps and the vertical line above $(0,0)$ is the supporting beam of the spiral staircase. This cannot be a picture of a continuous point.

Exercise 2.
Is the function

$$f(x,y)=\begin{cases}\dfrac{(2x+y)^3}{x^2+y^2} & \text{if } (x,y)\neq(0,0);\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

continuous in $(0,0)$?

Solution.
We take an arbitrary $\epsilon > 0$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ holds then $|f(x,y) - f(0,0)| < \epsilon$ is valid.

By looking at our function, we can translate this as follows. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{(2x+y)^3}{x^2+y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{(2x+y)^3}{x^2+y^2} - 0 \right| &\leq \frac{(|2x| + |y|)^3}{x^2+y^2} \\
&\leq \frac{(2\,|x| + |y|)^3}{x^2+y^2} \\
&\leq \frac{(2\sqrt{x^2+y^2} + \sqrt{x^2+y^2})^3}{x^2+y^2} \\
&\leq 27\,\frac{\sqrt{x^2+y^2}^{\,3}}{\sqrt{x^2+y^2}^{\,2}} \\
&\leq 27\sqrt{x^2+y^2}.
\end{aligned}$$

It is sufficient to prove that $27\sqrt{x^2+y^2} < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2+y^2} < \delta$. We can choose for example $\sqrt{x^2+y^2} < \delta$ with $\delta = \epsilon/27$. We have found our δ and the function is indeed continuous.

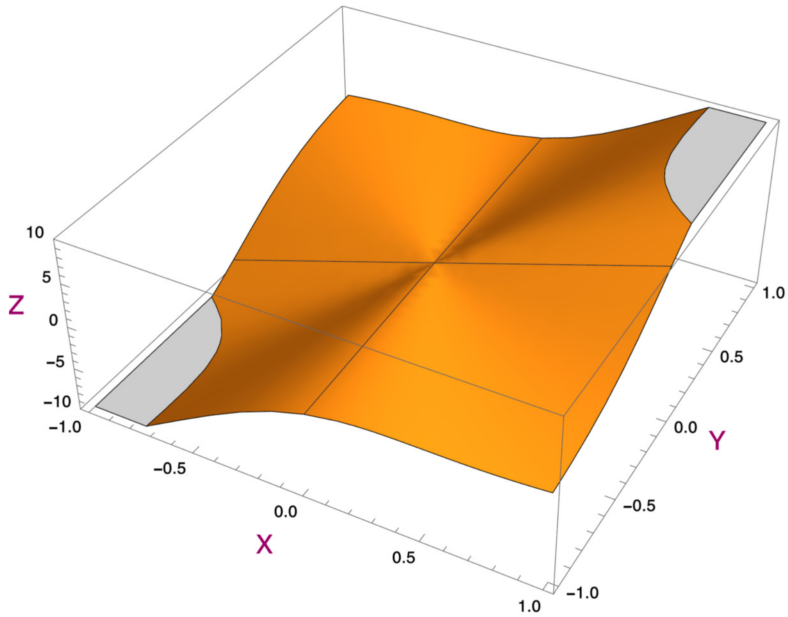


Figure 4. We see here a three dimensional figure of the graph of the function.

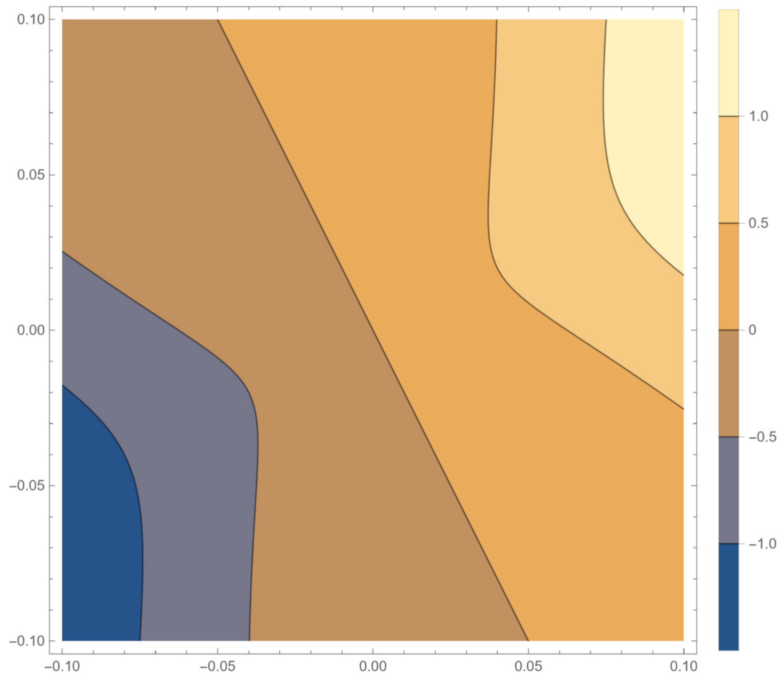


Figure 5. We see here a figure of the contour plot of the function. This looks like a classic picture of a continuous point. We see that the level lines that come close to $(0,0)$ tend to have a level approximating 0.

Exercise 3.

Is the function

$$f(x,y) = \begin{cases} \dfrac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous in $(0,0)$?

Solution.

We restrict the function to the X-axis. We have then the following function by restricting to $y = 0$:

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = \dfrac{1}{x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

It is clear that this restricted function has the following limit in $x = 0$

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

So this limit is not going to 0, and this function $f(x, 0)$ should have the limit 0 if it is continuous. The function is in fact unbounded in any neighbourhood of $(0, 0)$ and this is not possible if the function is continuous in $(0, 0)$. We conclude that the function is discontinuous in $(0, 0)$.

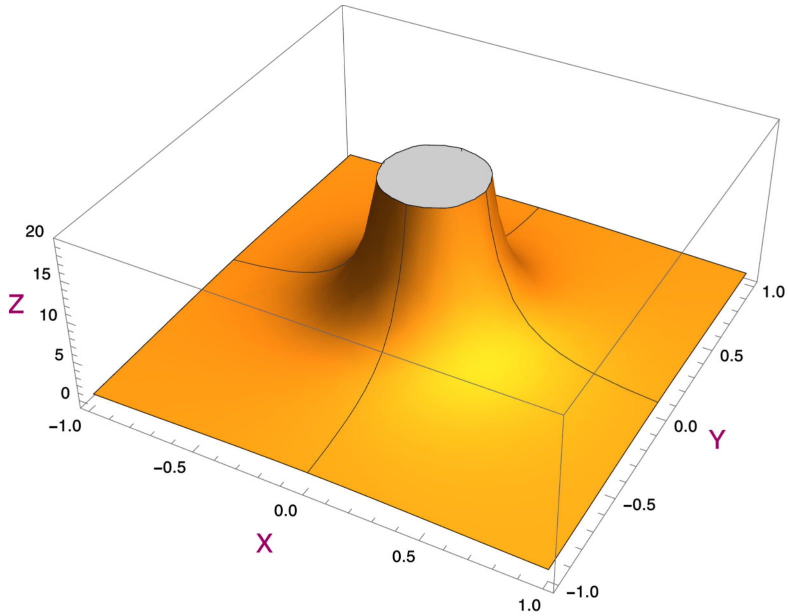


Figure 6. We see here a three dimensional figure of the graph of the function. Remark that the function is unbounded in any neighbourhood of $(0, 0)$. The horizontal intersection with a horizontal plane of the graph at level 20 is pictured as grey area.

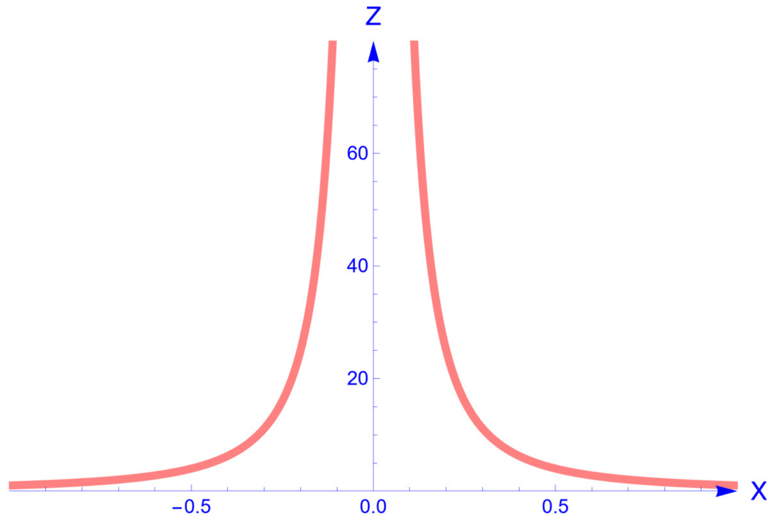


Figure 7. We see here a figure of the function restricted to the X-axis. Remark that the function is unbounded in any neighbourhood of $(0,0)$.

Exercise 4.

Is the function

$$f(x,y) = \begin{cases} \dfrac{1}{x-y} & \text{if } x \neq y; \\ 0 & \text{if } x = y. \end{cases}$$

continuous in $(0,0)$?

Solution.

We restrict the function to the X-axis. We have then the following function by restricting $y = 0$:

$$f|_{y=0}(x,y) = f(x,0) = \begin{cases} \dfrac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

It is clear that this restricted function has the following limits in $x = 0$

$$\lim_{x \xrightarrow{\leq} 0} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x \xrightarrow{\geq} 0} \frac{1}{x} = +\infty.$$

It is clear that the restricted function is not continuous in $x = 0$ because the function is not bounded there.

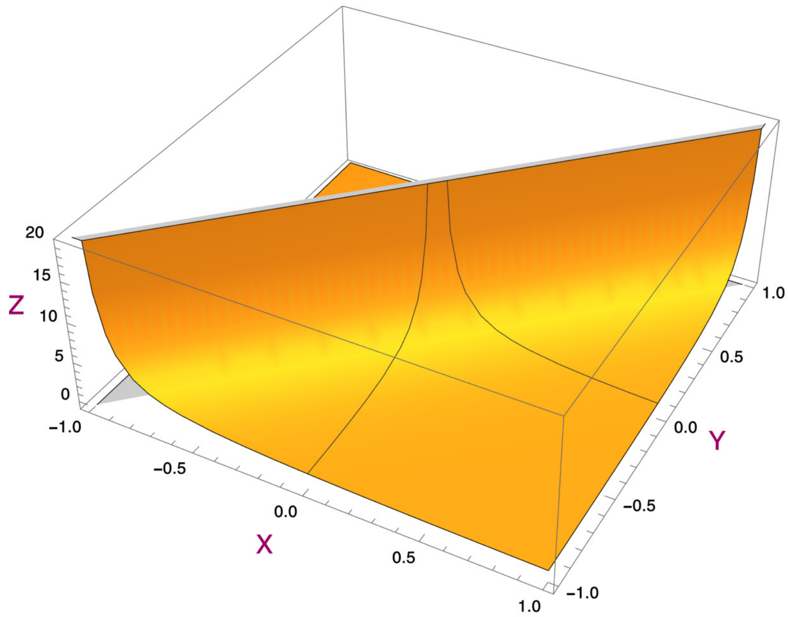


Figure 8. We see here a three dimensional figure of the graph of the function. We see that the function above the graph of the line $x - y = 0$ in the X-Y plane is infinite. The function is not continuous on any point of the line. The reason is that if the function would be continuous on any point of the line with equation $x - y = 0$, then it would be bounded on at least one closed neighbourhood of that point which is clearly not the case.

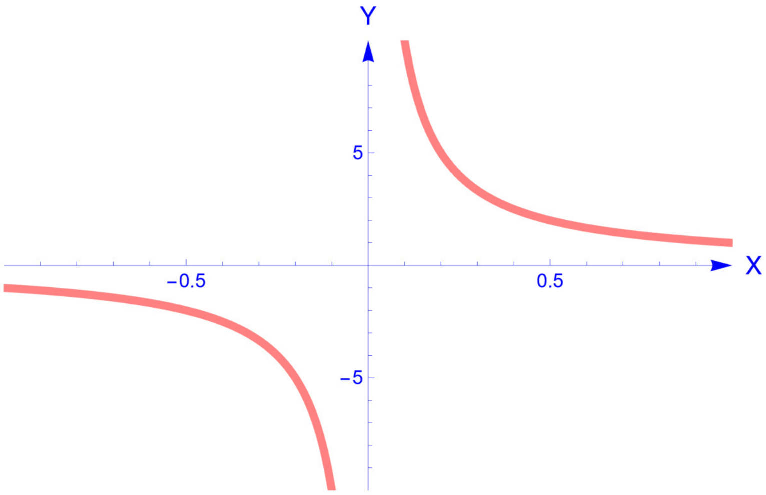


Figure 9. We see here a figure of the function restricted to the X-axis. Remark that the function is unbounded in a neighbourhood of $(0,0)$.

Exercise 5.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We choose an arbitrary $\epsilon > 0$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ holds then $|f(x,y) - f(0,0)| < \epsilon$ is valid.

By looking at our function, we have the following. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{x^2 y}{x^2+y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{x^2 y}{x^2+y^2}\right| &\leq \frac{x^2\,|y|}{x^2+y^2}\\ &\leq \frac{\sqrt{x^2+y^2}^2\sqrt{x^2+y^2}}{x^2+y^2}\\ &\leq \frac{\sqrt{x^2+y^2}^3}{x^2+y^2}\\ &\leq \sqrt{x^2+y^2}.\end{aligned}$$

We can manipulate now the value of δ in the inequality $\sqrt{x^2+y^2} < \delta$. In order to have $\sqrt{x^2+y^2} < \epsilon$, it is clearly enough to take $\delta = \epsilon$. We have found our δ and the function is continuous in $(0,0)$.

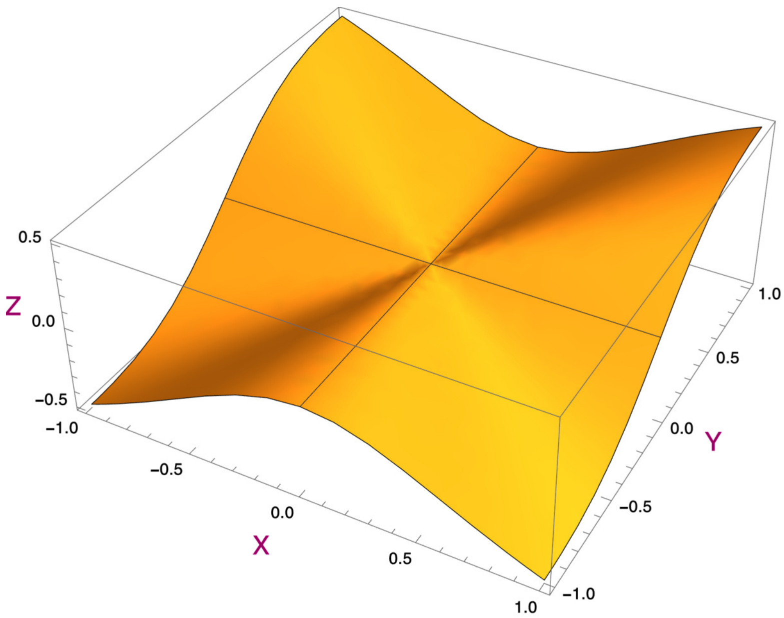


Figure 10. We see here a three dimensional figure of the graph of the function.

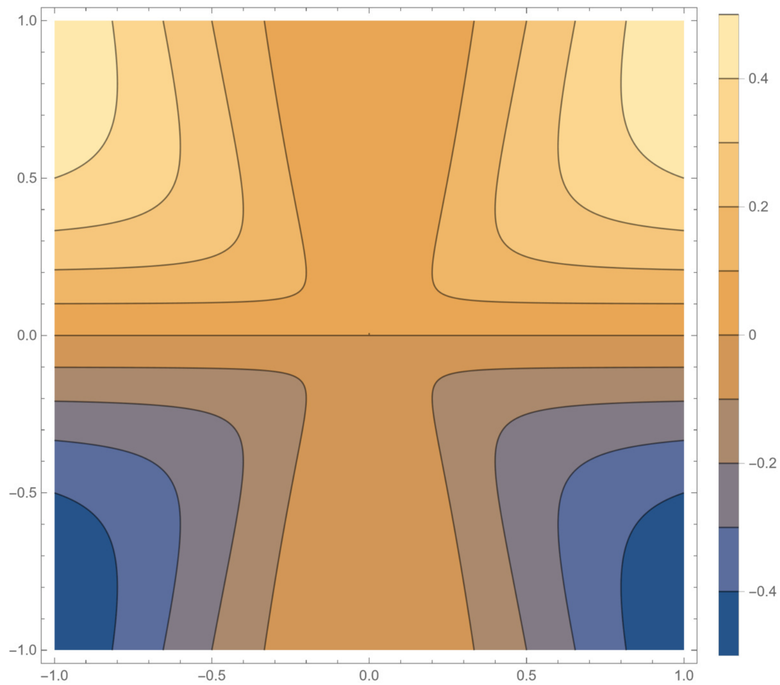


Figure 11. We see here a figure of the contour plot of the function. We see in this plot of the level curves that the level curve that come infinitesimally close to $(0,0)$ have a level close to 0. This looks like a classic picture the plot of the contour lines of a continuous point.

Exercise 6.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x\,y^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We look for an interesting curve on which the restriction of f behaves in a weird way. Let us take the standard parabola $x = \lambda\,y^2$. Our function is then

$$f|_{x=\lambda y^2}(x,y) = \begin{cases} f(\lambda y^2, y) = \dfrac{\lambda y^2 y^2}{\lambda^2 y^4 + y^4} = \dfrac{\lambda}{\lambda^2+1} & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So the limit for $y \to 0$ on this restricted function is

$$\lim_{y\to 0} f(\lambda y^2, y) = \frac{\lambda}{\lambda^2+1}$$

which takes many different values dependent on λ. But the limit should have the value 0 if the function is continuous. We conclude that the function is not continuous in $(0,0)$.

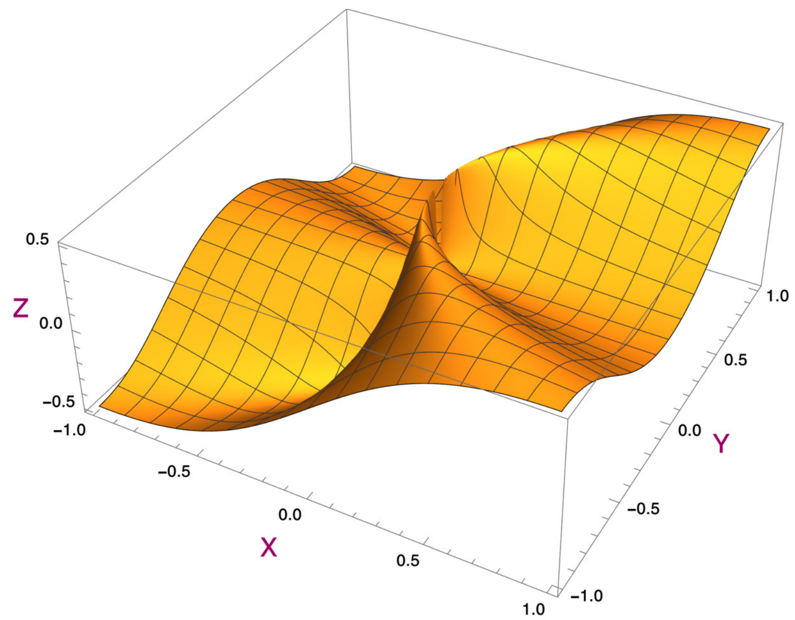


Figure 12. We see here a three dimensional figure of the graph of the function. The function behaves so weird that it is not easy to make a good drawing. We see again the vertical line above $(0,0)$. This line is caused by the limit points of the level curves.

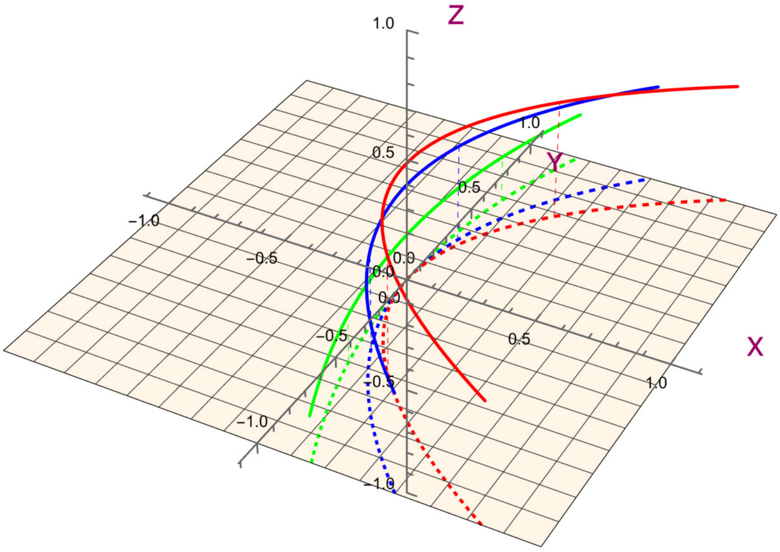


Figure 13. We have restricted the function here to $x = 4/5\,y^2$, $x = 1/5\,y^2$ and $x = 1/2\,y^2$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

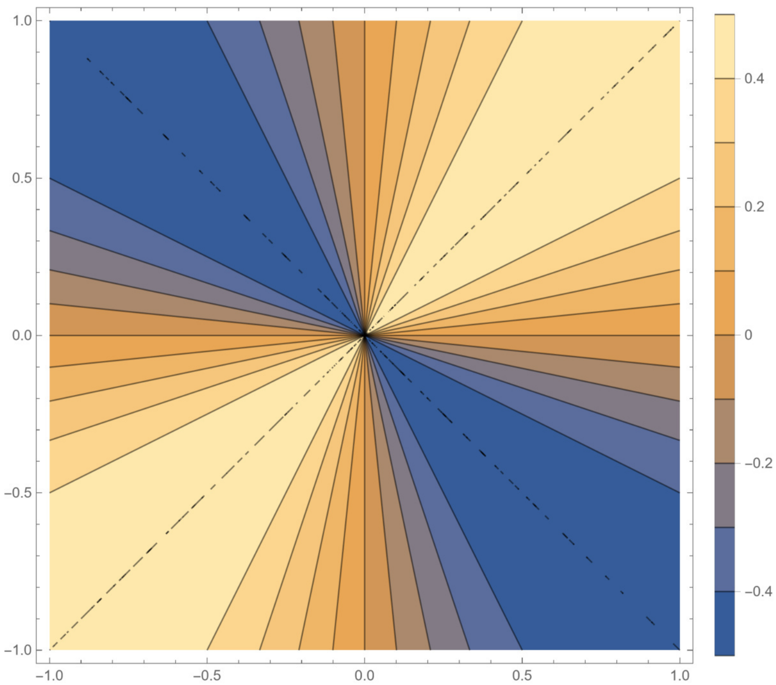


Figure 14. We see here a figure of the contour plot of the function. We see in this plot of the level curves that many level curves of different levels not close to zero come infinitesimally close to $(0,0)$. This makes continuity very suspicious.

Exercise 7.

Is the function

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^4} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We choose an arbitrary $\epsilon > 0$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

By looking at our function, we can translate this in the following. Try to find a δ such that if $||(x, y) - (0, 0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{x\,y^3}{x^2 + y^4} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

We observe for the inequality that we have
$a\,b \le (1/2)(a^2 + b^2)$ because this is equivalent with $0 \le (a - b)^2$.

$$\begin{aligned}
\left| \frac{x\,y^3}{x^2 + y^4} \right| &\le \left| \frac{x\,y^2\,y}{x^2 + y^4} \right| \\
&\le \left| \frac{(x^2 + y^4)\,y}{2\,(x^2 + y^4)} \right| \\
&\le \frac{|y|}{2} \\
&\le \frac{\sqrt{x^2 + y^2}}{2}.
\end{aligned}$$

We can manipulate now the value of δ in the inequality $\sqrt{x^2 + y^2} < \delta$. In order to have $1/2\sqrt{x^2 + y^2} < \epsilon$, it is clearly sufficient to take $\delta = 2\,\epsilon$. We have found a δ and the function is continuous in $(0, 0)$.

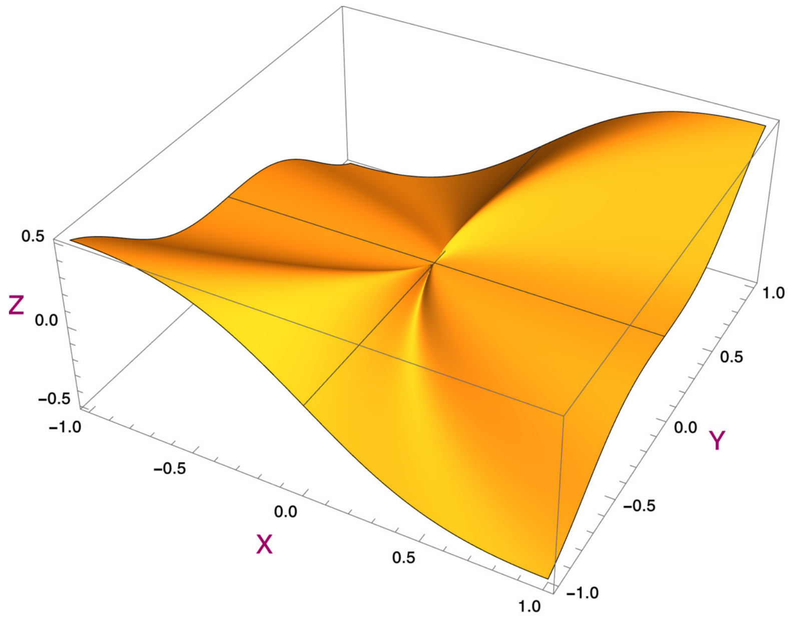


Figure 15. We see here a three dimensional figure of the graph of the function.

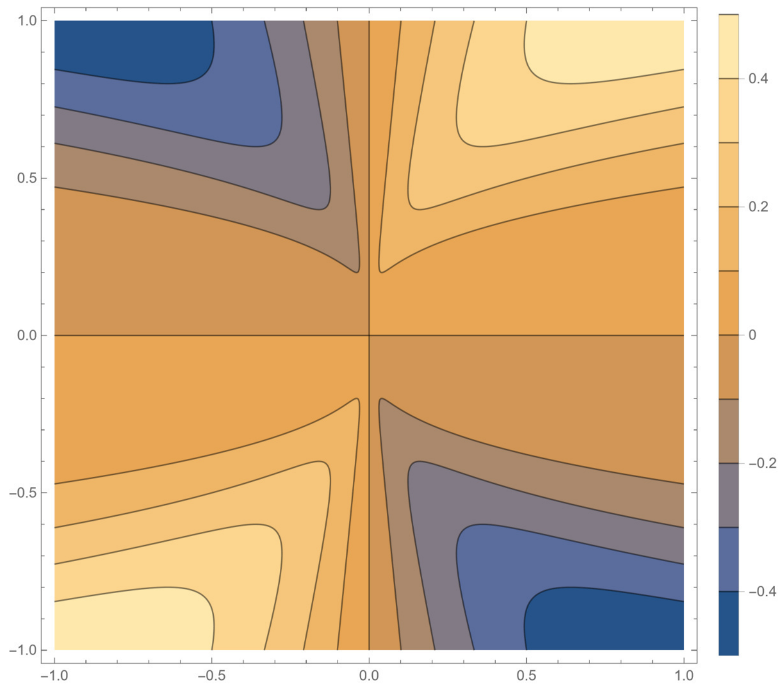


Figure 16. We see here a figure of the contour plot of the function. We see that level curves that come close to the point $(0,0)$ have levels that come close to 0. This looks like a classical continuity case.

Exercise 8.

Is the following function continuous in $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{x\,y}{3\,x^2 + y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Solution.

If we restrict in this case the function to the points on the continuous curve with equation $y = \lambda\,x$.

$$f\big|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda\,x) = \dfrac{\lambda\,x^2}{\lambda^2\,x^2 + 3\,x^2} = \dfrac{\lambda}{\lambda^2 + 3} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We have infinitely many different limit values for $x \to 0$. But we observe that if $f(x,y)$ is a continuous function, there should be only one solution for the limits: $f(0,0) = 0$. So the function is not continuous.

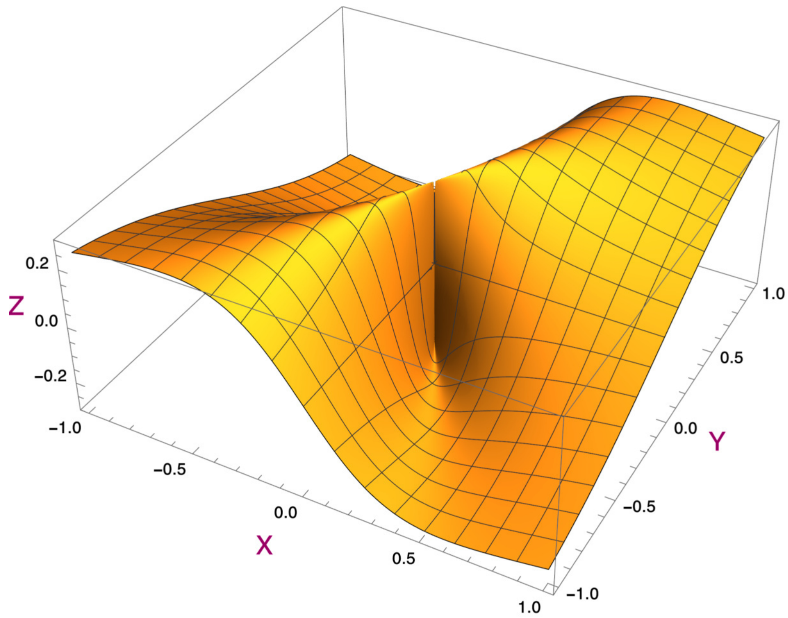


Figure 17. We see here a three dimensional figure of the graph of the function.

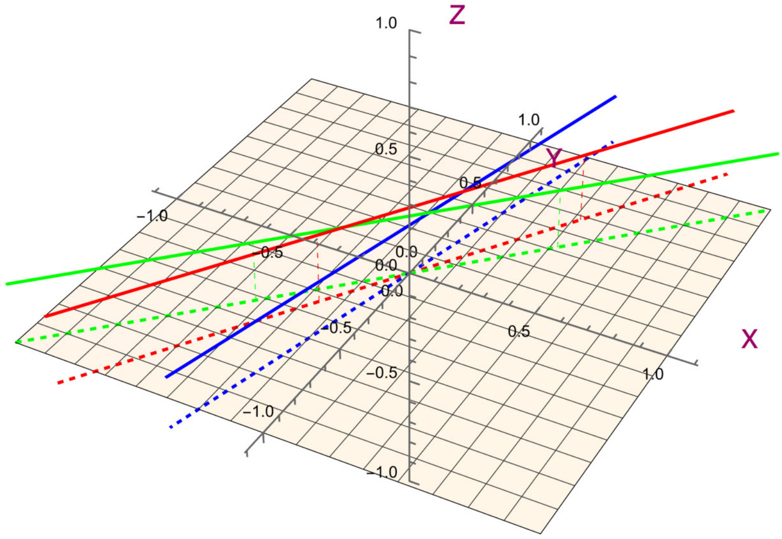


Figure 18. We have restricted the function here to $y = 3/2\,x$ and $y = x$ and $y = 4\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

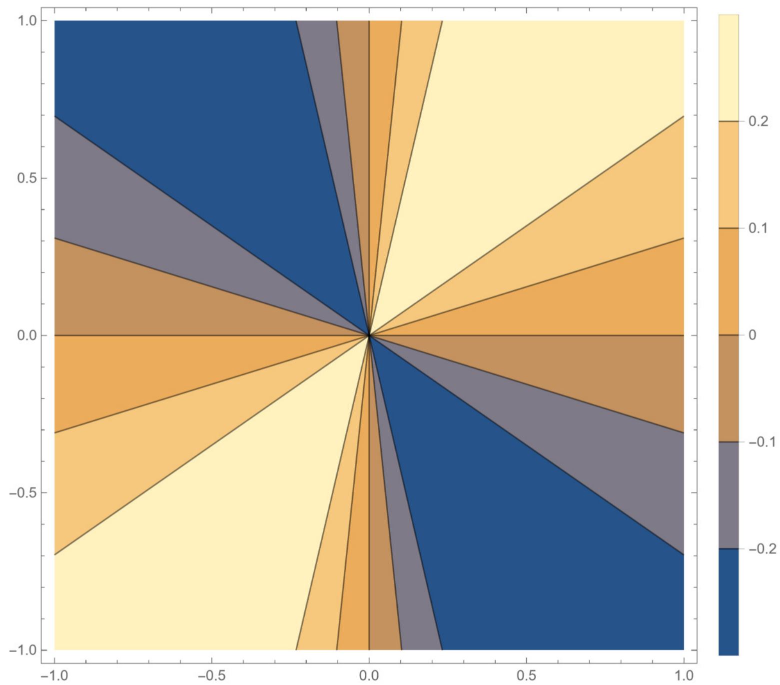


Figure 19. We see here a figure of the plot of the contour lines of the function. Many level curves of a level not equal to zero come infinitesimally close to $(0,0)$. This is very suspicious and the picture does not look like a picture of a continuous point.

Exercise 9.

Is the following function continuous in $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{x\,y\,\cos(y)}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Solution.

If we restrict this function to the continuous curves with equation $y = \lambda x$, then

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{\lambda\cos(\lambda x)}{\lambda^2+1} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So the limit is then

$$\lim_{x\to 0}(x,\lambda x) = \frac{\lambda}{\lambda^2+1}.$$

There are infinitely many limit values. They should be both equal to $f(0,0) = 0$ if the function f is a continuous function. So the function is discontinuous in $(0,0)$.

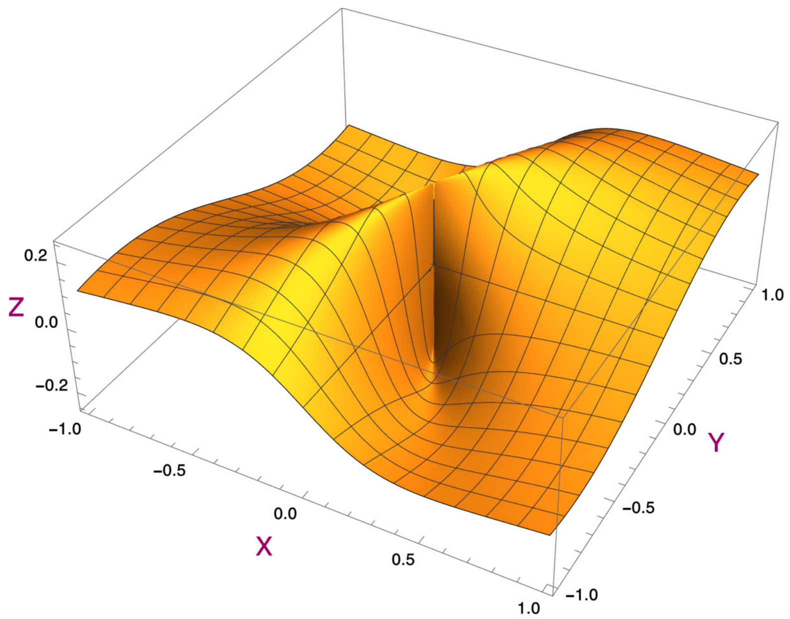


Figure 20. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ gives an indication of non continuity.

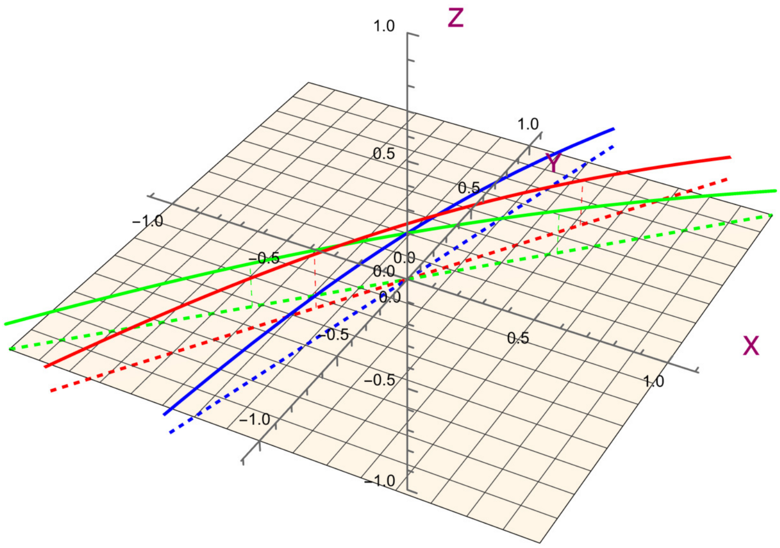


Figure 21. We have restricted the function here to $y = 1/2\,x$ and $y = x$ and $y = 2x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

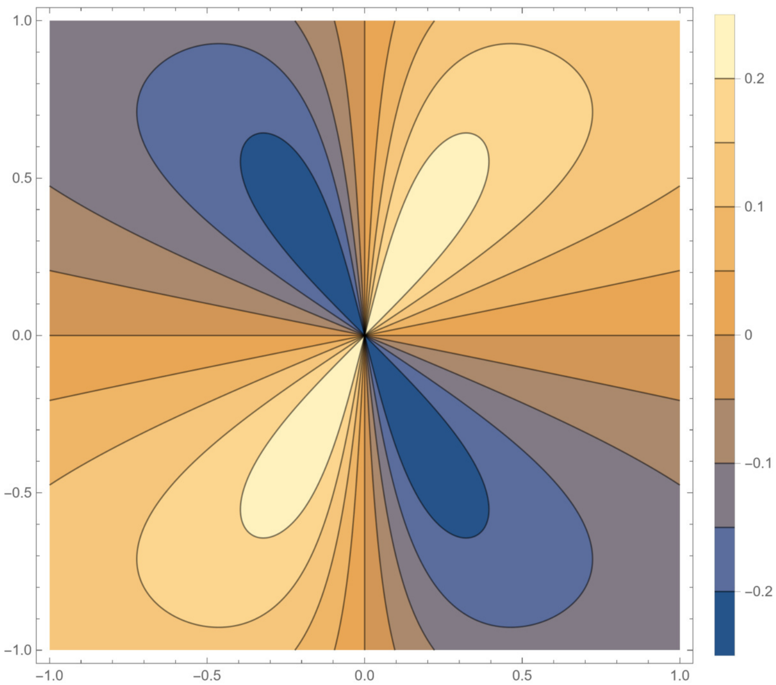


Figure 22. We see here a figure of the contour plot of the function. Many level curves of very different levels come arbitrarily close to $(0,0)$. This is exactly the phenomenon we described in our solution. This is intuitively an almost sure sign of non continuity.

Exercise 10.

Is the following function continuous in $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{e^{x^2+y^2}-1}{x+y} & \text{if } x \neq -y; \\ 0 & \text{if } x = -y. \end{cases}$$

It is allowed to use the traditional inequality: $e^x - 1 \geq x$ which says basically that the tangent line in $x = 0$ at the graph of $f(x) = e^x$ lies under the graph of $f(x)$. We remark that this inequality can be proven by elementary means and does not require the heavy machinery of derivatives. So no series are involved though one can surely produce an easy and elegant proof of this inequality with these tools. The reader can read more information about this inequality in the appendix of this text.

Solution.
We might expect that the behaviour of this function in a neighbourhood of $(0,0)$ is unbounded. In order to investigate that, we take a curve that is strongly tangent to the line with equation $x+y=0$. Let us choose the curve $y=-x+x^3$. Let us restrict the function $f(x,y)$ to this curve.

$$f|_{y=-x+x^3}(x,y)=\begin{cases} f(x,-x+x^3)=\frac{e^{x^2+(x-x^3)^2}-1}{x^3} & \text{if } (x,y)\neq(0,0);\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

We are going to take the limit for $x \stackrel{>}{\to} 0$ of this function. We use the inequality for $x \geq 0$.

$$0 \leq \frac{x^2+(x-x^3)^2}{x^3} \leq \frac{e^{x^2+(x-x^3)^2}-1}{x^3}.$$

We take the limits.

$$\lim_{x \stackrel{>}{\to} 0} 0 \leq \lim_{x \stackrel{>}{\to} 0} \frac{x^2+(x-x^3)^2}{x^3} \leq \lim_{x \stackrel{>}{\to} 0} \frac{e^{x^2+(x-x^3)^2}-1}{x^3}.$$

Because $\lim_{x \stackrel{>}{\to} 0} \frac{x^2+(x-x^3)^2}{x^3}=\infty$, we have unboundedness in any neighbourhood of $(0,0)$ and the function is not continuous in $(0,0)$.

Note that the exponent of the second term in x in the equation of the curve $y=-x+x^3$ is quite arbitrary. It can be changed at leisure. We can take any $n>2$. In that case we have $y=-x+x^n$. The larger the n, the stronger is the tangential behaviour and the behaviour of unboundedness.

Let us check these results with the plot of the level curves.

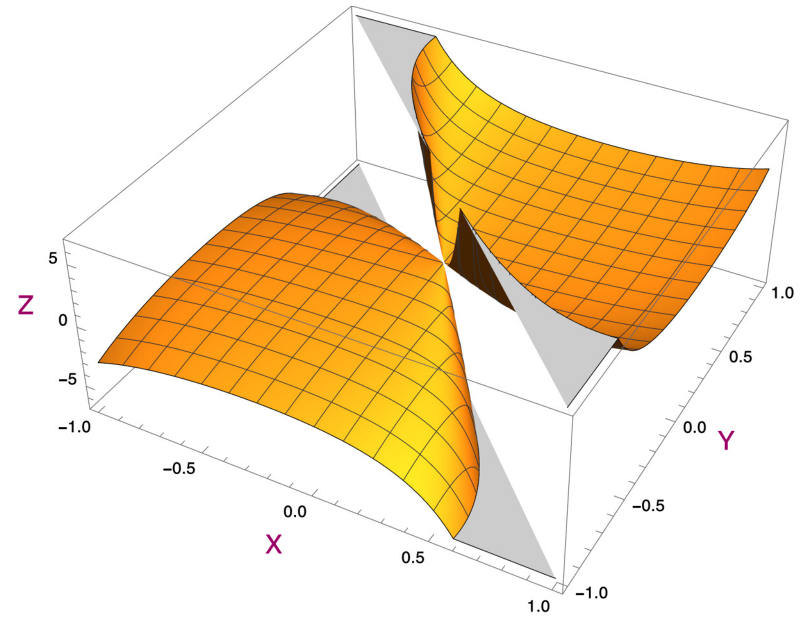


Figure 23. We see here a three dimensional figure of the graph of the function.

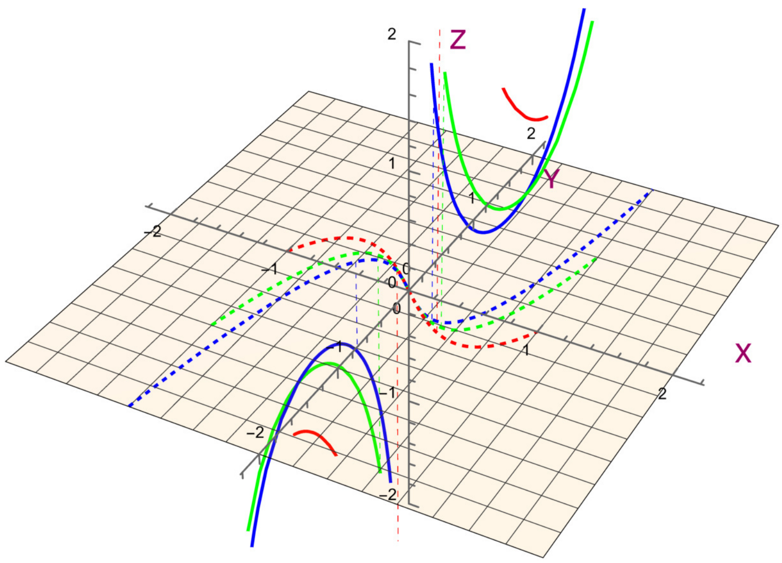


Figure 24. We have restricted the function here to $y = -x + x^2$ and $y = -x + 2x^2$ and $y = -x + 3x^2$. We see in this figure clearly that the restrictions of the function to these lines are functions that have an unbounded behaviour in 0.

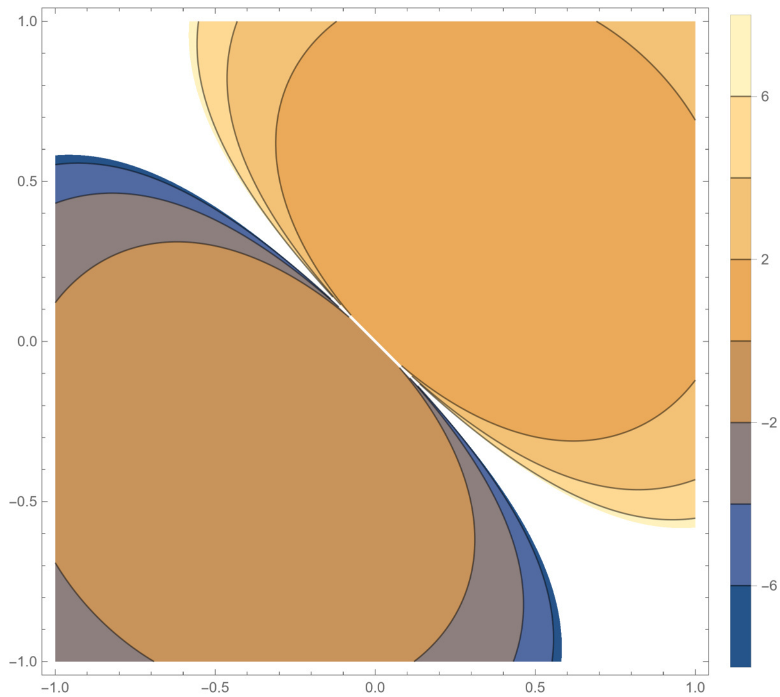


Figure 25. We see here a figure of the contour plot of the function. Many level curves of very different levels come arbitrarily close to $(0,0)$. This is exactly the phenomenon we described in our solution. This is intuitively an almost sure sign of discontinuity.

Exercise 11.

Is the function

$$f(x,y) = \begin{cases} \dfrac{5x^2y - y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying this to our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0, 0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{5\,x^2\,y - y^3}{x^2 + y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{5\,x^2\,y - y^3}{x^2 + y^2} - 0 \right| &\leq \left| \frac{y\,(5\,x^2 - y^2)}{x^2 + y^2} \right| \\
&\leq \left| \frac{y\,(5\,x^2 + y^2)}{x^2 + y^2} \right| \\
&\leq \left| \frac{y\,(5\,x^2 + 5\,y^2)}{x^2 + y^2} \right| \\
&\leq 5\,|y| \\
&\leq 5\sqrt{x^2 + y^2}.
\end{aligned}$$

So it is sufficient to prove that $5\sqrt{x^2 + y^2} < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2 + y^2} < \delta$. It is for example sufficient to take $\delta = \epsilon/5$. We can find a δ, so we conclude that the function is continuous.

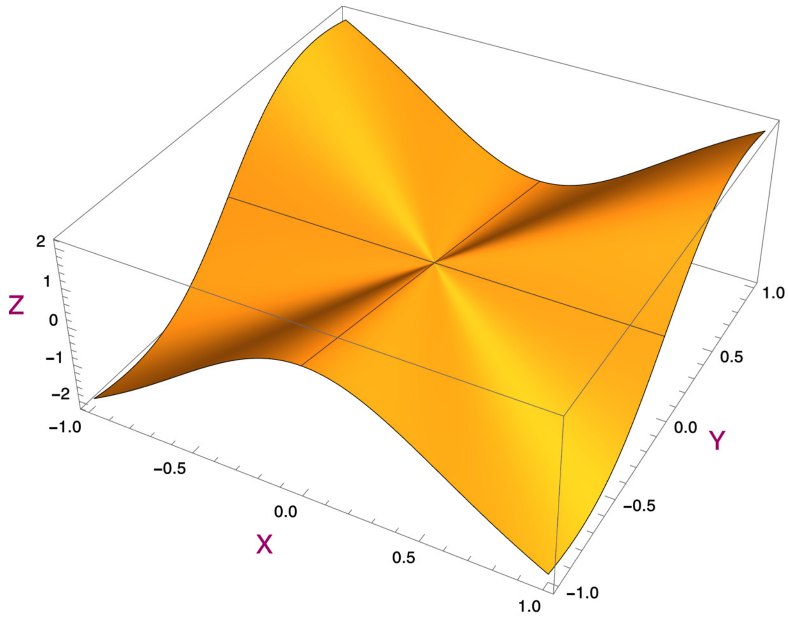


Figure 26. We see here a three dimensional figure of the graph of the function.

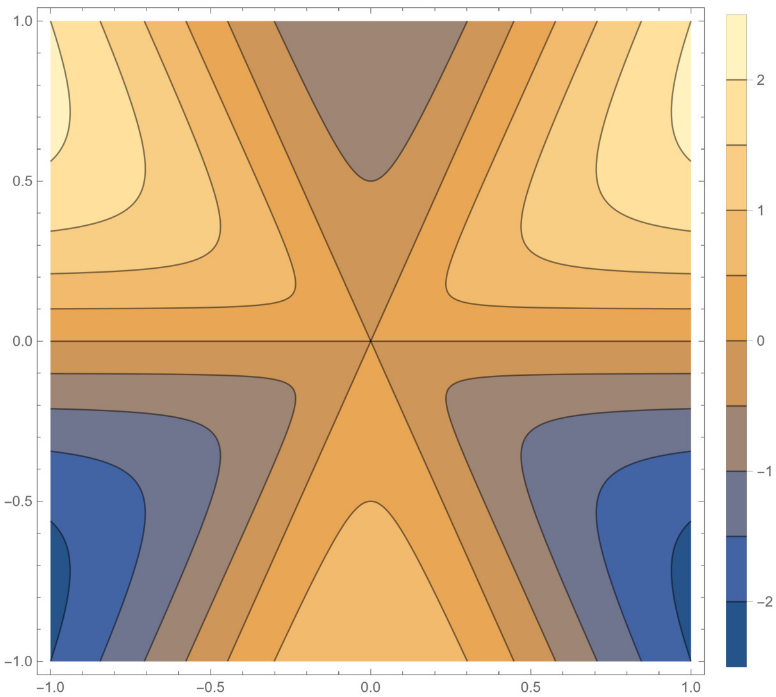


Figure 27. We see here a figure of the contour plot of the function. All level curves close to $(0,0)$ have a level close to level 0. This is intuitively an almost sure sign of continuity.

Exercise 12.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x^3y^2}{x^4+y^4} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0, 0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{x^3\, y^2}{x^4 + y^4} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{x^3\, y^2}{x^4 + y^4} - 0 \right| &\leq \left| \frac{x\, x^2\, y^2}{x^4 + y^4} \right| \\
&\leq \left| \frac{x\,(x^4 + y^4)^{2/4}\,(x^4 + y^4)^{2/4}}{x^4 + y^4} \right| \\
&\leq |x| \\
&\leq \sqrt{x^2 + y^2}.
\end{aligned}$$

We have made use of the fact that $|x| \leq (x^4 + y^4)^{1/4}$.

So it is sufficient to prove that $\sqrt{x^2 + y^2} < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2 + y^2} < \delta$. It is for example sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function is continuous.

There is another elegant way to prove it. We remark that $2\,a\,b \leq a^2 + b^2$. So

$$\begin{aligned}
\left| \frac{x^3\, y^2}{x^4 + y^4} - 0 \right| &\leq \left| \frac{x\, x^2\, y^2}{x^4 + y^4} \right| \\
&\leq \left| \frac{x\; 1/2\,(x^4 + y^4)}{x^4 + y^4} \right| \\
&\leq 1/2\,|x| \\
&\leq 1/2\,\sqrt{x^2 + y^2}.
\end{aligned}$$

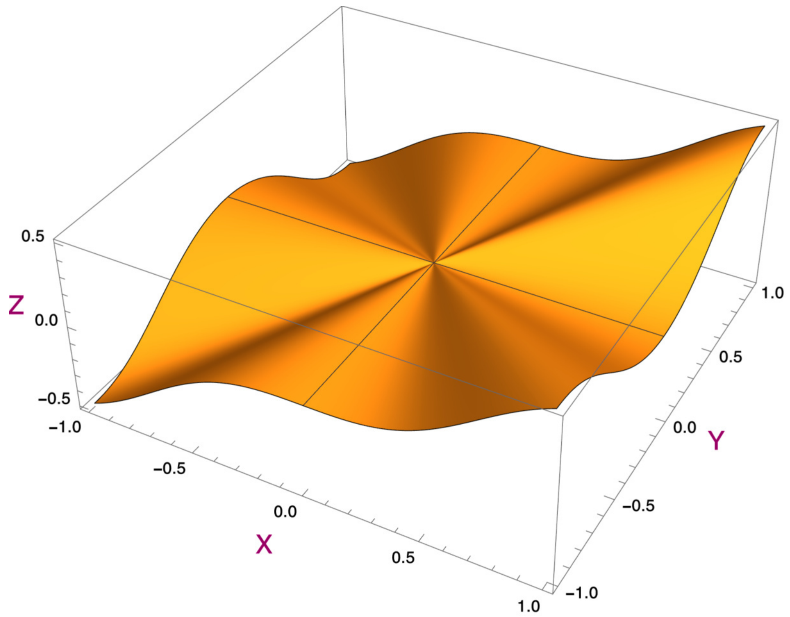


Figure 28. We see here a three dimensional figure of the graph of the function.

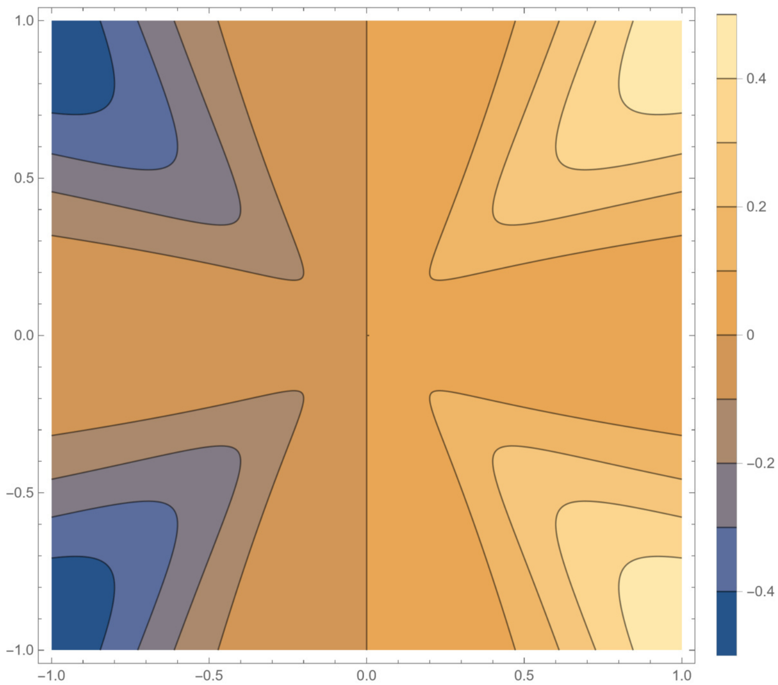


Figure 29. We see here a figure of the contour plot of the function. All level curves close to $(0, 0)$ have a level close to level 0. This is intuitively an almost sure sign of continuity.

Exercise 13.

Is the function

$$f(x, y) = \begin{cases} \dfrac{\sin(xy)}{x+y} & \text{if } x + y \neq 0; \\ 0 & \text{if } x = -y \end{cases}$$

continuous in $(0, 0)$?

Solution.

We restrict the function to the curve $y = -x + x^3$. This is a curve tangent to the line with equation $x + y = 0$. We compute now the restriction.

$$f|_{y=-x+x^3}(x,y) = \begin{cases} f(x,-x+x^3) = \dfrac{\sin(x\,(x^3-x))}{x^3} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Let us try to estimate the behaviour of $f|_{y=-x+x^3}(x,y)$. We know that $\lim_{\alpha\to 0}\sin(\alpha)/\alpha = 1$. We have consequently for small values of x that $|1/2\,x| \leq |\sin(x)|$, we see that

$$\begin{aligned} \left|\frac{\sin(x\,(x^3-x))}{x^3}\right| &\geq \left|\frac{x\,(x^3-x)}{2\,x^3}\right| \\ &\geq \left|\frac{x}{2} - \frac{1}{2\,x}\right|. \end{aligned}$$

We take the limit for $x \to 0$ in these inequalities and we see immediately that the function $f(x,-x+x^3)$ is unbounded in any small neighbourhood of $x = 0$. This function cannot be continuous because in that case the limit should be $f(0,0) = 0$.

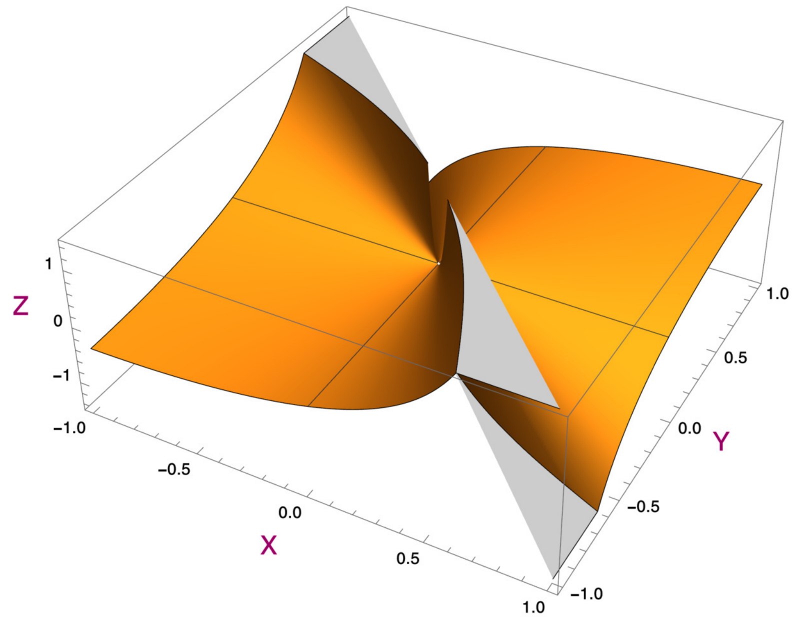


Figure 30. We see here a three dimensional figure of the graph of the function. The function is not bounded in a neighbourhood of $(0,0)$.

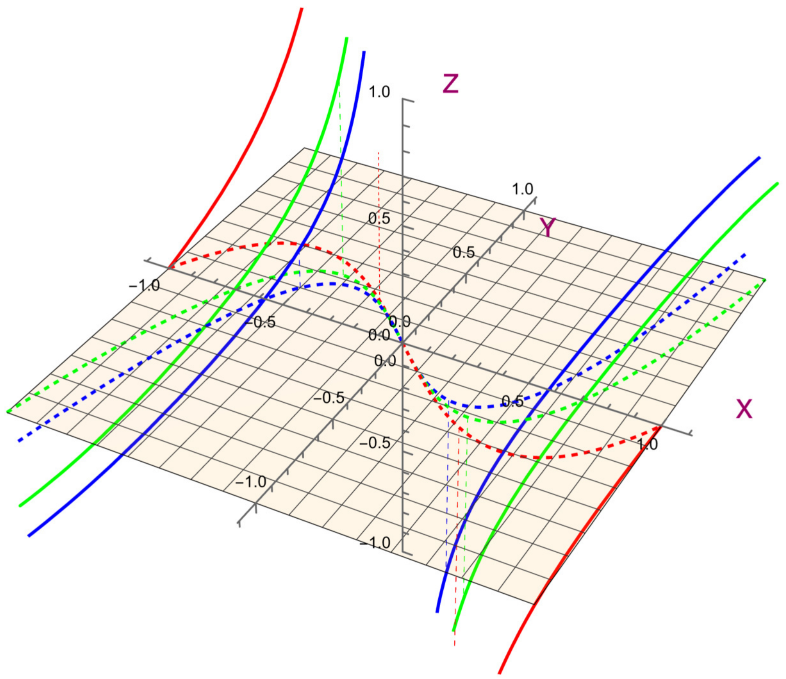


Figure 31. We have restricted the function here to $y = -x + x^2$ and $y = -x + 2x^2$ and $y = -x + 3x^2$. We see in this figure clearly that the restrictions of the function to these lines are functions that that have unbounded behaviour in 0.

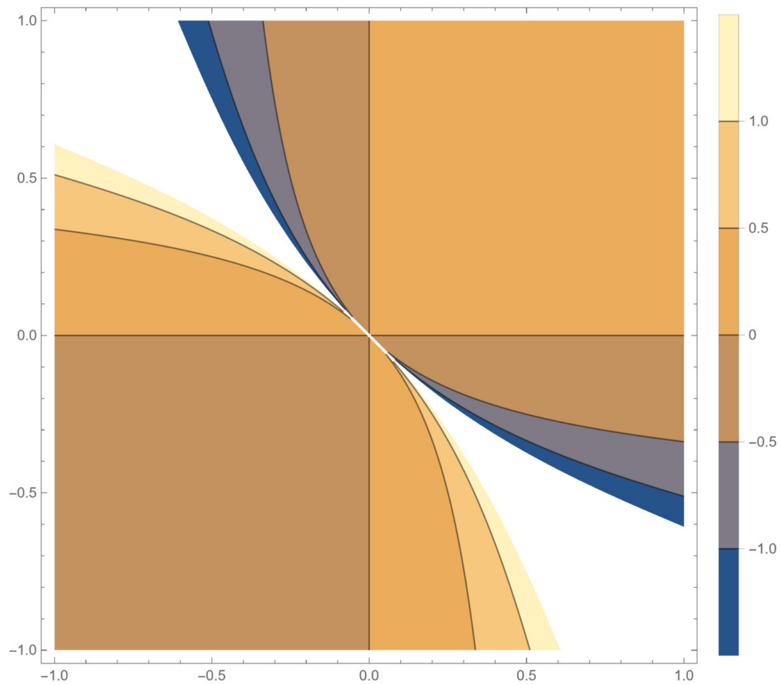


Figure 32. We see here a figure of the contour plot of the function. There are many level curves close to $(0, 0)$ of very different levels. This is intuitively an almost sure sign of non continuity.

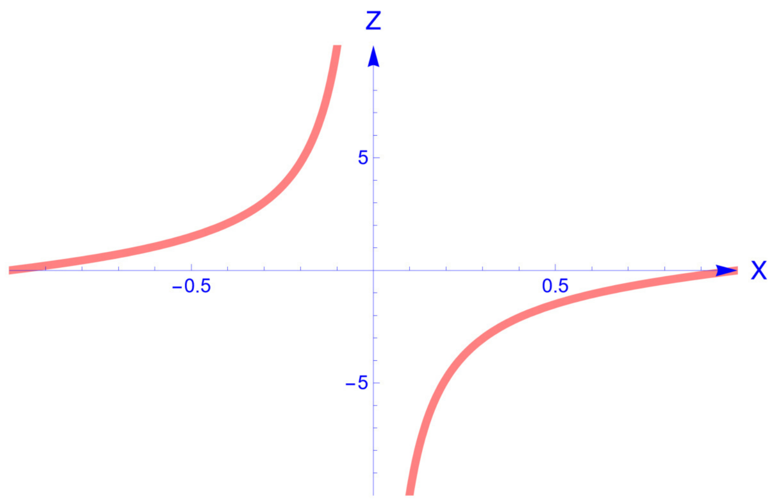


Figure 33. We see here a figure of the plot of the function $f|_{y=-x+x^3}(x, y)$. This illustrates the unbounded behaviour in a neighbourhood of 0.

Exercise 14.

Is the function

$$f(x,y) = \begin{cases} \dfrac{\sin(x\,y)}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

Because we know that $\lim_{\alpha\to 0}\sin(\alpha)/\alpha = 1$, we know that $\sin(\alpha) \approx \alpha$ for small values α and we see that we have essentially the function $x\,y/(x^2+y^2)$. We dealt with this function before. This is of course pure intuition and we have to use some stronger formal argument.

Let us restrict the function to lines $y = \lambda\,x$ through the origin. We have then

$$f\big|_{y=\lambda\,x}(x,y) = \begin{cases} f(x,\lambda\,x) = \dfrac{\sin(x\,\lambda\,x)}{x^2+\lambda^2\,x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Let us calculate the limit when $x \to 0$. We will use in the calculation that $\lim_{\alpha\to 0}\sin(\alpha)/\alpha = 1$.

$$\begin{aligned} \lim_{x\to 0}\frac{\sin(x\,\lambda\,x)}{x^2+\lambda^2\,x^2} &= \lim_{x\to 0}\frac{\sin(\lambda\,x^2)}{\lambda\,x^2}\,\frac{\lambda\,x^2}{x^2+\lambda^2\,x^2} \\ &= \lim_{x\to 0}\frac{\lambda\,x^2}{x^2+\lambda^2\,x^2} \\ &= \lim_{x\to 0}\frac{\lambda}{1+\lambda^2}. \end{aligned}$$

There are infinitely many limit values depending on the value of λ. If the function is continuous, there can be only one value and that is $f(0,0) = 0$. We conclude that the function is not continuous.

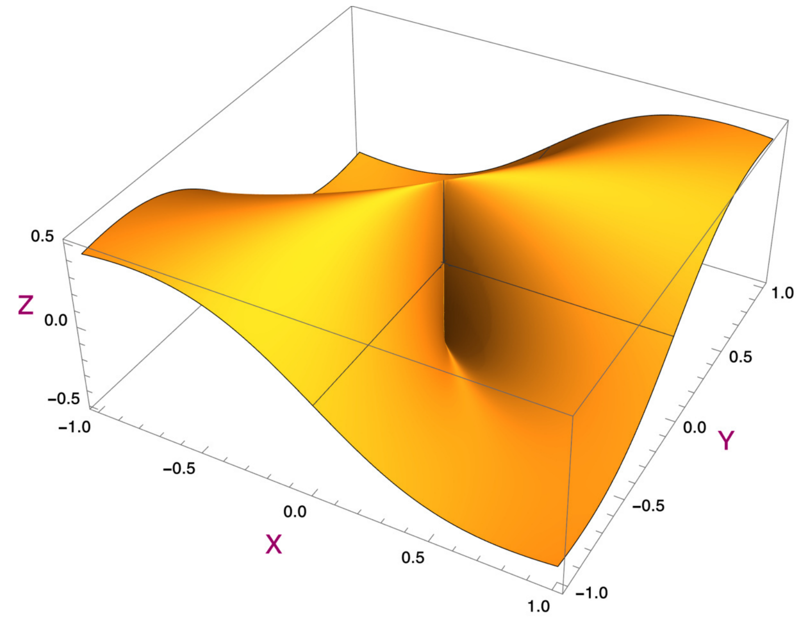


Figure 34. We see here a three dimensional figure of the graph of the function. The graph shows a vertical line above $(0, 0)$. This is an almost sure visual sign of discontinuity.

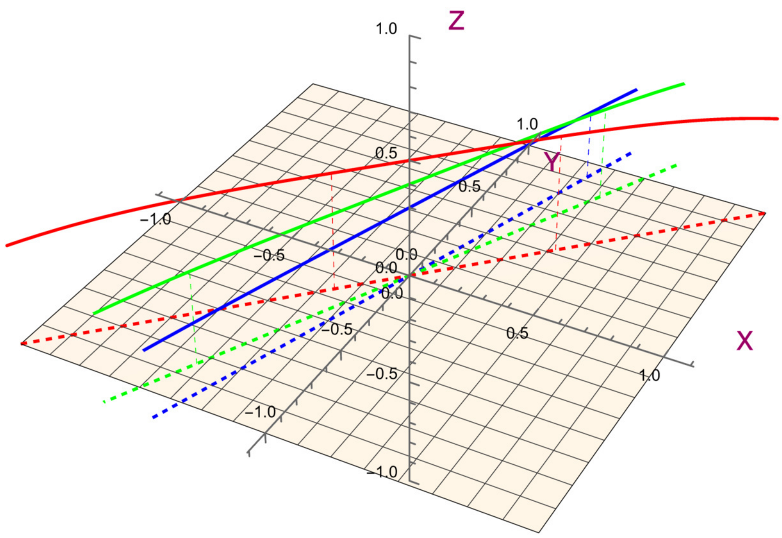


Figure 35. We have restricted the function here to $y = x$ and $y = 2x$ and $y = 3x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

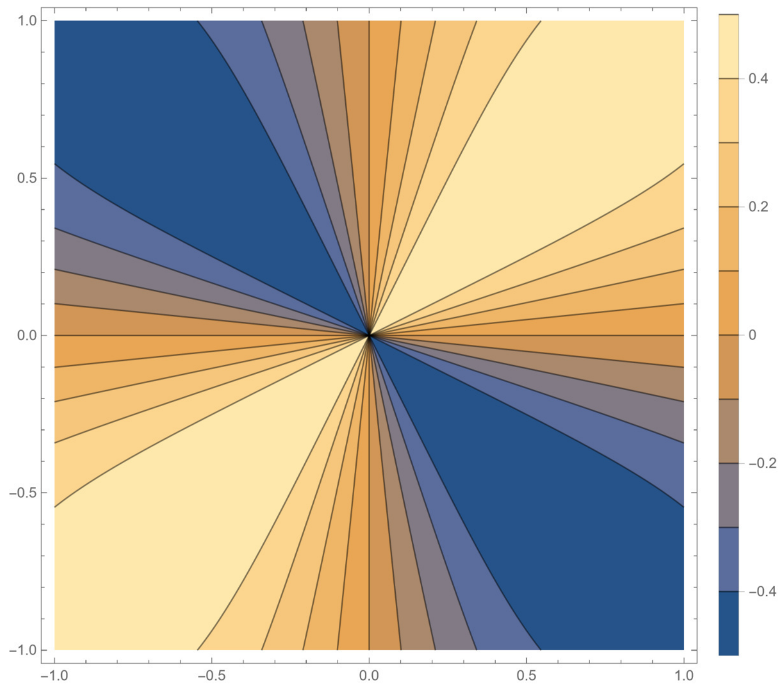


Figure 36. We see here a figure of the contour plot of the function. There are many level curves infinitesimally close to $(0, 0)$ of very different levels. This is intuitively an almost sure sign of non continuity.

Exercise 15.

Is the function

$$f(x, y) = \begin{cases} \dfrac{5x^2y - y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

continuous in $(0, 0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x, y) - f(0, 0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x, y) - (0, 0)|| < \delta$ it follows that $|f(x, y) - f(0, 0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left|\frac{5\,x^2\,y - y^3}{x^2+y^2} - 0\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left|\frac{5\,x^2\,y - y^3}{x^2+y^2} - 0\right| &\le \left|\frac{y\,(5\,x^2 - y^2)}{x^2+y^2}\right| \\
&\le \left|\frac{y\,(5\,x^2 + y^2)}{x^2+y^2}\right| \\
&\le \left|\frac{y\,(5\,x^2 + 5\,y^2)}{x^2+y^2}\right| \\
&\le 5\,|y| \\
&\le 5\sqrt{x^2+y^2}.
\end{aligned}$$

So it is sufficient to prove that $5\sqrt{x^2+y^2} < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2+y^2} < \delta$. It is for example sufficient to take $\delta = \epsilon/5$. We can find a δ, so we conclude that the function is continuous.

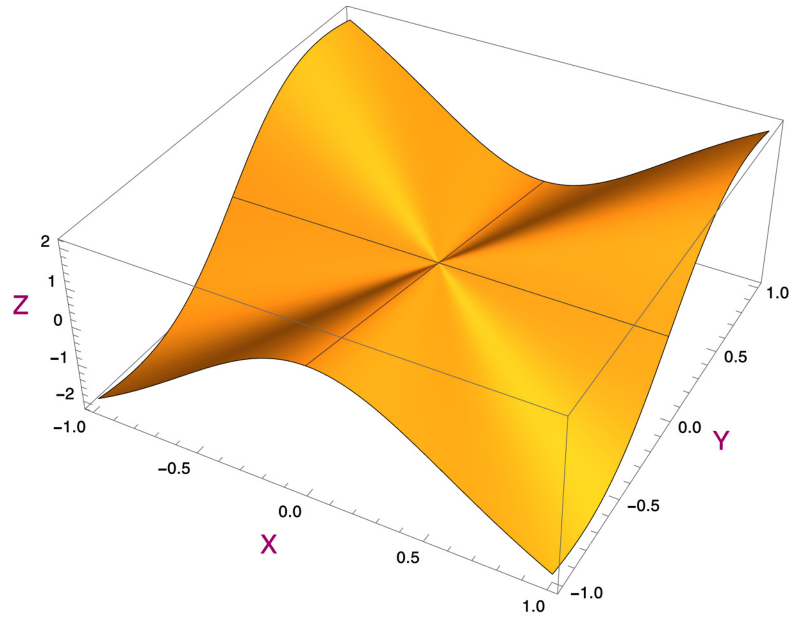


Figure 37. We see here a three dimensional figure of the graph of the function.

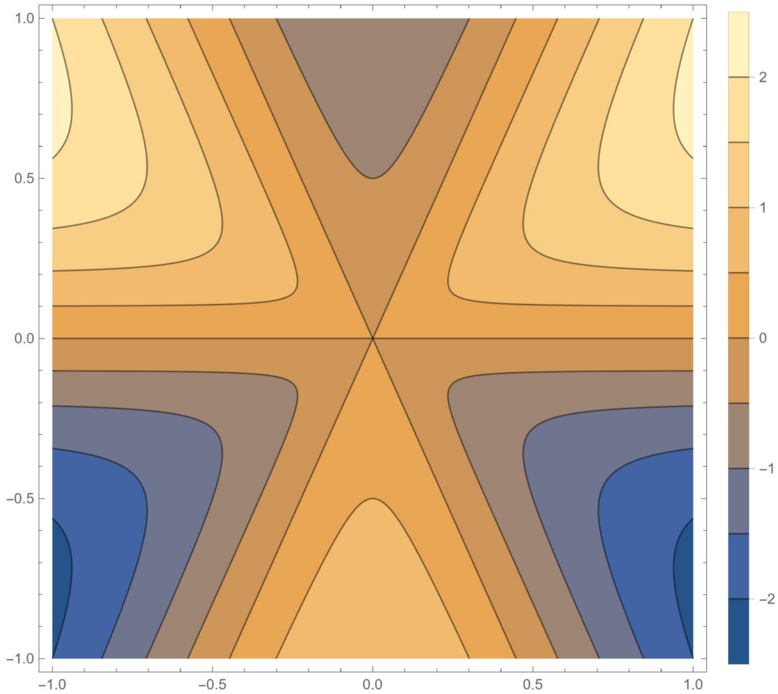


Figure 38. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 16.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left|\frac{x^2y^2}{x^2+y^2} - 0\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left|\frac{x^2y^2}{x^2+y^2} - 0\right| &\leq \left|\frac{\sqrt{x^2+y^2}^2\sqrt{x^2+y^2}^2}{x^2+y^2}\right| \\
&\leq \left|\frac{\sqrt{x^2+y^2}^4}{x^2+y^2}\right| \\
&\leq \sqrt{x^2+y^2}^2.
\end{aligned}$$

We have made use of the fact that $|x| \leq (x^2+y^2)^{1/2}$.

So it is sufficient to prove that $\sqrt{x^2+y^2}^2 < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2+y^2} < \delta$. It is for example sufficient to take $\delta = \sqrt{\epsilon}$. We can find a δ, so we conclude that the function is continuous.

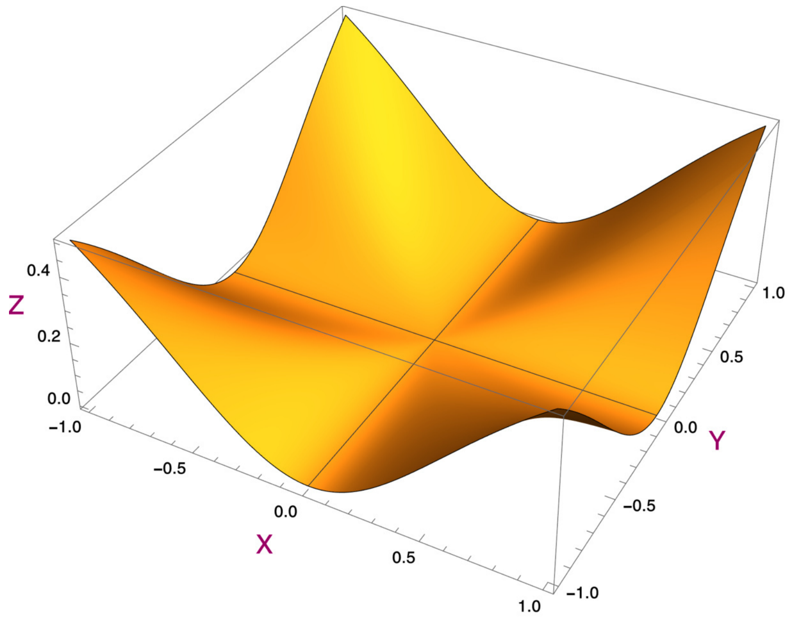


Figure 39. We see here a three dimensional figure of the graph of the function.

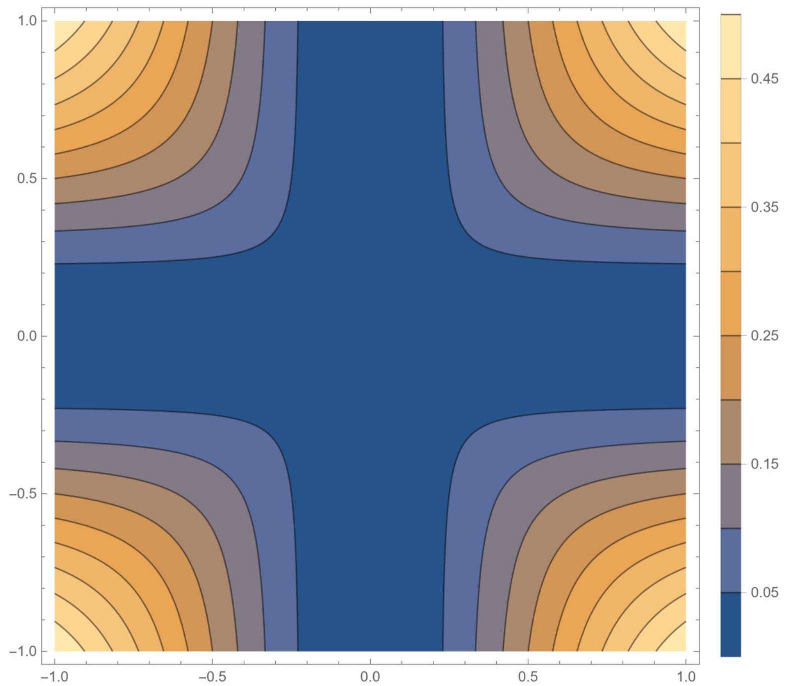


Figure 40. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 17.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{x^3 + y^3}{x^2 + y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{x^3 + y^3}{x^2 + y^2} - 0 \right| &\le \frac{|x|^3 + |y|^3}{x^2 + y^2} \\
&\le \frac{\sqrt{x^2 + y^2}^3 + \sqrt{x^2 + y^2}^3}{x^2 + y^2} \\
&\le \frac{2\sqrt{x^2 + y^2}^3}{x^2 + y^2} \\
&\le 2\sqrt{x^2 + y^2}.
\end{aligned}$$

We have made use of the fact that $|x| \le (x^2 + y^2)^{1/2}$.

So it is sufficient to prove that $2\sqrt{x^2 + y^2}^2 < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2 + y^2} < \delta$. It is for example sufficient to take $\delta = \epsilon/2$. We can find a δ, so we conclude that the function is continuous.

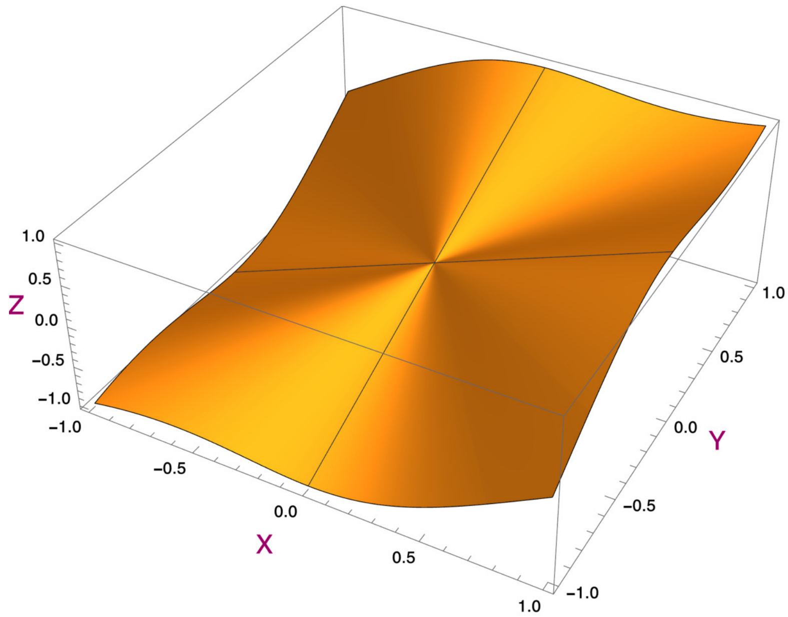
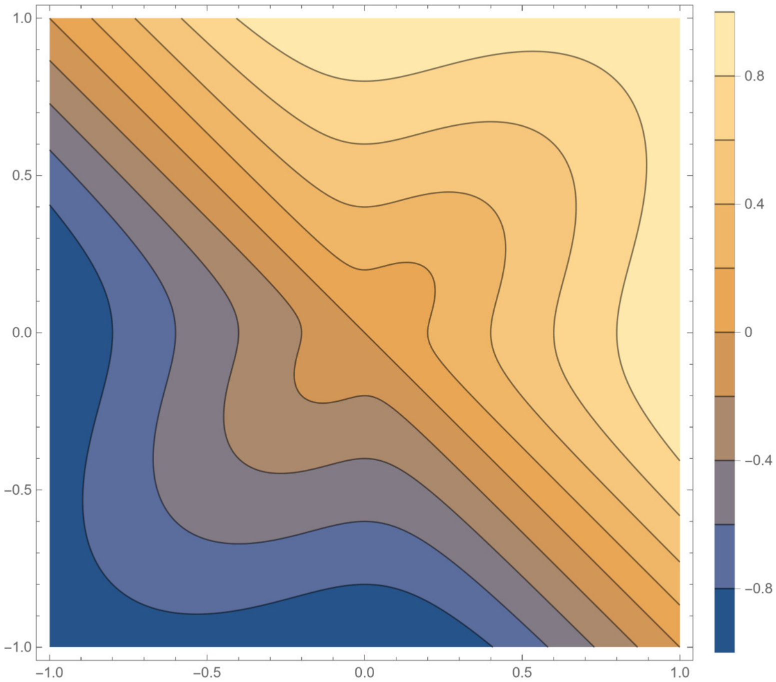


Figure 41. We see here a three dimensional figure of the graph of the function.

Figure 42. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.



Exercise 18.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^4} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We restrict the function to the curve $y = \lambda x$.

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{x^3 \lambda x}{x^4 + \lambda^4 x^4} = \dfrac{\lambda}{1+\lambda^4} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

$$\lim_{x\to 0} \frac{\lambda\, x^4}{x^4\,(1+\lambda^4)} = \frac{\lambda}{1+\lambda^4}.$$

We have an infinite amount of limit values depending upon the value of λ.

This function cannot be continuous because in that case all those limits should be $f(0,0) = 0$.

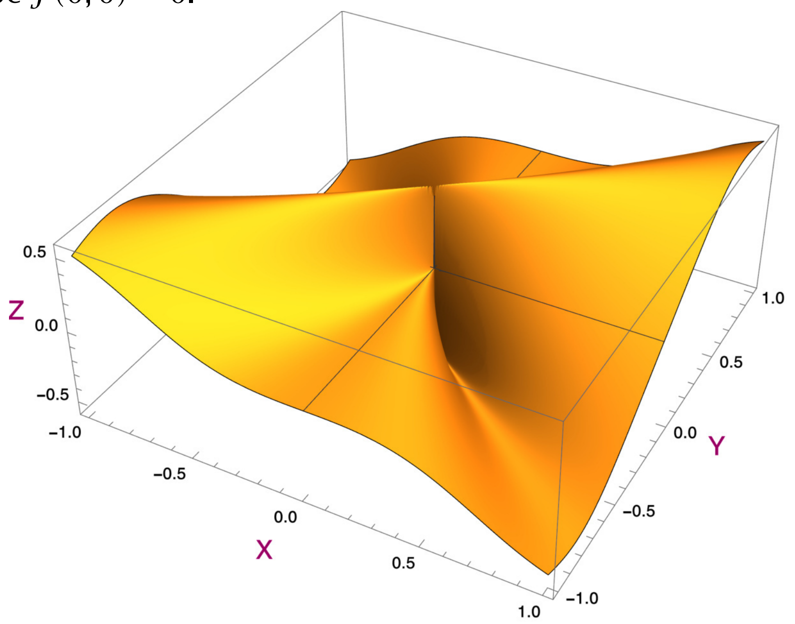


Figure 43. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks very suspicious. This does not seem to be a graph of a continuous function.

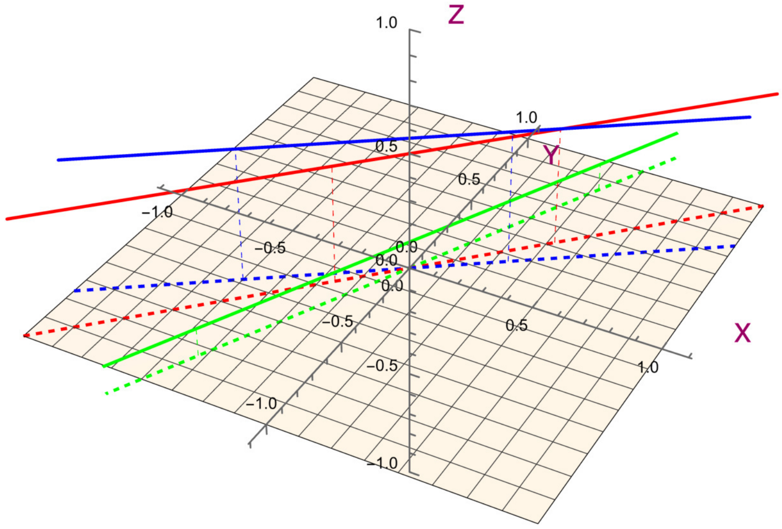


Figure 44. We have restricted the function here to $y = x$ and $y = 2x$ and $y = 7/10\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

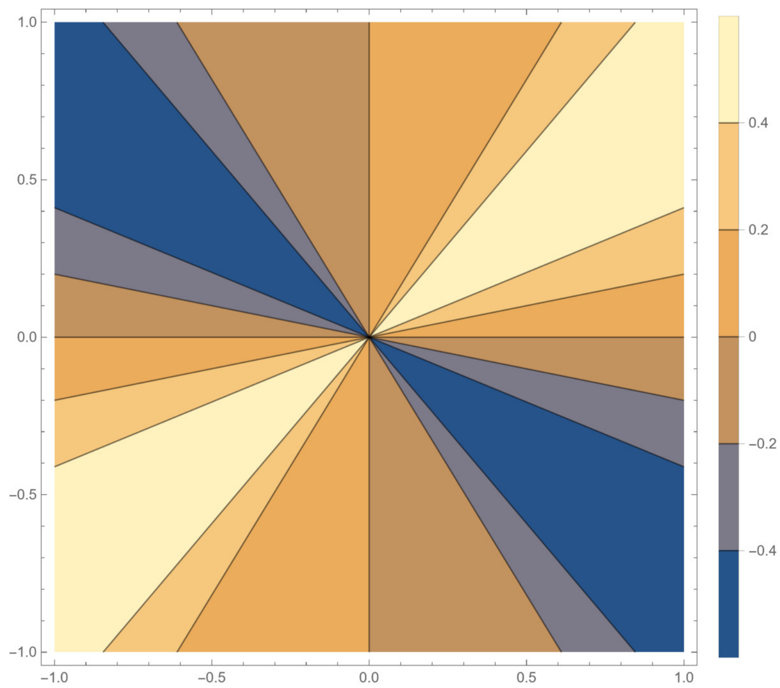


Figure 45. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0, 0)$. This looks very discontinuous indeed.

Exercise 19.

Is the function

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

continuous in $(0, 0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$. The problem is now to find a $\delta > 0$ such that if $||(x, y) - (0, 0)|| < \delta$ it follows that $|f(x, y) - f(0, 0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0, 0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{x^3 - y^3}{x^2 + y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned} \left| \frac{x^3 - y^3}{x^2 + y^2} - 0 \right| &\leq \left| \frac{|x|^3 + |y|^3}{x^2 + y^2} \right| \\ &\leq \left| \frac{\sqrt{x^2 + y^2}^3 + \sqrt{x^2 + y^2}^3}{x^2 + y^2} \right| \\ &\leq 2\sqrt{x^2 + y^2}. \end{aligned}$$

We have made use of the fact that $|x| \leq (x^2 + y^2)^{1/2}$.

So it is sufficient to prove that $2\sqrt{x^2 + y^2} < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2 + y^2} < \delta$. It is for example sufficient to take $\delta = \epsilon/2$. We can find a δ, so we conclude that the function is continuous.

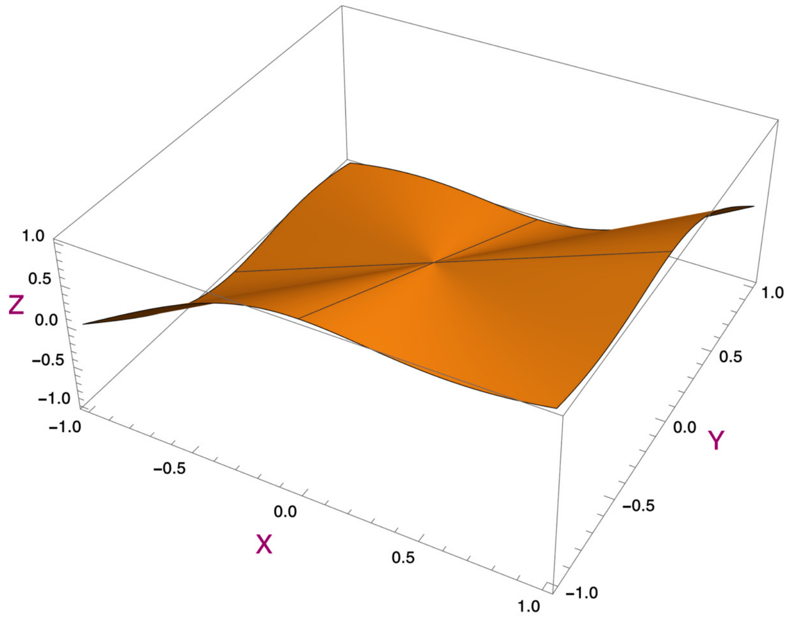


Figure 46. We see here a three dimensional figure of the graph of the function.

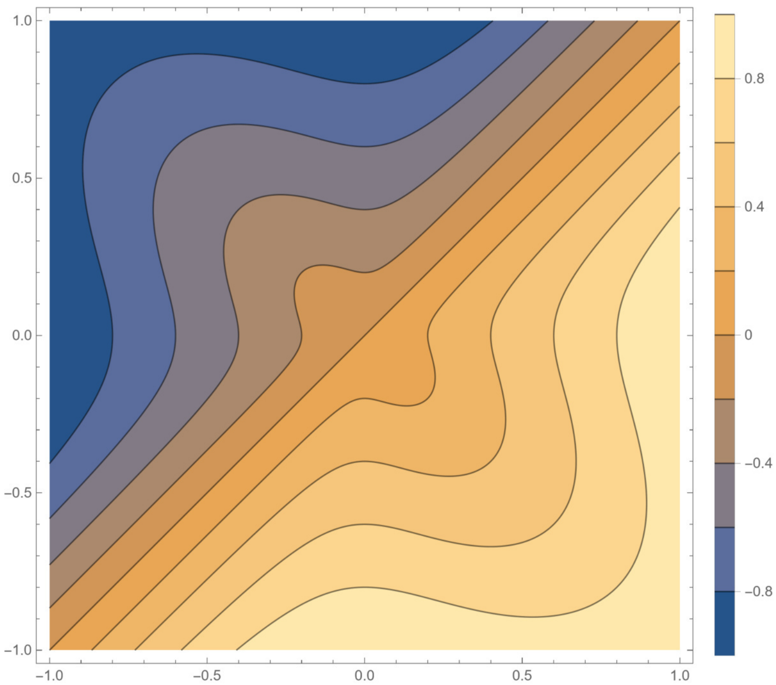


Figure 47. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 20.

Is the function

$$f(x,y) = \begin{cases} \dfrac{2x - y}{x^2 + y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

This seems to be a discontinuous function. Restricting the function to the X-axis or putting $y = 0$ gives

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = \dfrac{2x}{x^2} = \dfrac{2}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

This function is unbounded in any neighbourhood of $x = 0$ and cannot have the limit 0 as it should have if the function $f(x, y)$ is continuous.

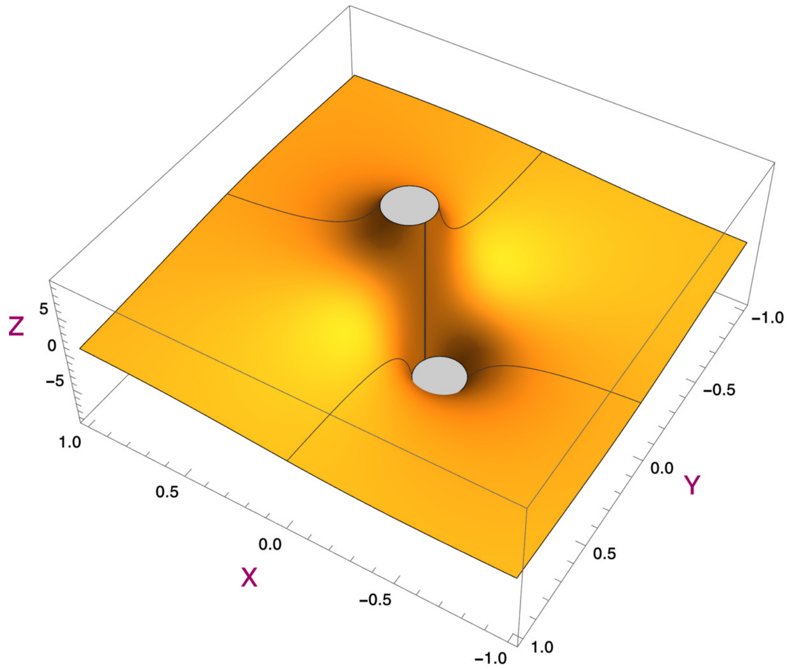


Figure 48. We see here a three dimensional figure of the graph of the function. This does not seem to be a graph of a continuous function.

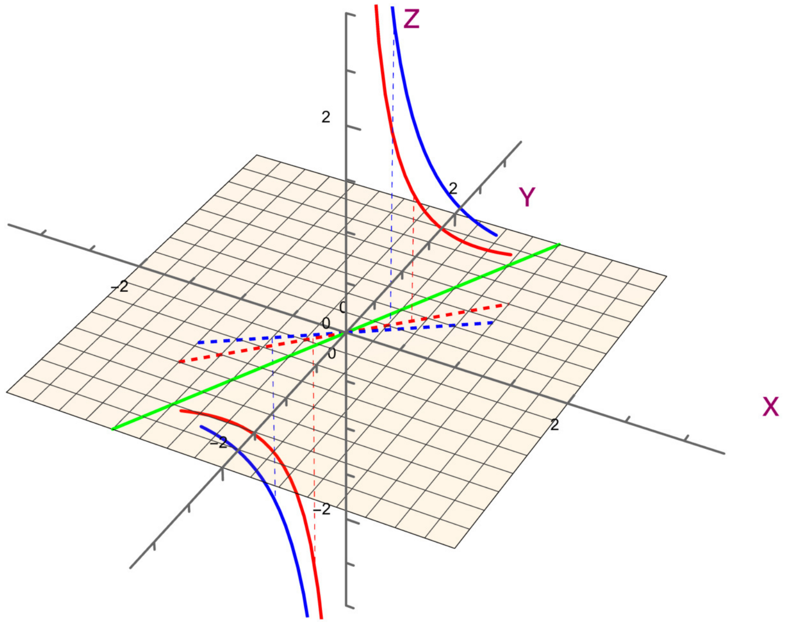


Figure 49. We have restricted the function here to $y = x$ and $y = 2x$ and $y = 7/10\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have unbounded behaviour in 0.

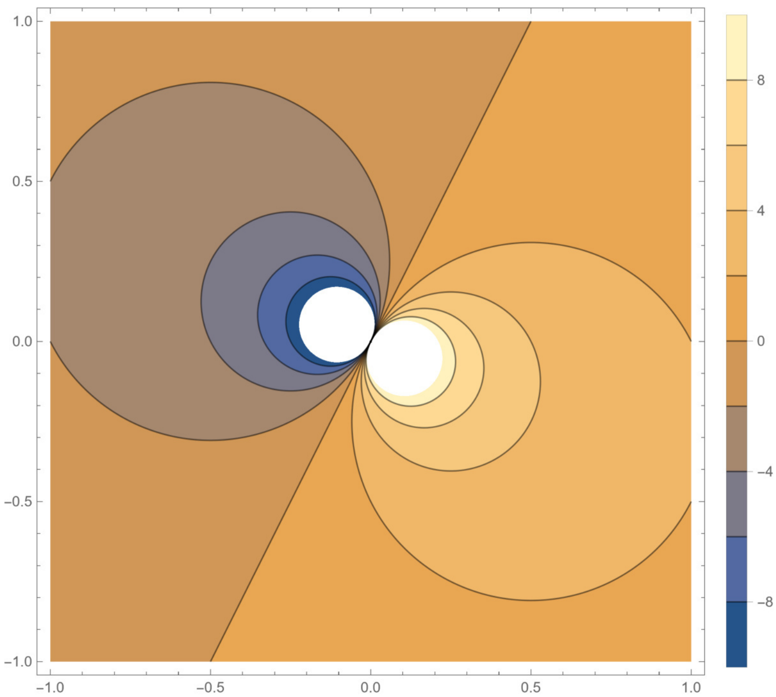


Figure 50. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks very discontinuous indeed.

Exercise 21.

Is the function

$$f(x,y) = \begin{cases} \dfrac{(x-1)^2 \ln(x)}{(x-1)^2 + y^2} & \text{if } (x,y) \neq (1,0); \\ 0 & \text{if } (x,y) = (1,0) \end{cases}$$

continuous in $(0,0)$? One may assume that one knows that the function $\ln(x)$ is continuous.

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(1,0)|| < \delta$ it follows that $|f(x,y) - f(1,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(1,0)|| = \sqrt{(x-1)^2+y^2} < \delta$, we have that

$$\left| \frac{(x-1)^2 \ln(x)}{(x-1)^2+y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned} \left| \frac{(x-1)^2 \ln(x)}{(x-1)^2+y^2} - 0 \right| &\leq \left| \frac{(x-1)^2\,|\ln(x)|}{(x-1)^2} \right| \\ &\leq |\ln(x)|. \end{aligned}$$

Because we may assume that $\ln(x)$ is continuous, we may assume that there exists a δ_1 such that if $|x-1| < \delta_1$, then $|\ln(x)-0| < \epsilon$. So if we take $\sqrt{(x-1)^2+y^2} < \delta_1$, we have also that $|x-1| \leq \sqrt{(x-1)^2+y^2} < \delta_1$, it follows that $|\ln(x)-0| < \epsilon$. It suffices to choose $\delta = \delta_1$. We conclude that the function is continuous.

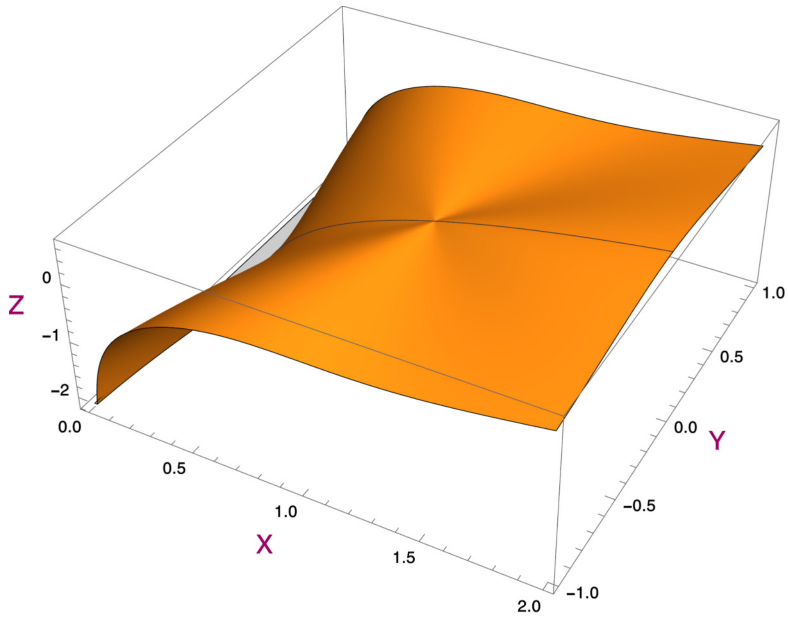


Figure 51. We see here a three dimensional figure of the graph of the function.

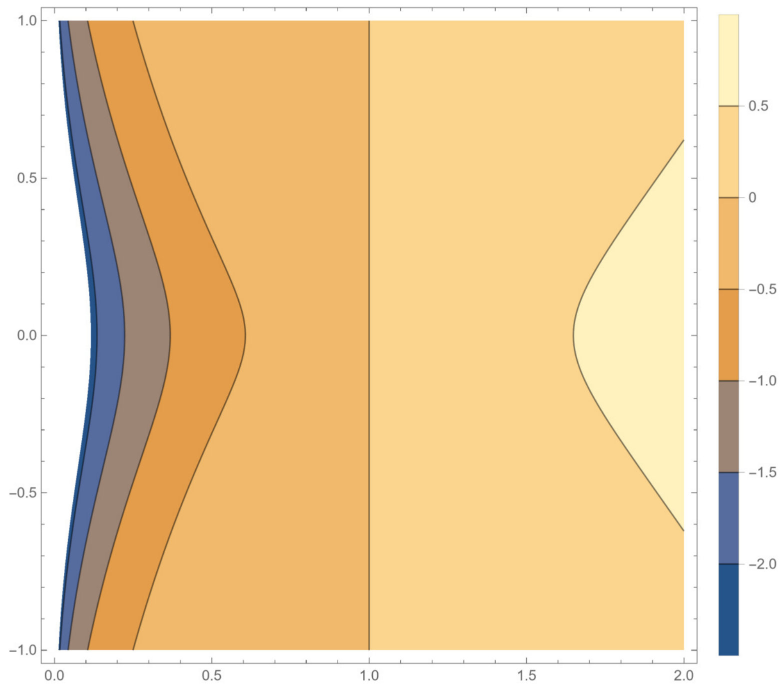


Figure 52. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 22.

Is the function

$$f(x,y) = \begin{cases} \dfrac{\sin((x-y)^2)}{|x|+|y|} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

Let us investigate this with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0, 0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{\sin((x-y)^2)}{|x| + |y|} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

We know that $\lim_{\alpha \to 0} (\sin(\alpha))/\alpha = 1$. So we may assume that $(\sin(\alpha))/\alpha$ is bounded from above in at least one neighbourhood of 0. We have then $|(\sin(\alpha))/\alpha| < 3/2$ for $|\alpha| < \delta_1$.

$$\begin{aligned}
\left| \frac{\sin((x-y)^2)}{|x| + |y|} - 0 \right| &\le \left| \frac{\sin((x-y)^2)}{(x-y)^2} \frac{(x-y)^2}{|x| + |y|} \right| \\
&\le \frac{3}{2} \frac{(x-y)^2}{|x| + |y|} \\
&\le \frac{3}{2} \frac{(|x| + |y|)^2}{|x| + |y|} \\
&\le \frac{3}{2} (|x| + |y|) \\
&\le \frac{3}{2} 2 \sqrt{x^2 + y^2} \\
&\le 3 \sqrt{x^2 + y^2}.
\end{aligned}$$

We have made use of the fact that e.g. $|x| \le (x^2 + y^2)^{1/2}$ and $|y| \le (x^2 + y^2)^{1/2}$.

So it is sufficient to prove that $\sqrt{x^2 + y^2} < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2 + y^2} < \delta$. It is for example sufficient to take $\delta_2 = \epsilon/3$. We can find a δ such that the two demands both are satisfied, we can take $\delta = \min\{\delta_1, \delta_2\}$. So we conclude that the function is continuous.

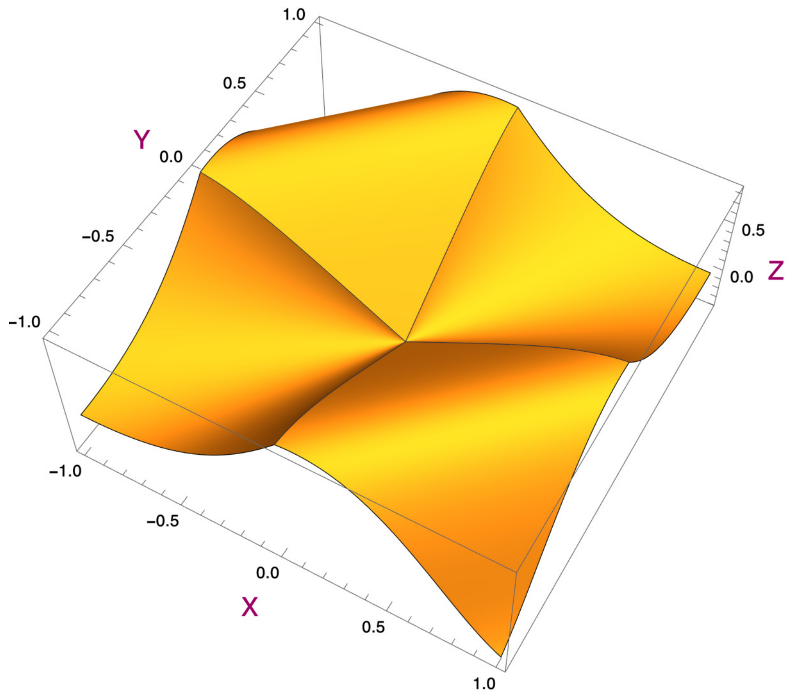


Figure 53. We see here a three dimensional figure of the graph of the function.

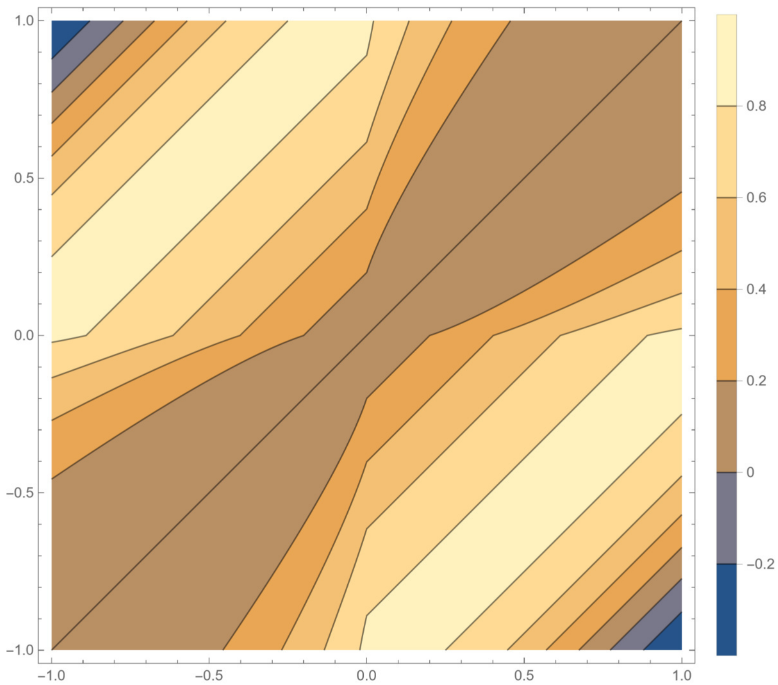


Figure 54. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0, 0)$.

Exercise 23.

Is the function

$$f(x,y) = \begin{cases} \dfrac{1}{x^2 - y} & \text{if } y \neq x^2; \\ 0 & \text{if } y = x^2 \end{cases}$$

continuous in $(0, 0)$?

Solution.

We restrict the function on the X-axis, so $y = 0$.
We have

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = \dfrac{1}{x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

This restricted function is unbounded in any neighbourhood of $x = 0$, so it cannot be continuous. If $f(x,y)$ is continuous, then the restricted function is also continuous with value 0 in $x = 0$. So the function $f(x,y)$ is not continuous.

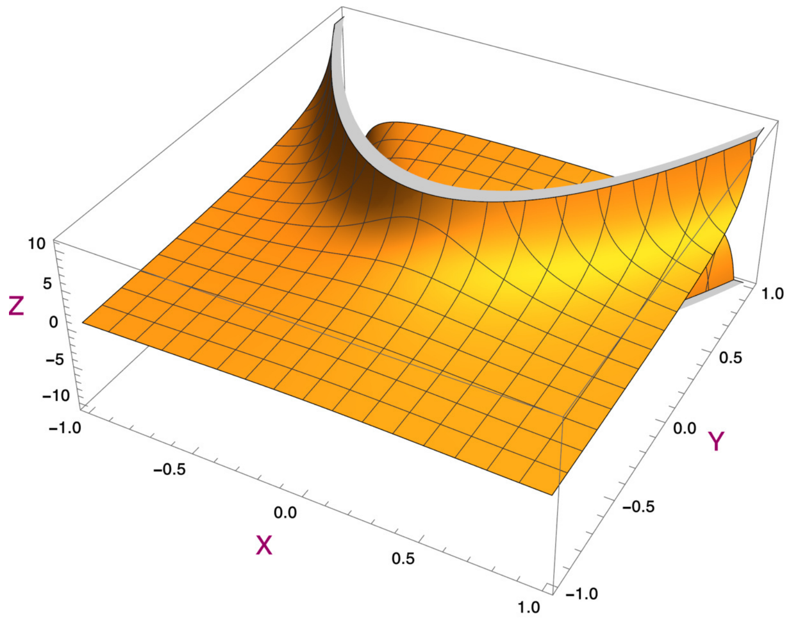


Figure 55. We see here a three dimensional figure of the graph of the function. The function shows unbounded behaviour in a neighbourhood of $(0,0)$.

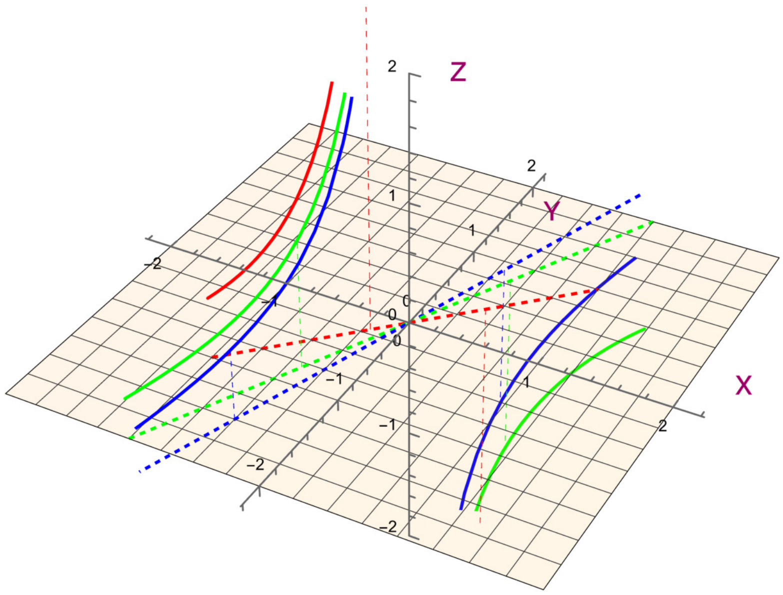


Figure 56. We have restricted the function here to $y = x$ and $y = 2x$ and $y = 3x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have unbounded behaviour in 0.

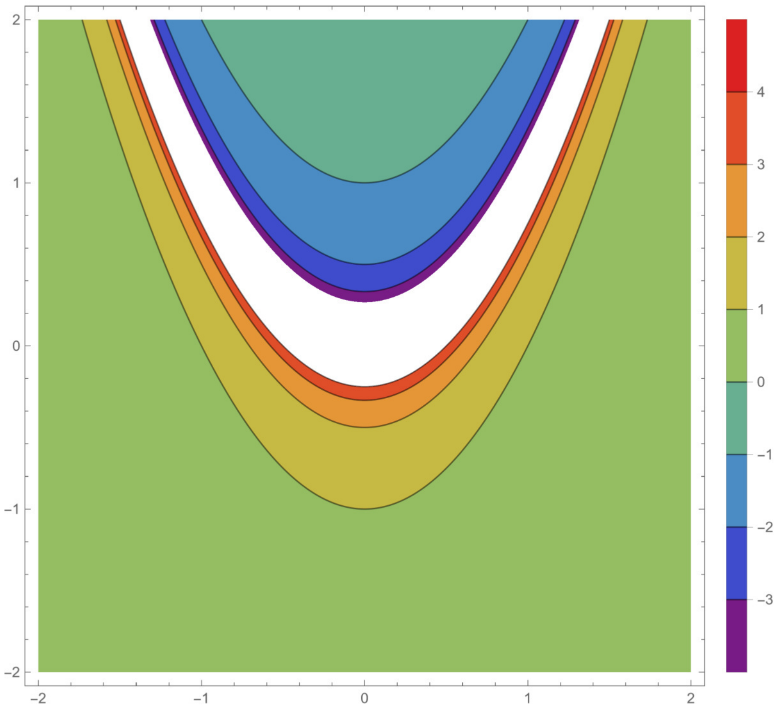


Figure 57. We see here a figure of the contour plot of the function. The function has unbounded behaviour above points in the neighbourhood of $y = x^2$.

Exercise 24.

Is the function

$$f(x, y) = \begin{cases} \arctan\left(\dfrac{x\,y^2}{x+y}\right) & \text{if } x \neq -y; \\ 0 & \text{if } x = -y \end{cases}$$

continuous in $(0, 0)$? We may assume that the one variable function $\arctan(x)$ is proven to be continuous.

Solution.

Let us try to restrict the function to curves through the origin with equation $x = -y + \lambda\,y^3$.

$$f|_{x=-y+\lambda y^3}(x,y) = \begin{cases} f(-y+\lambda y^3, y) = \arctan\left(y^2 - \dfrac{1}{\lambda}\right) & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

We see that this function has many limits dependent on the value of λ. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function is not continuous.

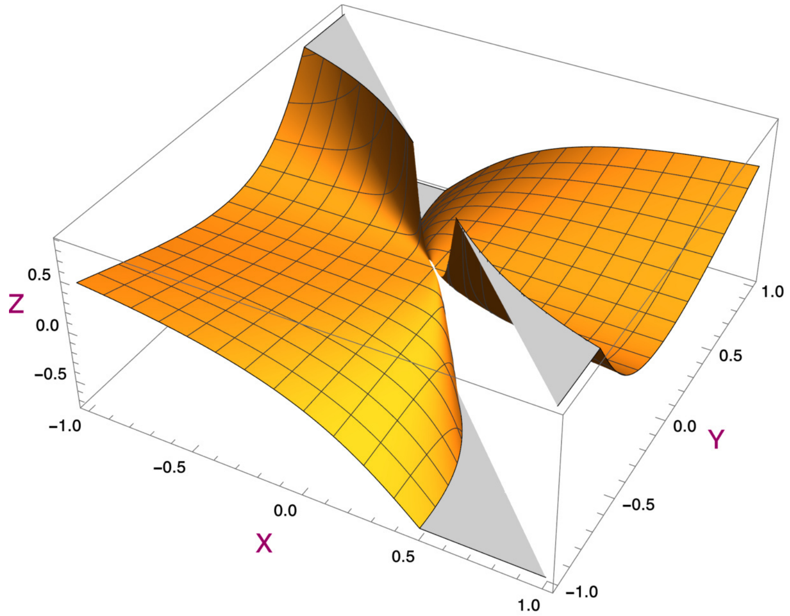


Figure 58. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks very suspicious. This does not seem to be a graph of a continuous function.

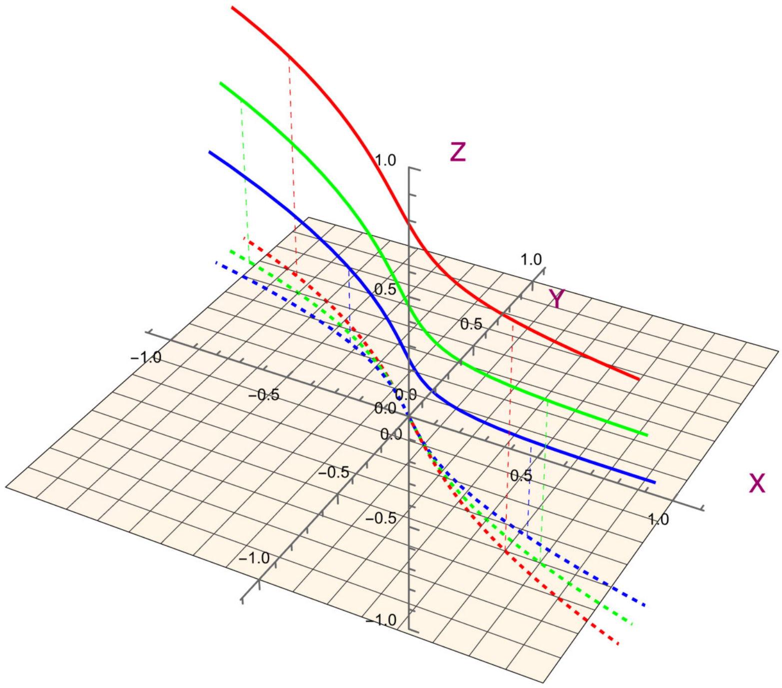


Figure 59. We have restricted the function here to $x = -y - y^3$ and $x = -y - 2y^3$ and $x = -y - 4y^3$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

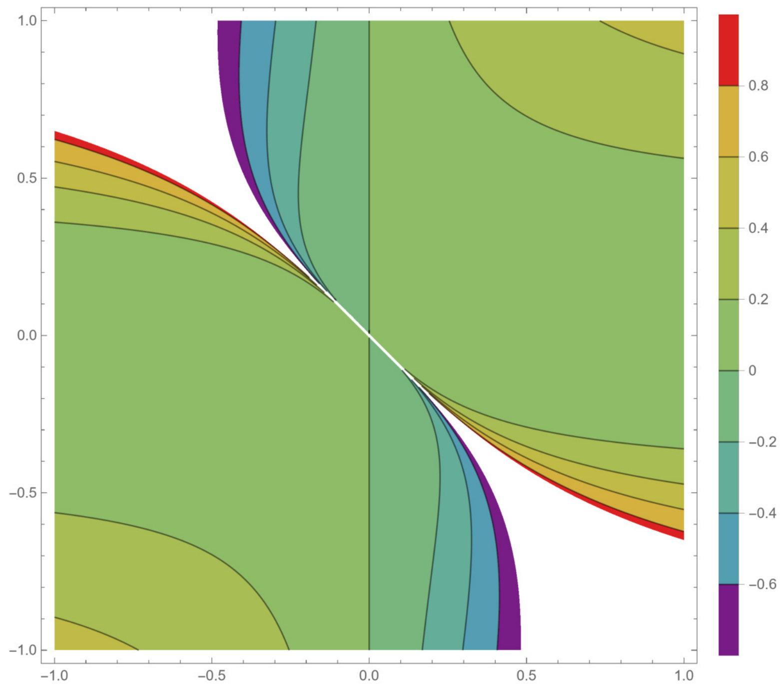


Figure 60. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 25.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

Let us restrict this function to straight lines through the origin with equation $y = \lambda x$.

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{x^2 - \lambda^2 x^2}{\lambda^2 x^2 + x^2} = \dfrac{1-\lambda^2}{\lambda^2+1} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have many different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function is not continuous.

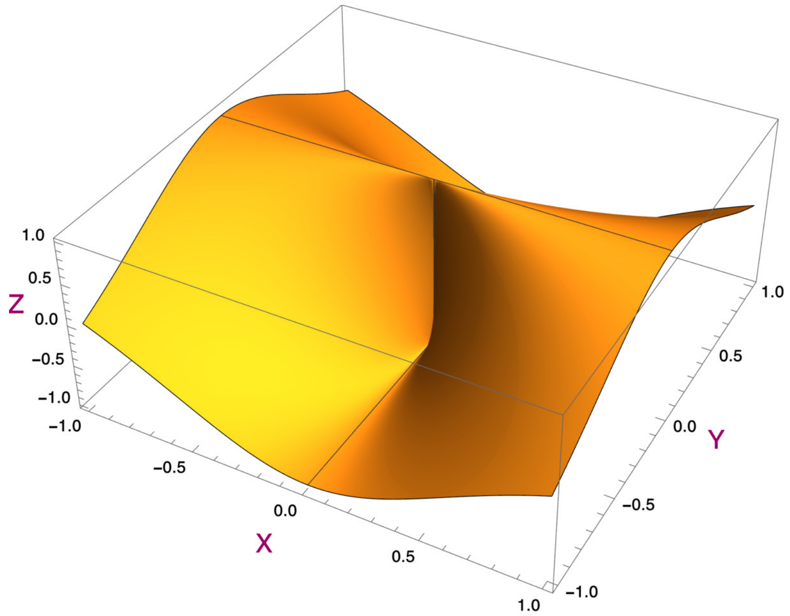


Figure 61. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks very suspicious. This does not seem to be a graph of a continuous function.

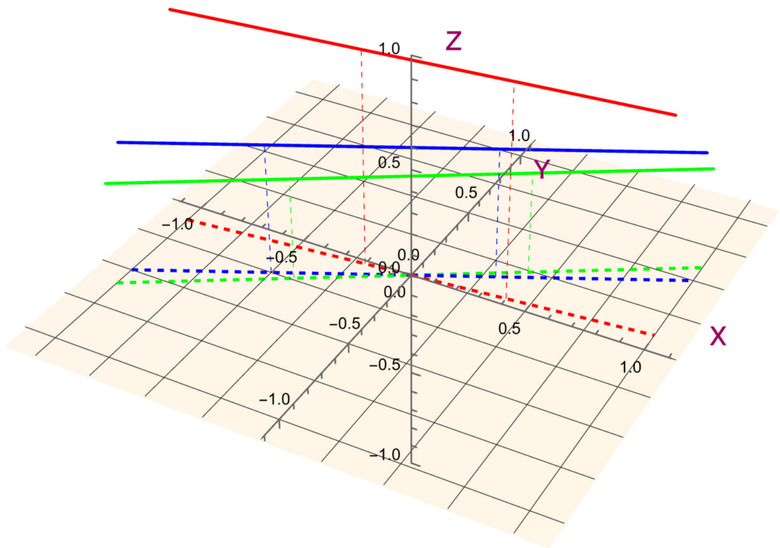


Figure 62. We have restricted the function here to $y = 1/10\,x$ and $y = 3/5\,x$ and $y = 1/2\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

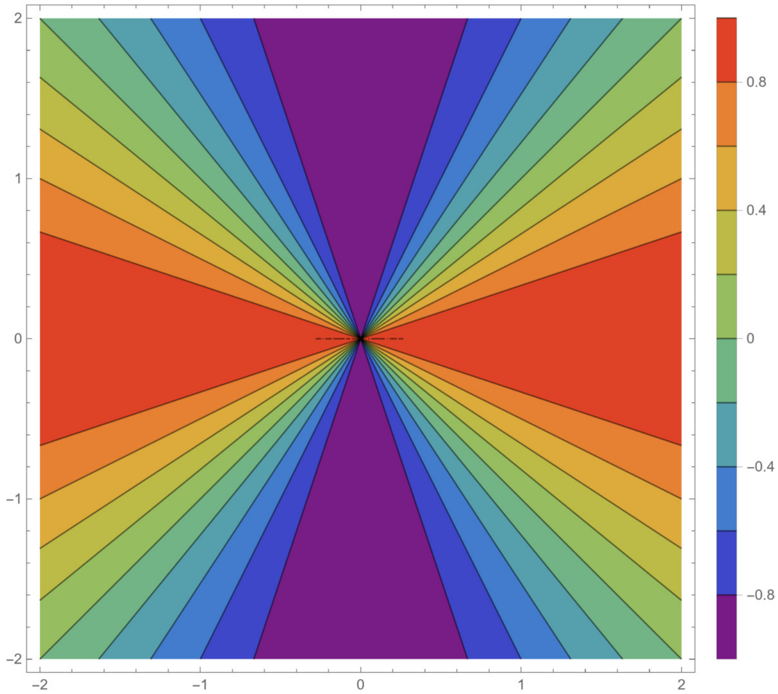


Figure 63. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks very discontinuous indeed.

Exercise 26.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

Let us restrict this function to the X-axis with $y = 0$, then we have

$$f|_{x=\lambda y^2}(x,y) = \begin{cases} f(x,0) = \dfrac{x}{\sqrt{x^2}} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So we have

$$f(x,0) = \begin{cases} -1 & \text{if } x < 0; \\ 0 & \text{if } x = 0; \\ 1 & \text{if } x > 0. \end{cases}$$

This function has no limit in $x = 0$ and it should have the limit $f(0,0) = 0$ if the function $f(x,y)$ is continuous. The function $f(x,y)$ cannot be continuous.

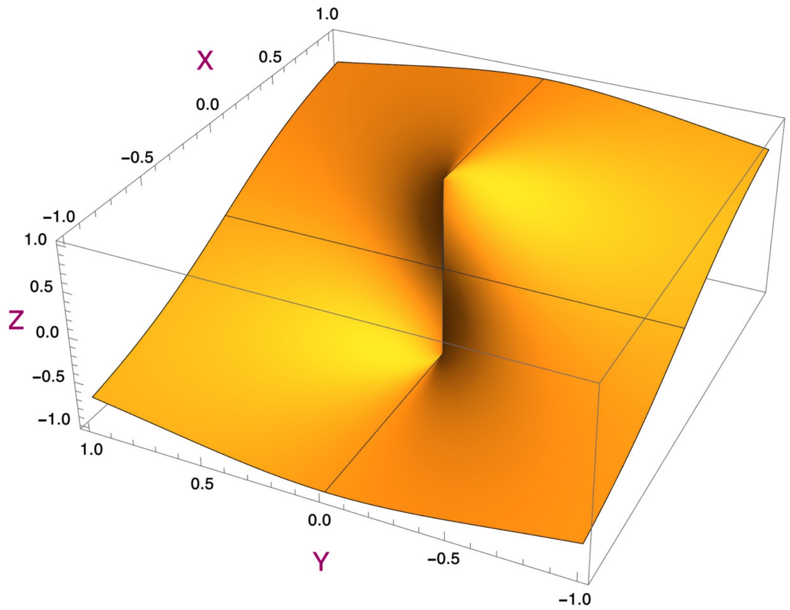


Figure 64. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks very suspicious. This does not seem to be a graph of a continuous function.

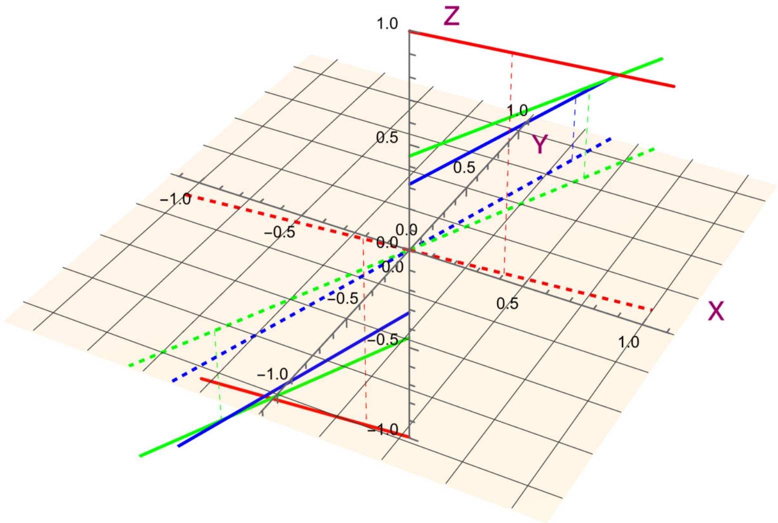


Figure 65. We have restricted the function here to $y = 1/10\,x$ and $y = 2\,x$ and $y = 3\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

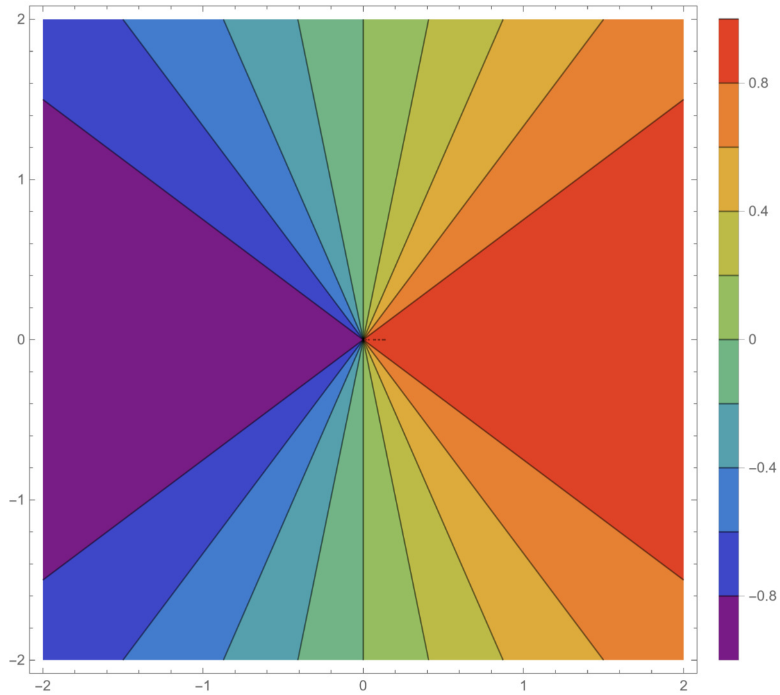


Figure 66. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 27.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x-y}{x+y} & \text{if } x \neq -y; \\ 0 & \text{if } x = -y \end{cases}$$

continuous in $(0,0)$?

Solution.

Let us restrict this function to lines through the origin with equation $y = \lambda x$ with $\lambda \neq -1$.

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{x-\lambda x}{\lambda x + x} = \dfrac{1-\lambda}{\lambda+1} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have many different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function is not continuous.

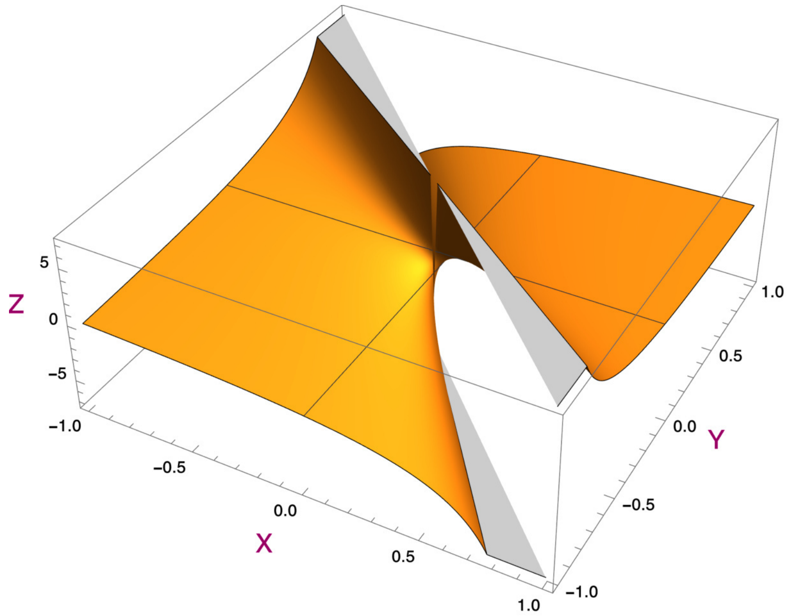


Figure 67. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

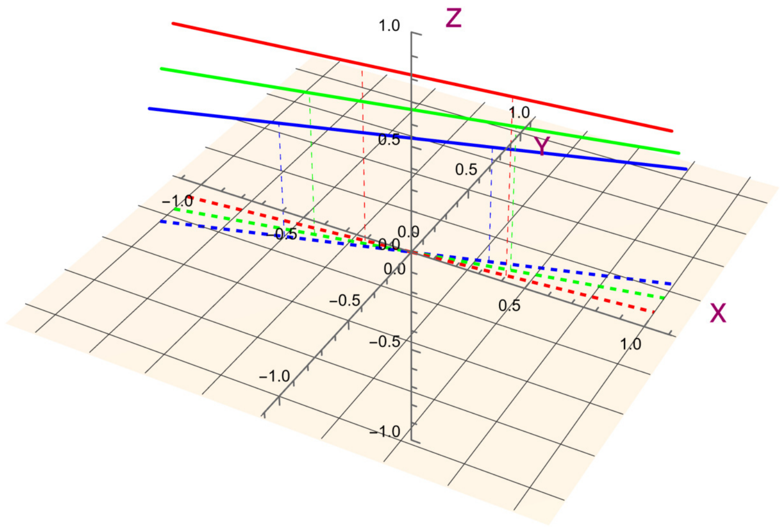


Figure 68. We have restricted the function here to $y = 1/2\,x$ and $y = x$ and $y = 3/2\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

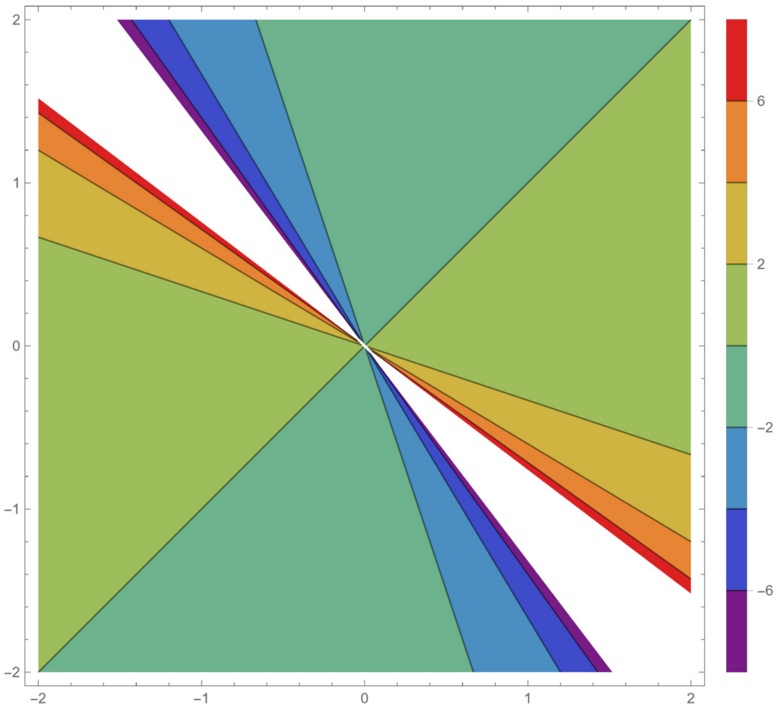


Figure 69. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 28.

Is the function

$$f(x,y)=\begin{cases}\dfrac{x^2+y}{y} & \text{if } y\neq 0;\\ 0 & \text{if } y=0.\end{cases}$$

continuous in $(0,0)$?

Solution.

Let us restrict this function to the line through the origin with equation $x=0$.

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = \dfrac{y}{y} = 1 & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

In this case the limit

$$\lim_{y\to 0} f|_{x=0}(x,y) = 1.$$

Let us restrict this function to the continuous curves through the origin with equation $y = \lambda x^2$.

$$f|_{y=\lambda x^2}(x,y) = \begin{cases} f(x,\lambda x^2) = \dfrac{\lambda x^2 + x^2}{\lambda x^2} = \dfrac{1+\lambda}{\lambda} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

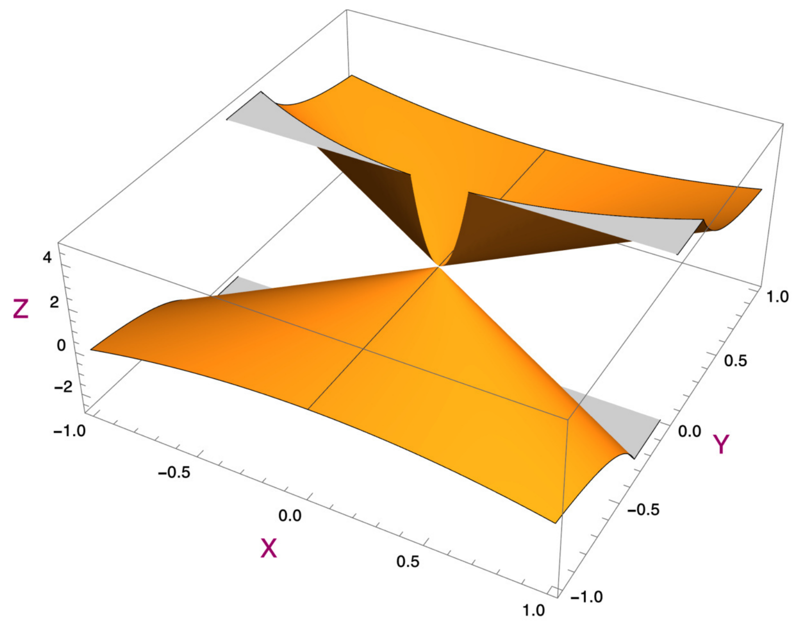


Figure 70. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

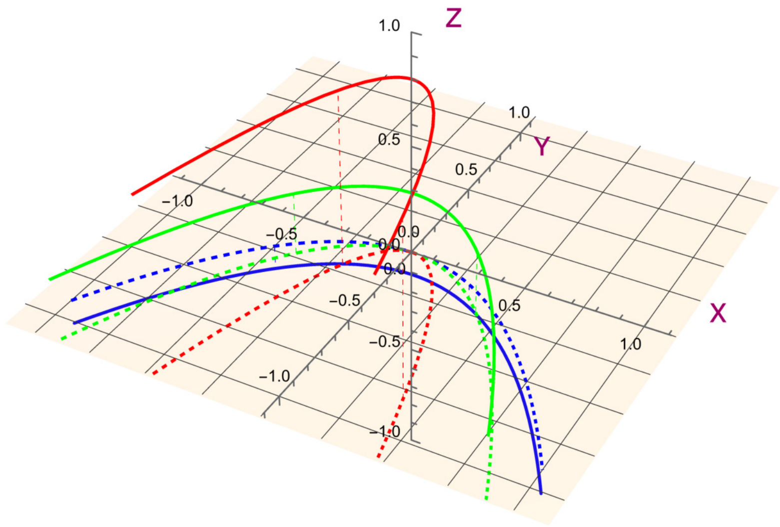


Figure 71. We have restricted the function here to $y = -5x^2$ and $y = -7/5\,x^2$ and $y = -9/10\,x^2$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

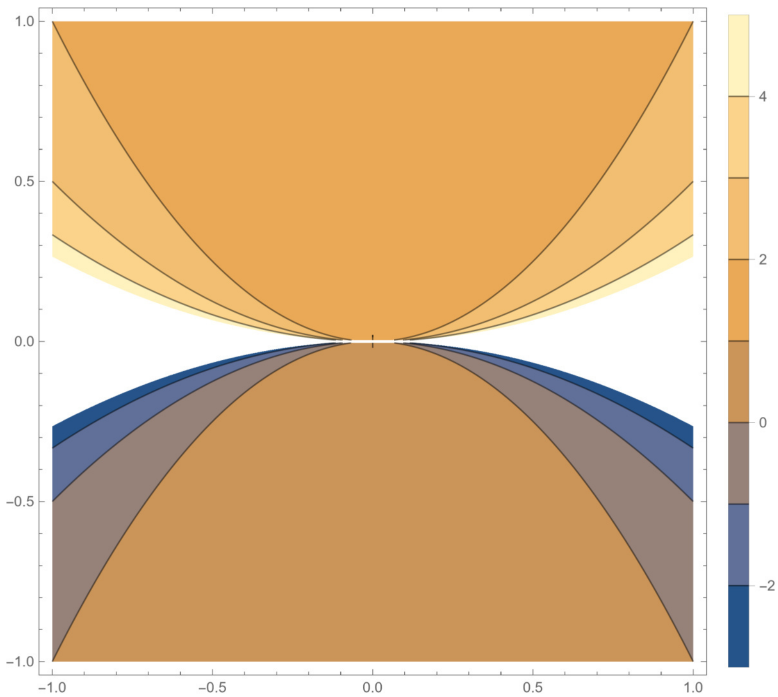


Figure 72. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 29.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x^3 - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{x^3 - x\,y^2}{x^2 + y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{x^3 - x\,y^2}{x^2 + y^2} - 0 \right| &\leq \left| \frac{x\,(x^2 - y^2)}{x^2 + y^2} \right| \\
&\leq \frac{|x|\,(x^2 + y^2)}{x^2 + y^2} \\
&\leq |x| \\
&\leq \sqrt{x^2 + y^2}.
\end{aligned}$$

So it is sufficient to prove that $\sqrt{x^2 + y^2} < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2 + y^2} < \delta$. It is for example sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function is continuous.

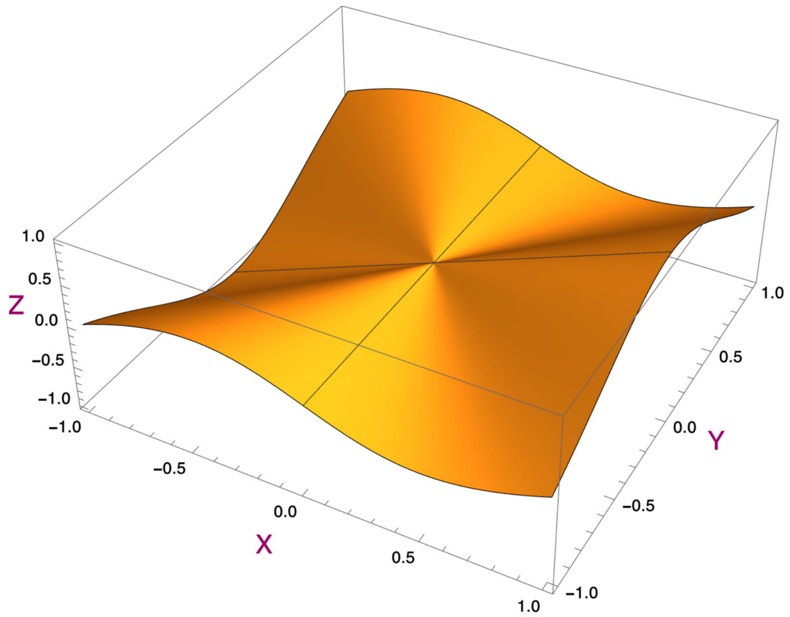


Figure 73. We see here a three dimensional figure of the graph of the function.

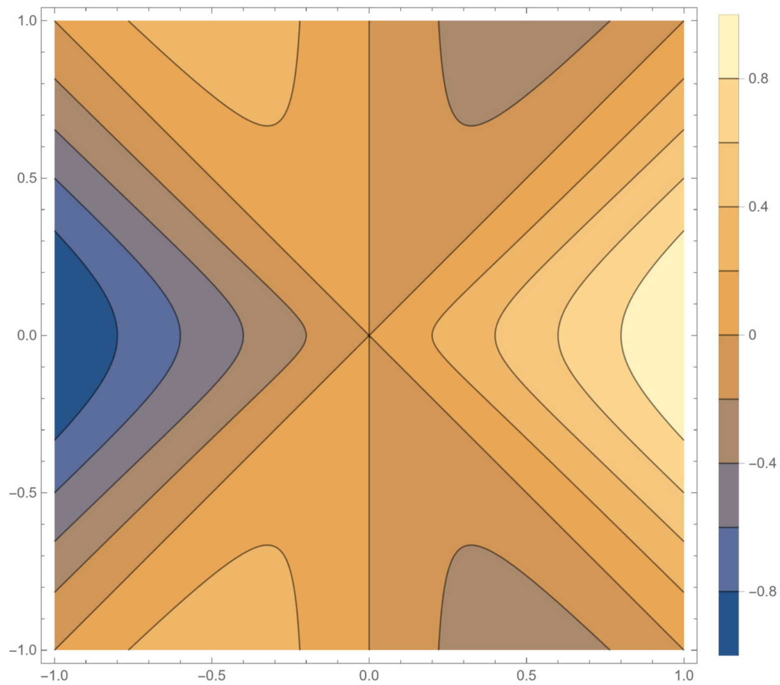


Figure 74. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 30.

Is the function

$$f(x,y)=\begin{cases}\dfrac{x^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0);\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

continuous in $(0,0)$?

Solution.

We restrict the function to the continuous curves with equations $y=\lambda x$. We observe then that

$$f|_{y=\lambda x}(x,y)=\begin{cases} f(x,\lambda x)=\dfrac{x^2}{\lambda^2x^2+x^2}=\dfrac{1}{\lambda^2+1} & \text{if } x\neq 0;\\ 0 & \text{if } x=0.\end{cases}$$

We see that these restricted functions have different limits. But if $f(x, y)$ is continuous, all these limit values should be $f(0, 0) = 0$. So this function $f(x, y)$ is discontinuous.

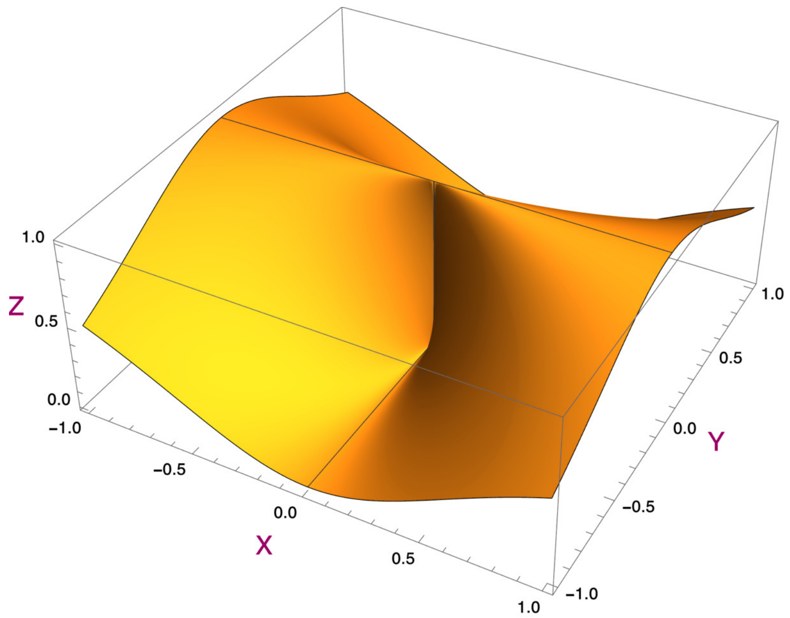


Figure 75. We see here a three dimensional figure of the graph of the function. The vertical line above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

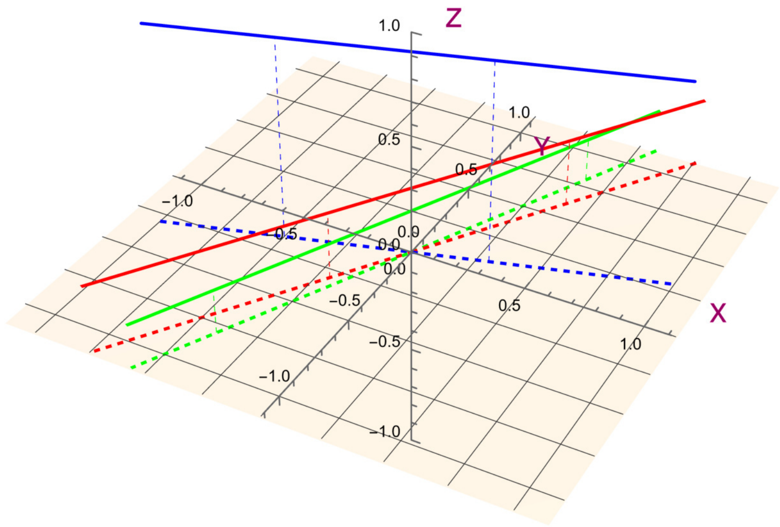


Figure 76. We have restricted the function here to $y = 3/2\,x$ and $y = 2\,x$ and $y = 3\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

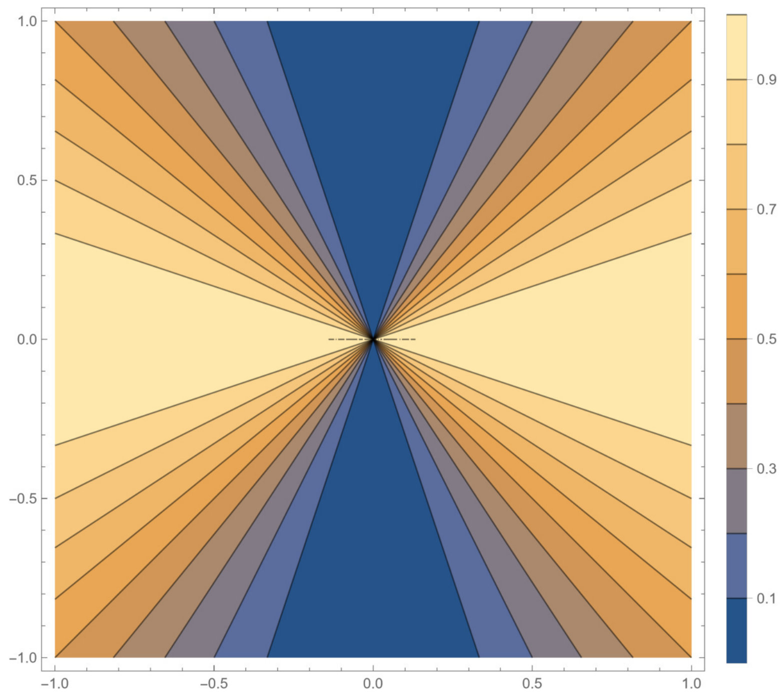


Figure 77. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This is typically a function with a discontinuity.

Exercise 31.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x^2+y^2-x^3y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We observe that proving that $f(x,y)$ is continuous in $(0,0)$ with $f(0,0) = 1$, is equivalent with proving that $g(x,y) = f(x,y) - 1$ is continuous in $(0,0)$ with $g(0,0) = 0$.

$$g(x,y) = f(x,y) - 1 = \frac{-x^3y^3 + x^2 + y^2}{x^2+y^2} - 1 = -\frac{x^3y^3}{x^2+y^2}.$$

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to proof that $|g(x,y) - g(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|g(x,y) - g(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| -\frac{x^3y^3}{x^2+y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| -\frac{x^3y^3}{x^2+y^2} - 0 \right| &\leq \frac{\sqrt{x^2+y^2}^3\sqrt{x^2+y^2}^3}{x^2+y^2} \\
&\leq \frac{(x^2+y^2)^3}{x^2+y^2} \\
&\leq (x^2+y^2)^2 \\
&\leq \sqrt{x^2+y^2}^4.
\end{aligned}$$

So it is sufficient to prove that $\sqrt{x^2+y^2} < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2+y^2} < \delta$. It is for example sufficient to take $\delta = \epsilon^{1/4}$. We can find a δ, so the function $g(x,y)$ is continuous and consequently also the function $f(x,y)$.

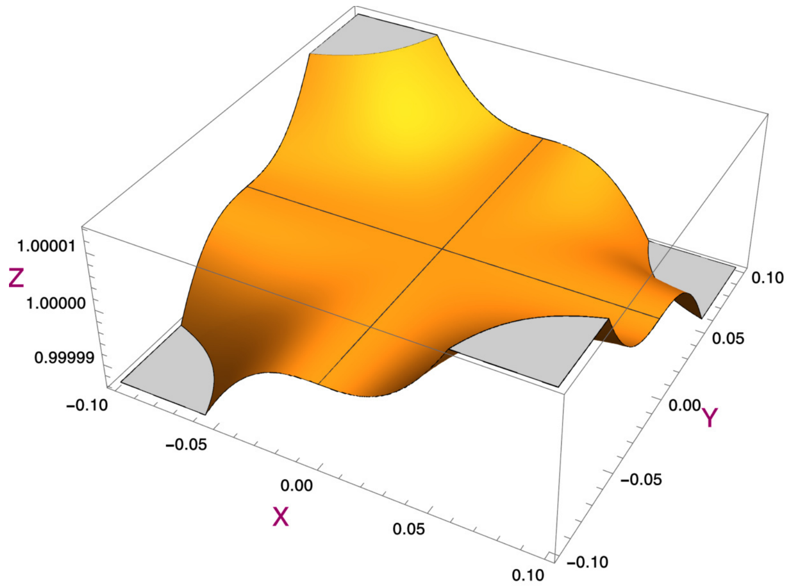


Figure 78. We see here a three dimensional figure of the graph of the function.

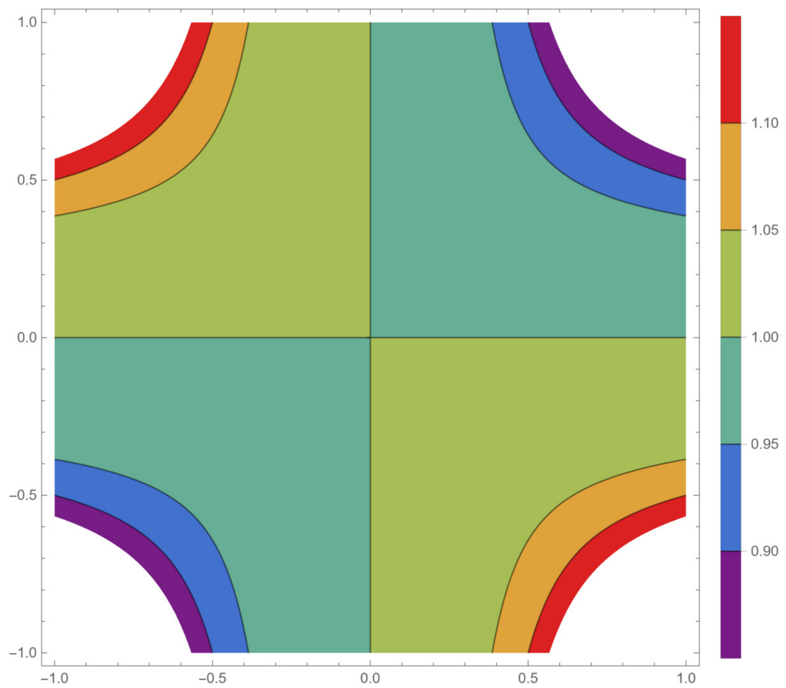


Figure 79. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0, 0)$.

Exercise 32.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^4} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We restrict the function to the continuous curves with equations $y = \lambda x^2$. We observe then that

$$f|_{y=\lambda x^2}(x,y) = \begin{cases} f(x,\lambda x^2) = \dfrac{\lambda x^4}{\lambda^4 x^8 + x^4} = \dfrac{\lambda}{\lambda^4 x^4 + 1} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

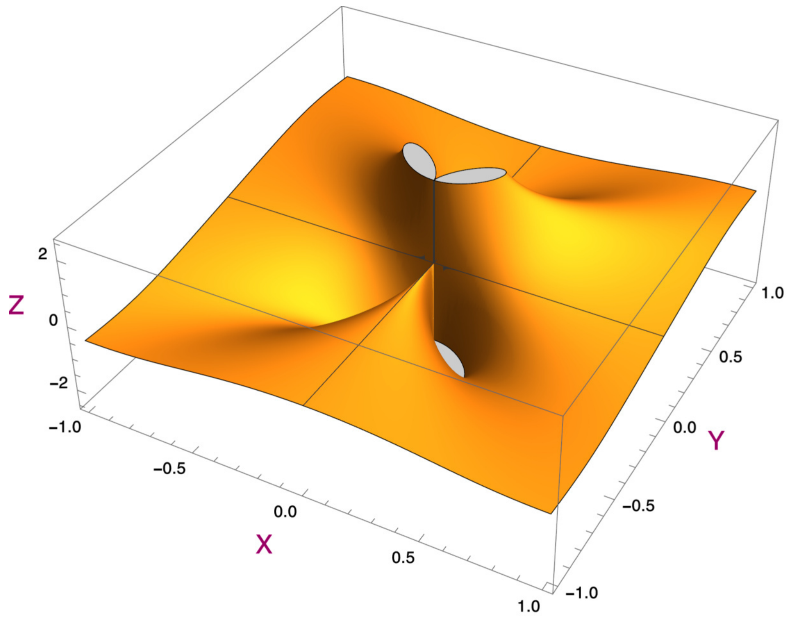


Figure 80. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

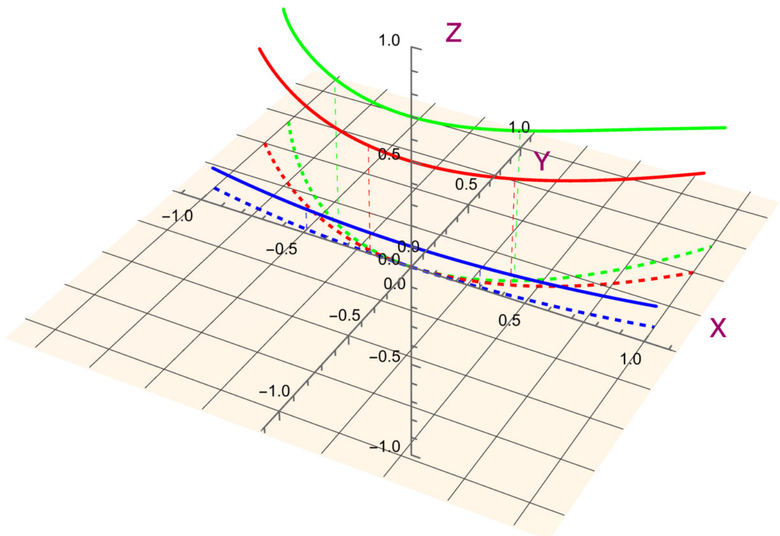


Figure 81. We have restricted the function here to $y = 1/2\,x^2$ and $y = 7/10\,x^2$ and $y = x^2$. We see in this figure clearly that the restrictions of the function to these curves are functions that have different limits in 0.

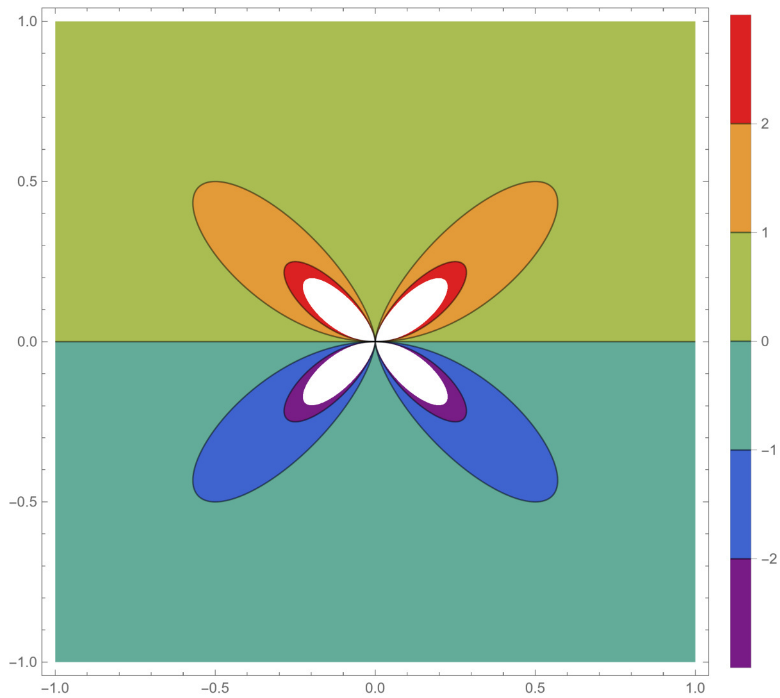


Figure 82. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 33.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{x^2 y^2}{x^2+y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{x^2 y^2}{x^2+y^2} - 0 \right| &\leq \frac{x^2 y^2}{x^2+y^2} \\
&\leq \frac{\sqrt{x^2+y^2}^{\,2}\sqrt{x^2+y^2}^{\,2}}{x^2+y^2} \\
&\leq \frac{(x^2+y^2)^2}{x^2+y^2} \\
&\leq \sqrt{x^2+y^2}^{\,2}.
\end{aligned}$$

So it is sufficient to prove that $\sqrt{x^2+y^2}^{\,2} < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2+y^2} < \delta$. In order to do that it is sufficient to take $\delta = \sqrt{\epsilon}$. We can find a δ, so we conclude that the function is continuous.

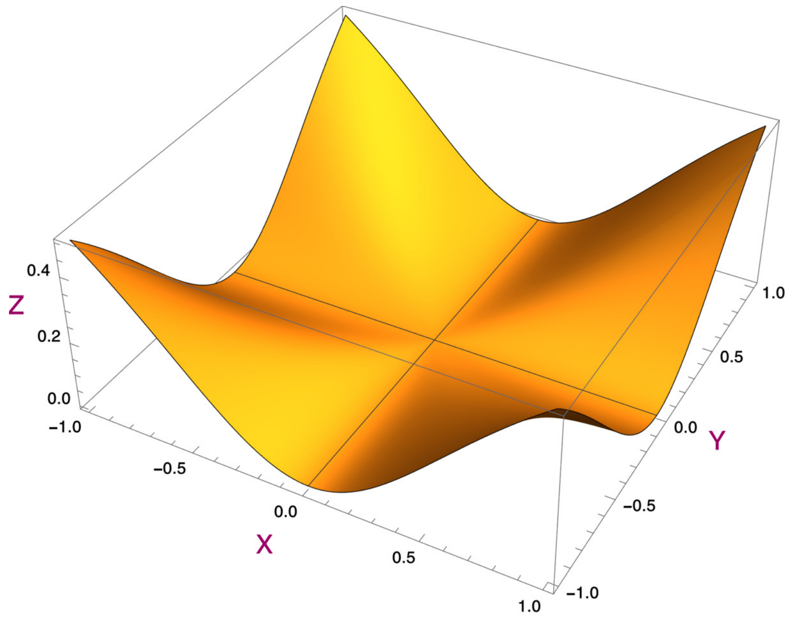


Figure 83. We see here a three dimensional figure of the graph of the function.

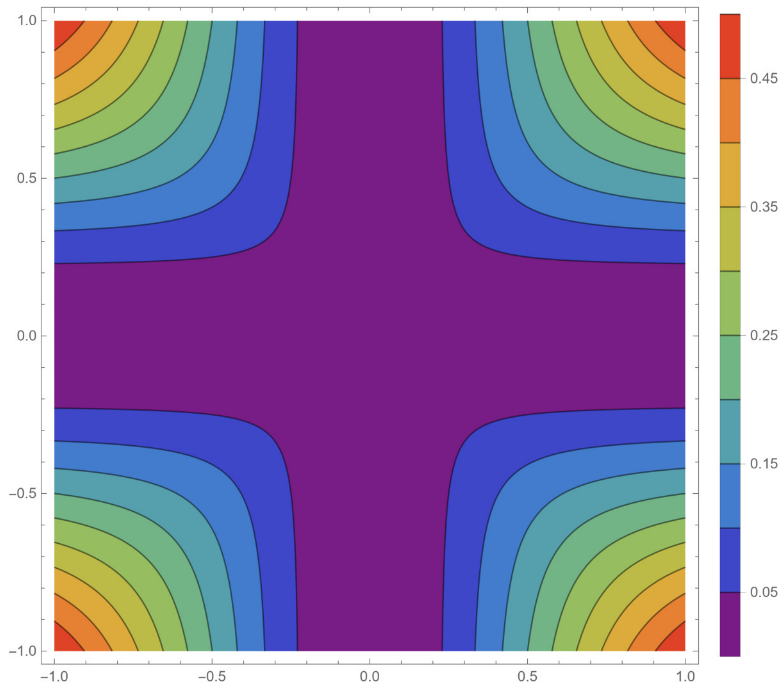


Figure 84. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 34.

Let a, b and c be three positive real numbers. Suppose also that $a+b>2c$. Is the function

$$f(x,y)=\begin{cases}\dfrac{|x|^a\,|y|^b}{(x^2+y^2)^c} & \text{if } (x,y)\neq(0,0);\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

continuous in $(0,0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon>0$. We try to find a $\delta>0$ such that if $||(x,y)-(0,0)||<\delta$ it follows that $|f(x,y)-f(0,0)|<\epsilon$.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{|x|^a \, |y|^b}{(x^2 + y^2)^c} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{|x|^a \, |y|^b}{(x^2 + y^2)^c} - 0 \right| &\leq \frac{|x|^{\,a} \, |y|^b}{(x^2 + y^2)^{\,c}} \\
&\leq \frac{\sqrt{x^2 + y^2}^{\,a} \sqrt{x^2 + y^2}^{\,b}}{(x^2 + y^2)^{\,c}} \\
&\leq \frac{\sqrt{x^2 + y^2}^{\,a+b}}{\sqrt{x^2 + y^2}^{\,2c}} \\
&\leq \sqrt{x^2 + y^2}^{\,a+b-2c}.
\end{aligned}$$

So it is sufficient to prove that $\sqrt{x^2 + y^2}^{\,a+b-2c} < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2 + y^2} < \delta$. In order to do that it is sufficient to take $\delta = \epsilon^{1/(a+b-2c)}$. We can find a δ, so we conclude that the function is continuous.

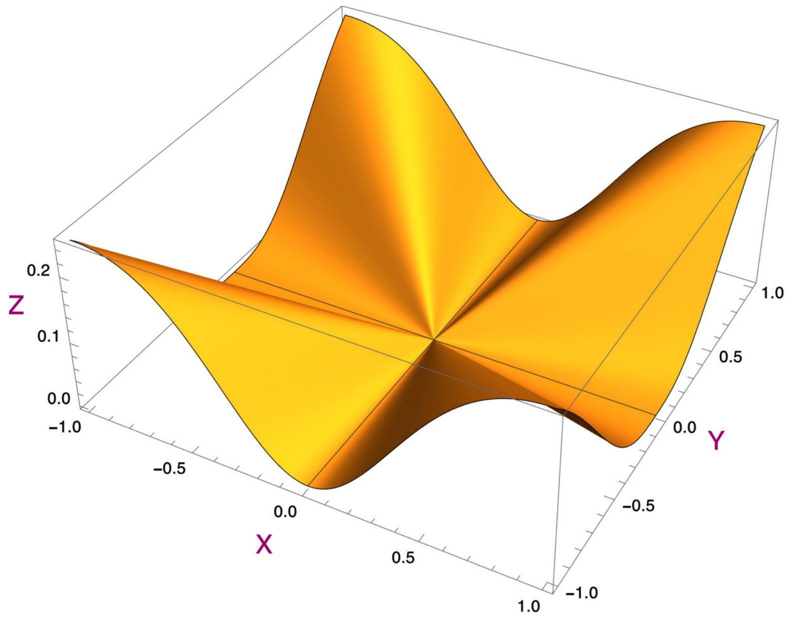


Figure 85. We see here a three dimensional figure of the graph of the function. This does not seem at first sight to be a graph of a continuous function. If we look at it closely, we see that functions above $y = \lambda x$ seem to dive very steeply to level 0. The function is drawn here for the values $a = 2$, $b = 3$ and $c = 2$.

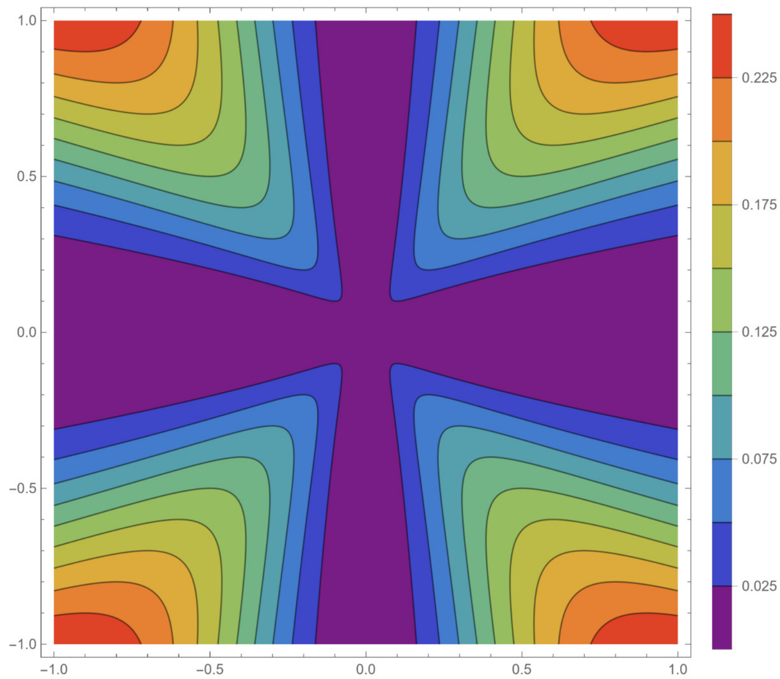


Figure 86. We see here a figure of the contour plot of the function. We see that the levels of curves in a neighbourhood are approximately of level 0. The function is drawn here for the values $a = 2$, $b = 3$ and $c = 2$.

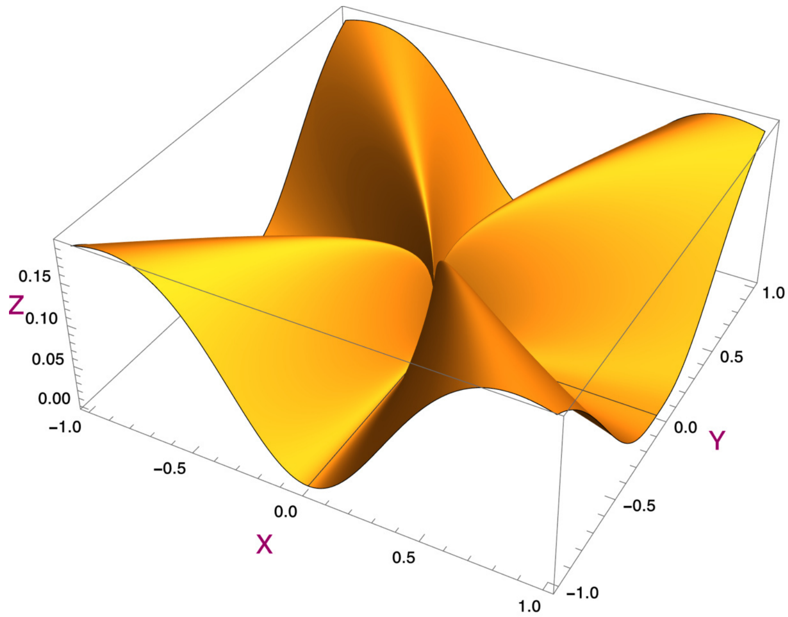


Figure 87. We see here a three dimensional figure of the graph of the function. This does not seem at first sight to be a graph of a continuous function. If we look at it closely, we see that functions above $y = \lambda x$ seem to dive very steeply to level 0. The function is drawn here for the values $a = 2$, $b = 3$ and $c = 2.4$.

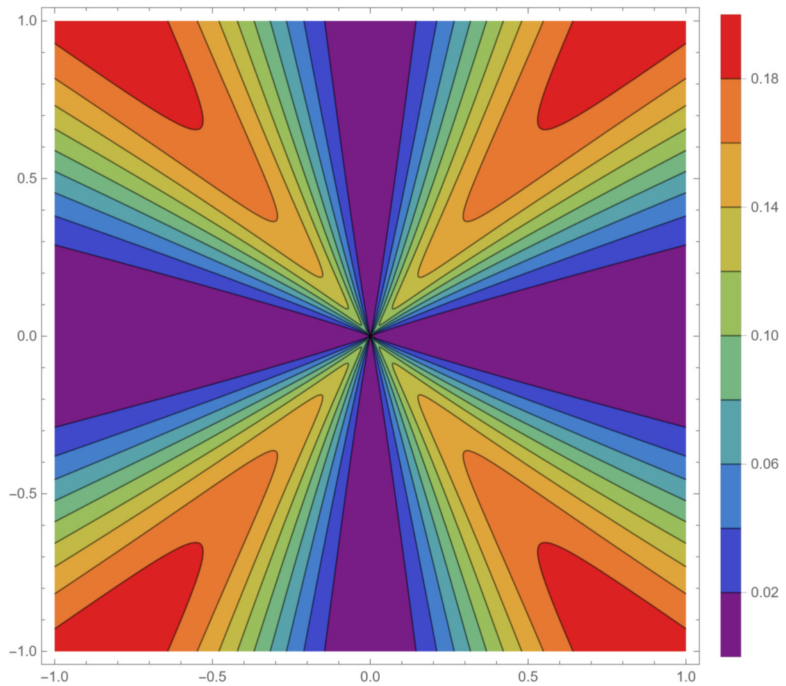


Figure 88. We see here a figure of the contour plot of the function. We see that the levels of curves in a neighbourhood of $(0,0)$ are still approximately of level 0. Many differently coloured curves in a neighbourhood of $(0,0)$ stop abruptly before reaching $(0,0)$. More explanation for this behaviour follow in the next pictures. The function is drawn here for the values $a = 2$, $b = 3$ and $c = 2.4$.

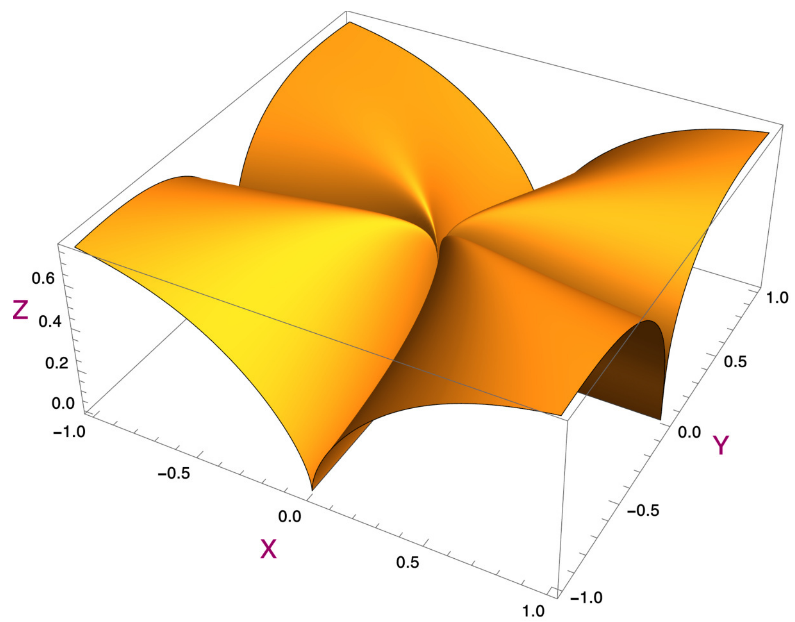


Figure 89. We see here a three dimensional figure of the graph of the function. This does not seem at first sight to be a graph of a continuous function. If we look at it closely, we see that functions above $y = \lambda x$ seem to dive very steeply to level 0. The function is drawn here for the values $a = 0.6$, $b = 0.3$ and $c = 0.4$.

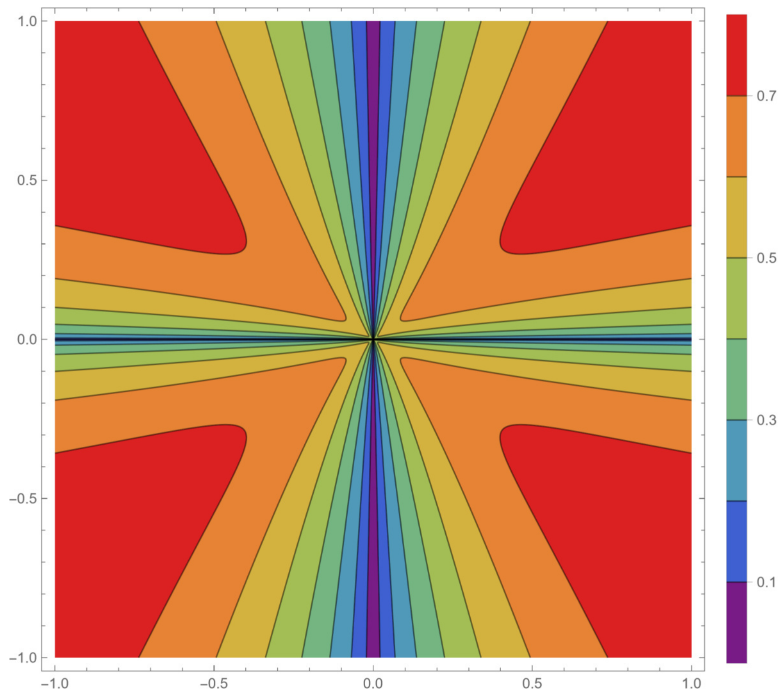


Figure 90. We see here a figure of the contour plot of the function. We see that the levels of curves in a neighbourhood are still approximately of level 0. Many coloured curves in a neighbourhood of $(0, 0)$ stop abruptly before reaching $(0, 0)$. More explanation for this behaviour follow in the next pictures. The function is drawn here for the values $a = 0.6$, $b = 0.3$ and $c = 0.4$.

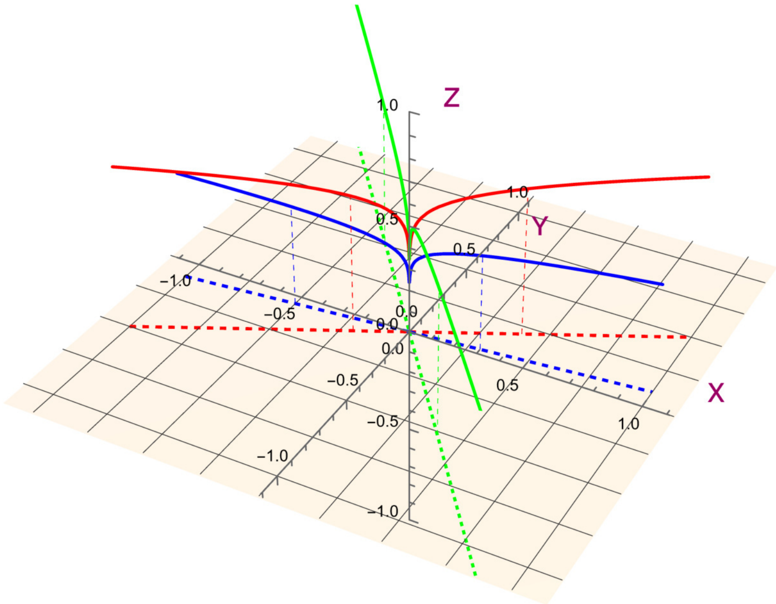


Figure 91. We see here a figure of the contour plot of the function. We see that lines with equation $y = \lambda x$ are mapped to curves that take a very steep dive to 0 when approaching the Z-axis. The function is drawn here for the values $a = 0.6$, $b = 0.3$ and $c = 0.4$.

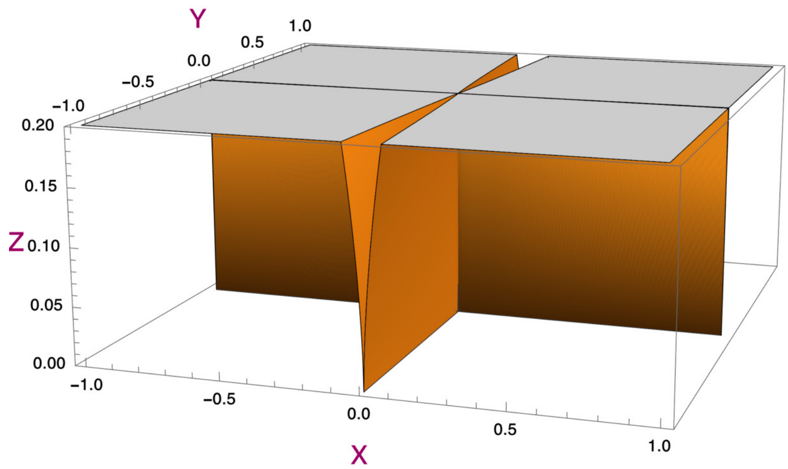


Figure 92. We see here a figure showing more clearly what happens close to $z = 0$. The function is drawn here for the values $a = 0.6$, $b = 0.3$ and $c = 0.4$.

We see that if we restrict the function $f(x, y)$ to $y = \lambda x$, then the main part of the function definition is

$$f\,|_{y=\lambda x}(x,y) = \begin{cases} \dfrac{|\lambda|^b\,|x|^{a+b}}{(|\lambda|^2+1)\,|x|^{2c}} = \dfrac{|\lambda|^b\,|x|^{a+b-2c}}{|\lambda|^2+1} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

The phenomenon that we observe is the following. Consider the sequence of functions $g_n(x) = x^{1/n}$ that converge pointwise but not uniformly to the function $g(x) = 1$ on $(0,1]$ and $g(0) = 0$. When the argument x of the function $g_n(x)$ is in a very small neighbourhood of 0 and very close to 0, the function takes a sudden deep dive from approximately level 1 to level 0 with a vertical tangent line in $x = 0$. See the following plot.

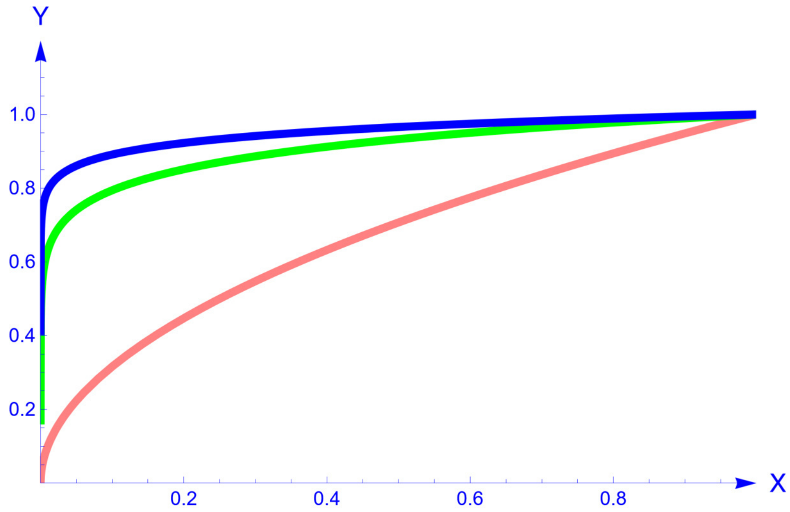


Figure 93. We see here a figure of the graphs of the function $g_n(x)$ described above. We pictured the three functions $g_2(x) = x^{1/2}$, $g_{10}(x) = x^{1/10}$, $g_{20}(x) = x^{1/20}$. We see that they tend to have function value 1 away from $x = 0$ and then descend steeply to function height 0 in a neighbourhood of $x = 0$. They have all vertical tangent lines in $x = 0$.

Exercise 35.

Is the function

$$f(x,y) = \begin{cases} (x+y)\,\sin(1/x)\,\sin(1/y) & \text{if } x\,y \neq 0; \\ 0 & \text{if } x\,y = 0 \end{cases}$$

continuous in $(0,0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left|(x+y)\,\sin(1/x)\,\sin(1/y) - 0\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left|(x+y)\,\sin(1/x)\,\sin(1/y) - 0\right| &\leq \left|(x+y)\,\sin(1/x)\,\sin(1/y)\right| \\
&\leq \left|x + y\right| \\
&\leq |x| + |y| \\
&\leq \sqrt{x^2 + y^2} + \sqrt{x^2 + y^2} \\
&\leq 2\sqrt{x^2 + y^2}.
\end{aligned}$$

So it is sufficient to prove that $2\sqrt{x^2 + y^2} < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2 + y^2} < \delta$. In order to do that it is sufficient to take $\delta = \epsilon/2$. We can find a δ, so we conclude that the function is continuous.

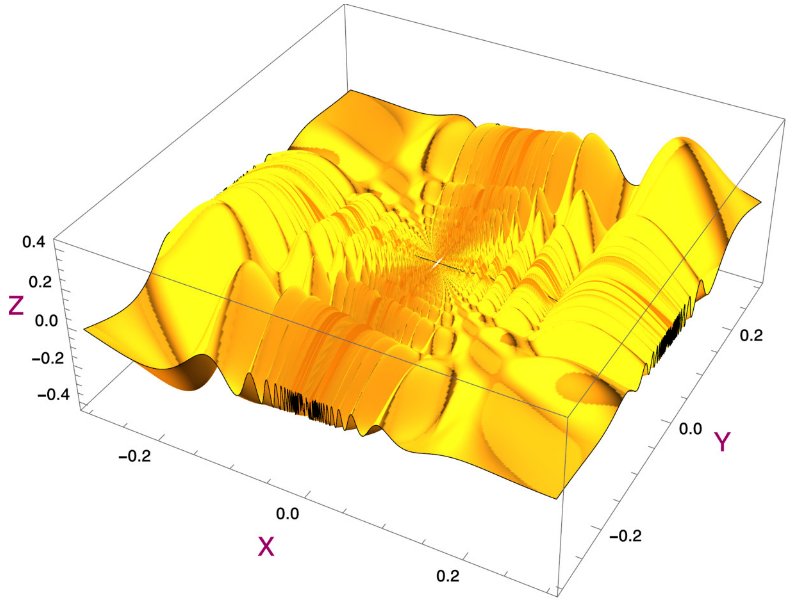


Figure 94. We see here a three dimensional figure of the graph of the function. The continuity is caused by the factor $x + y$. This flattens out the graph around $(0, 0)$. Remark that this type of functions is notoriously difficult to draw due to the high frequency of the oscillations.

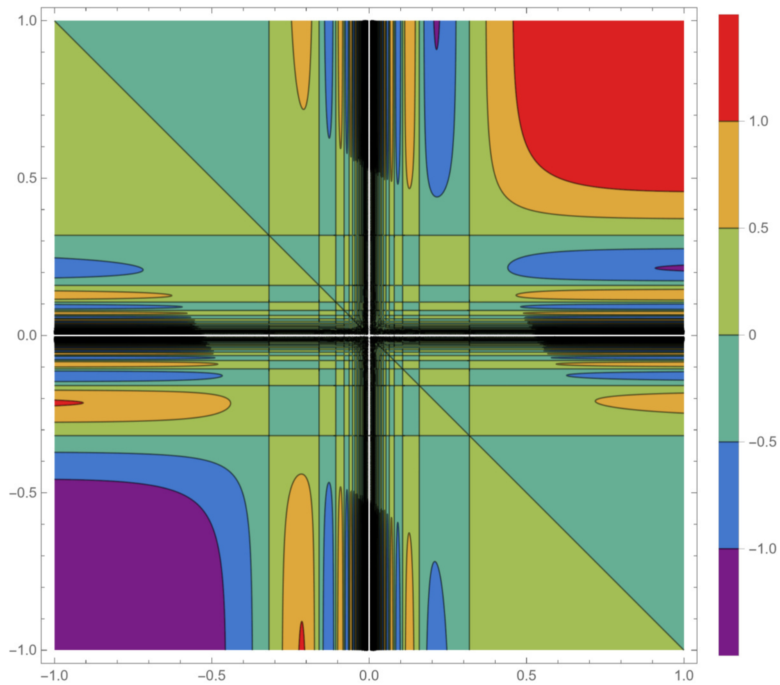


Figure 95. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$. Remark that when many different colours come close together caused by the wild oscillations, the end effect is black colour. The continuity is caused by the factor $x+y$. This flattens out the graph around $(0,0)$. Remark that this type of functions is notoriously difficult to draw.

We remark here that this function restricted to vertical lines with equation $x=\lambda$, $\lambda \neq 0$ and $\sin(1/\lambda) \neq 0$ will not be continuous. The restriction is again with $\lambda \neq 0$ and $\sin(1/\lambda) \neq 0$ is then

$$f|_{x=\lambda}(x,y) = \begin{cases} f(\lambda, y) = (\lambda + y)\sin\left(\dfrac{1}{\lambda}\right)\sin\left(\dfrac{1}{y}\right) & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

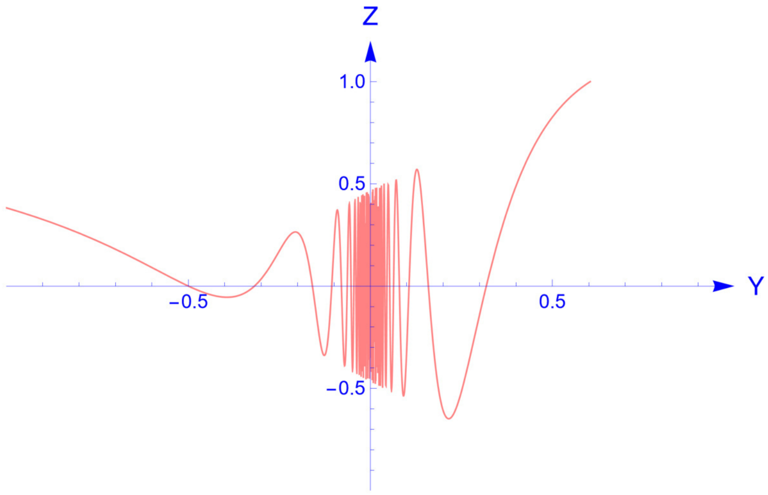


Figure 96. A typical graph of this restricted function to a line with equation $x = \lambda$. This figure is made with $\lambda = 1/2$. Figures of this type are notoriously difficult to make due to the high frequency of the oscillations.

But the function restricted to a line with equation $y = \lambda x$ with $\lambda \neq -1$ is evidently continuous. The restriction is

$$f|_{y=\lambda x}(x, y) = \begin{cases} f(x, \lambda x) = (\lambda x + x)\sin\left(\frac{1}{x}\right)\sin\left(\frac{1}{\lambda x}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

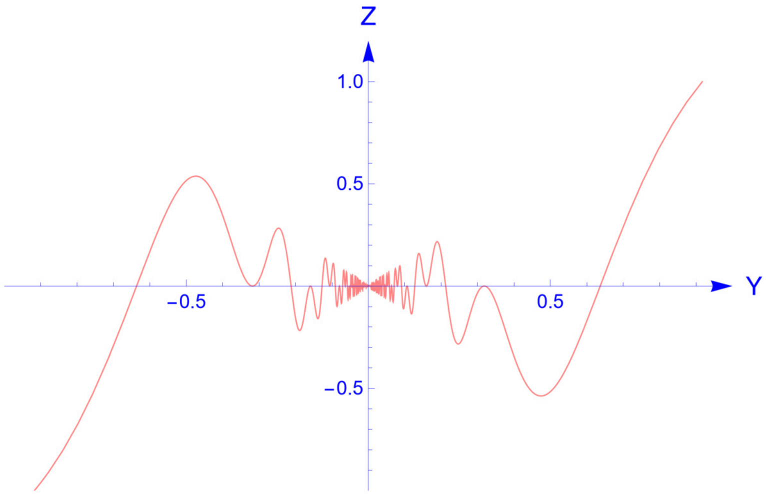


Figure 97. A typical graph of this restricted function to a line with equation $y = \lambda x$. This figure is made with $\lambda = 1/2$. Figures of this type are notoriously difficult to make due to the high frequency of the oscillations.

Exercise 36.

Is the function

$$f(x, y) = \begin{cases} \dfrac{x\,y}{x^2 + y^2} + y\,\sin(1/x) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0 \end{cases}$$

continuous in $(0, 0)$?

Solution.

We have in exercise 1 proven that $\frac{x\,y}{x^2+y^2}$ is not continuous in $(0, 0)$. So if we can prove that $y\,\sin(1/x)$ is continuous in $(0, 0)$, then the sum of these two functions cannot be continuous.

We set out to prove now that $y\,\sin(1/x)$ is continuous.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x, y) - f(0, 0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x, y) - (0, 0)|| < \delta$ it follows that $|f(x, y) - f(0, 0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| y \sin(1/x) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| y \sin(1/x) - 0 \right| &\leq |y|\,|\sin(1/x)| \\
&\leq |y| \\
&\leq \sqrt{x^2+y^2}.
\end{aligned}$$

So it is sufficient to prove that $\sqrt{x^2+y^2} < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2+y^2} < \delta$. In order to do that It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function is continuous.

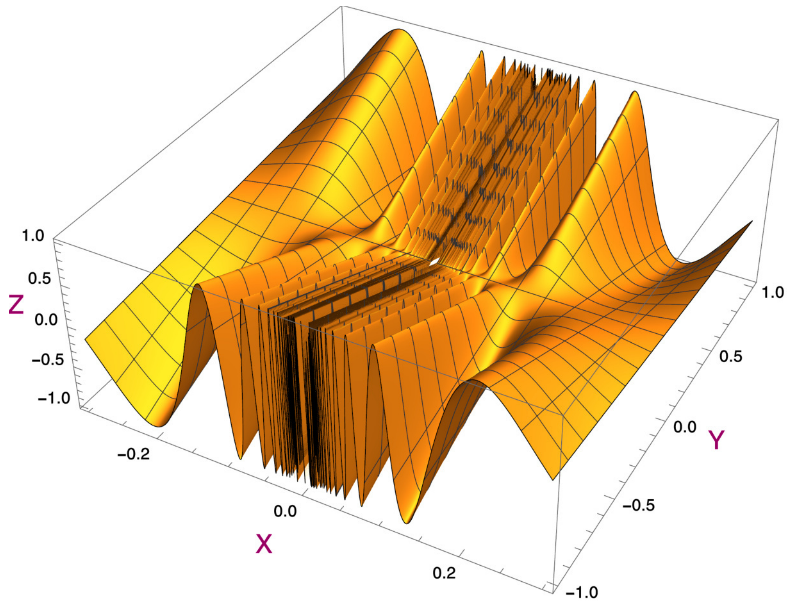


Figure 98. We see here a three dimensional figure of the graph of the function y sin$(1/x)$. Wild oscillations cause the overall colour to blacken. This type of functions is notoriously difficult to draw due to the high frequency of the oscillations.

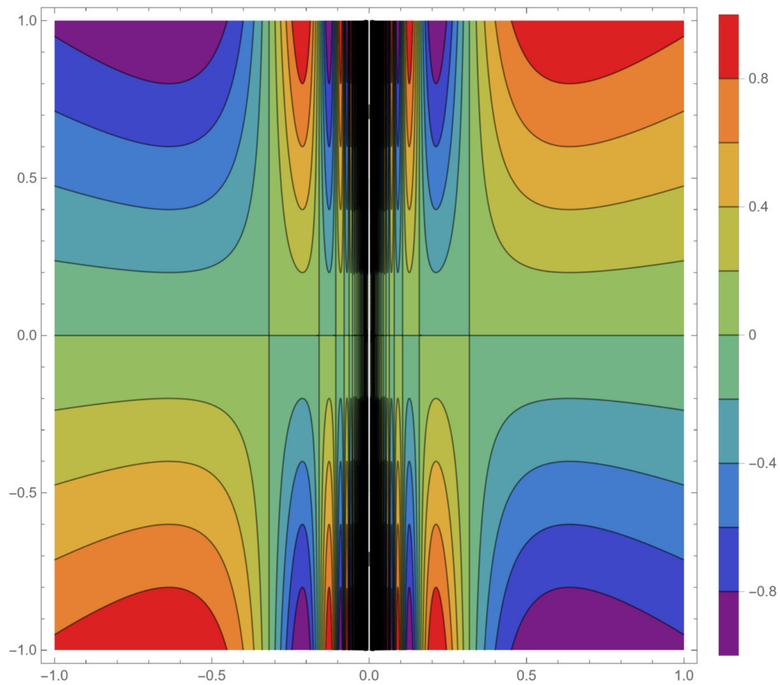


Figure 99. We see here a figure of the contour plot of the function $y\sin(1/x)$. Only level curves of level around 0 come close to $(0,0)$. Many differently coloured regions next to each other cause the overall colour to blacken. This is what we observe here. This type of functions is notoriously difficult to draw due to the high frequency of the oscillations.

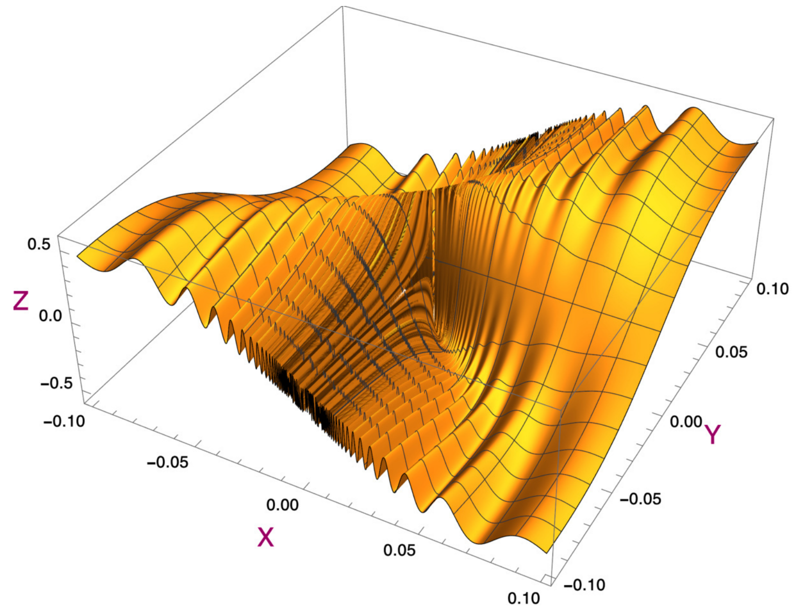


Figure 100. We see here a three dimensional figure of the graph of the function $\frac{x\,y}{x^2+y^2} + y\,\sin\left(\frac{1}{x}\right)$. The vertical line above $(0,0)$ looks very suspicious. This does not seem to be a graph of a continuous function.

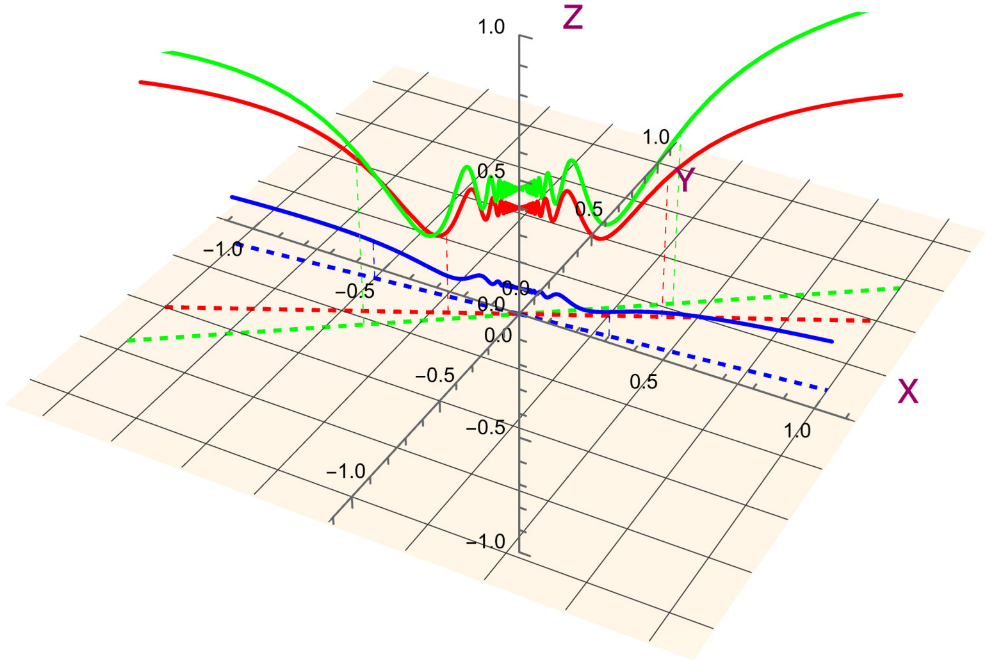


Figure 101. We have restricted the function here to $y = 1/2\,x$ and $y = 7/10\,x$ and $y = x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

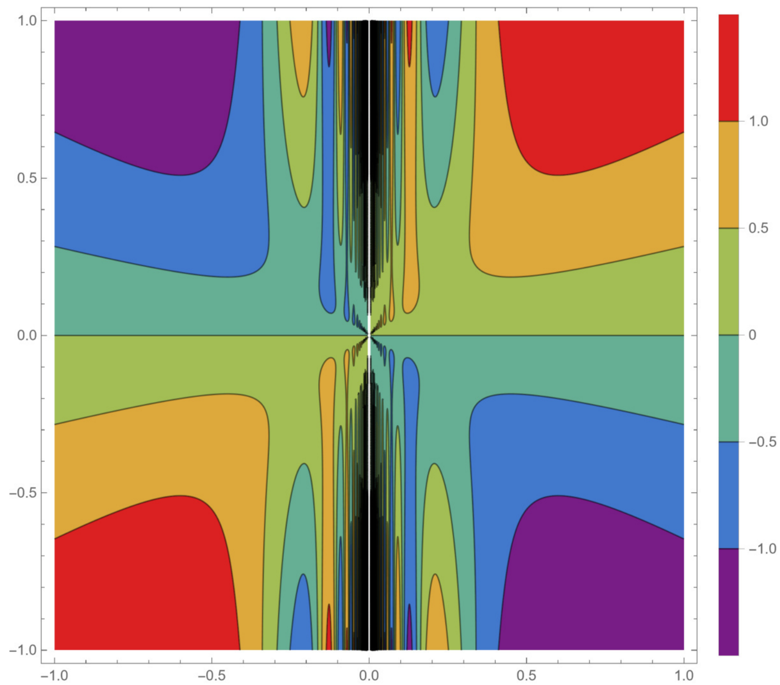


Figure 102. This is the level curves plot of $\frac{xy}{x^2+y^2} + y \sin\left(\frac{1}{x}\right)$. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks very discontinuous indeed. Many differently coloured regions next to each other cause the overall colour to blacken. Figures of this type are notoriously difficult to draw due to the high frequency of the oscillations.

Let us take a look at the restriction of $f(x,y) = y \sin\left(\frac{1}{x}\right)$ on a line with equation $y = \lambda x$.

We have then

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \lambda x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

This restricted function must be continuous. Let us take a look at it.

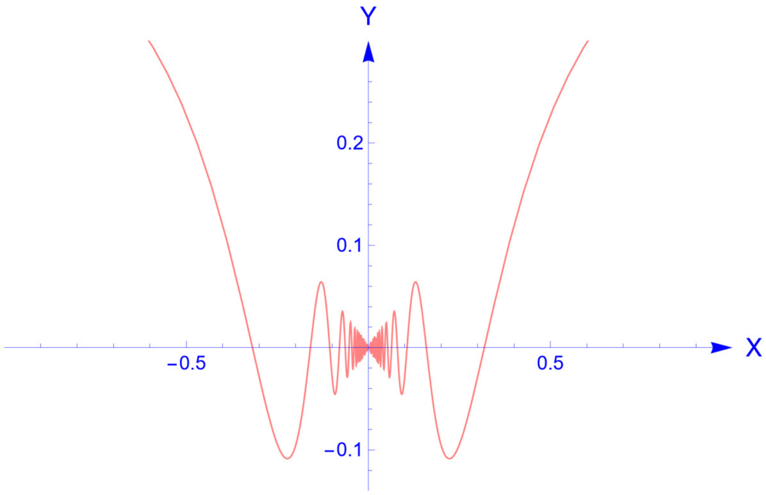


Figure 103. We have restricted the function here to $y = 1/2\,x$. We see here a typical graph of a continuous function. Figures of this type are notoriously difficult to draw due to the high frequency of the oscillations.

Exercise 37.

Is the function

$$f(x,y) = \begin{cases} x\ \sin(1/y) + y\ \sin(1/x) & \text{if } x\,y \neq 0; \\ 0 & \text{if } x\,y = 0 \end{cases}$$

continuous in $(0,0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left|x\ \sin(1/y) + y\ \sin(1/x) - 0\right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$
\begin{aligned}
\left|x \sin(1/y) + y \sin(1/x) - 0\right| &\leq \left|x \sin(1/y) + y \sin(1/x)\right| \\
&\leq |x|\,|\sin(1/y)| + |y|\,|\sin(1/x)| \\
&\leq \sqrt{x^2 + y^2} + \sqrt{x^2 + y^2} \\
&\leq 2\sqrt{x^2 + y^2}.
\end{aligned}
$$

So it is sufficient to prove that $2\sqrt{x^2 + y^2} < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2 + y^2} < \delta$. In order to do that It is sufficient to take $\delta = \epsilon/2$. We can find a δ, so we conclude that the function is continuous.

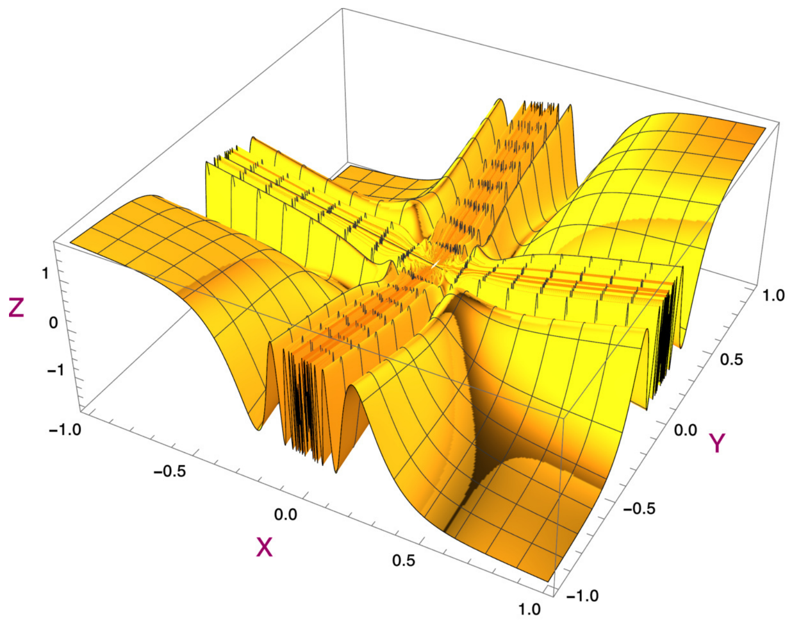


Figure 104. We see here a three dimensional figure of the graph of the function. This type of pictures is notoriously difficult to draw due to the high frequency of the oscillations.

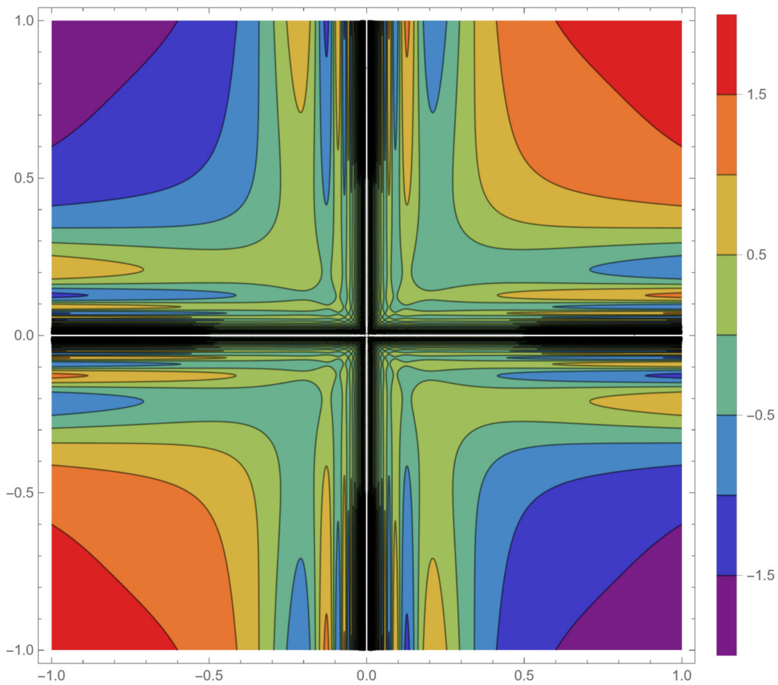


Figure 105. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

Let us illustrate the wild behaviour of this function by looking at the functions restricted on vertical lines with equation $x = \lambda$ with $\lambda \neq 0$. We have then the following function

$$f(x,y) = f(\lambda,y) = \lambda \sin(1/y) + y \sin(1/\lambda).$$

Here is a plot of this restricted function.

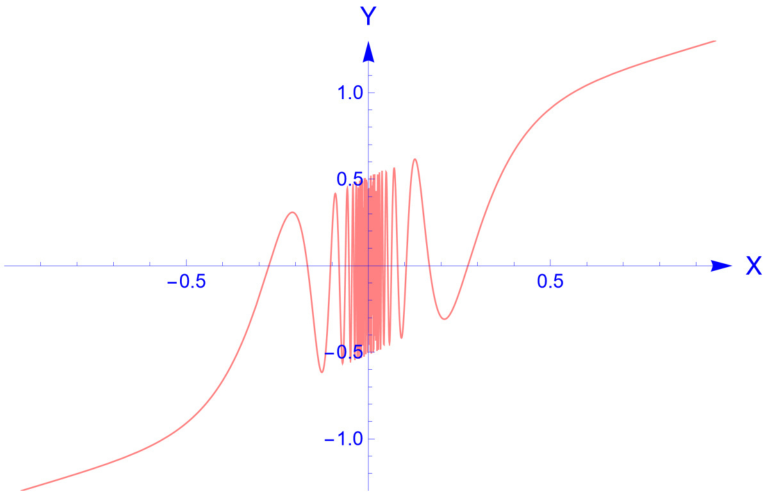


Figure 106. We see here a figure of the graph of the function $\lambda \sin(1/y) + y \sin(1/\lambda)$ with $\lambda = 1/2$. The behaviour is highly non-continuous. This function is a classical one and is in any textbook. It is a standard example of a non continuous function. We conclude that this function is not continuous on any vertical line, except on the Y-axis. We can by symmetry also make the analogue statement for horizontal lines.

But the function $f(x, y)$ is evidently continuous on every line going through the origin. Let us take such a line with equation $y = \lambda x$ with $\lambda \neq 0$. The restriction of the function on this line is

$$f|_{y=\lambda x}(x, y) = \begin{cases} f(x, \lambda x) = \lambda x \sin\left(\frac{1}{x}\right) + x \sin\left(\frac{1}{\lambda x}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

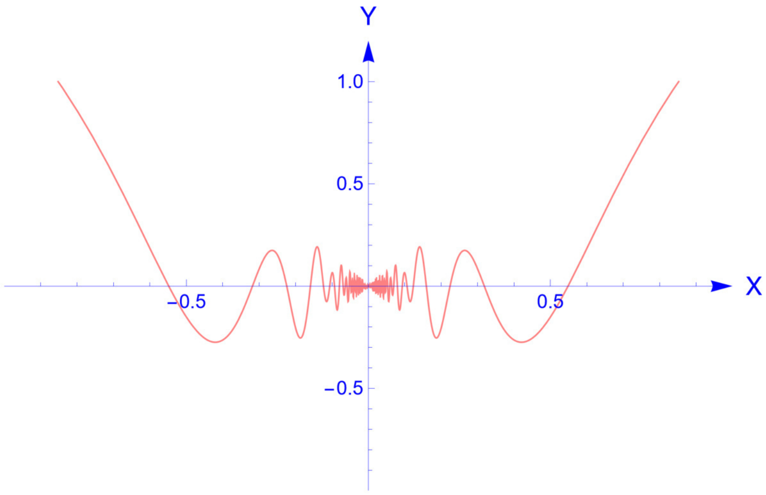


Figure 107. We see here a figure of the graph of the function $\lambda\, x \sin\left(\frac{1}{x}\right) + x \sin\left(\frac{1}{\lambda\, x}\right)$ with $\lambda = 1/2$. The behaviour is continuous. This function is a classical one and can be found in many textbooks. It is a standard example of a continuous function. We conclude that this function is continuous on almost any line ($\lambda \neq 0$) through the origin.

Exercise 38.

Is the function

$$f(x, y) = \begin{cases} \dfrac{x^2\, y^2}{x^3 + y^3} & \text{if } x \neq -y; \\ 0 & \text{if } x = -y. \end{cases}$$

continuous in $(0, 0)$?

Solution.

Let us restrict this function to the line through the origin with equation $y = -x + \lambda\, x^2$ to a small neighbourhood of $(0, 0)$.

$$f\,|_{y=-x+\lambda\, x^2}(x, y) = \begin{cases} \dfrac{x^2\, (\lambda x^2 - x)^2}{(\lambda\, x^2 - x)^3 + x^3} = \dfrac{(\lambda\, x - 1)^2}{\lambda\, (\lambda^2\, x^2 - 3\,\lambda\, x + 3)} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So this case the limit

$$\lim_{x \to 0} f(x, y) = \frac{1}{3\lambda}.$$

We see that these restricted functions have different limits. But if $f(x, y)$ is continuous, all these limit values should be $f(0, 0) = 0$. So this function $f(x, y)$ is not continuous.

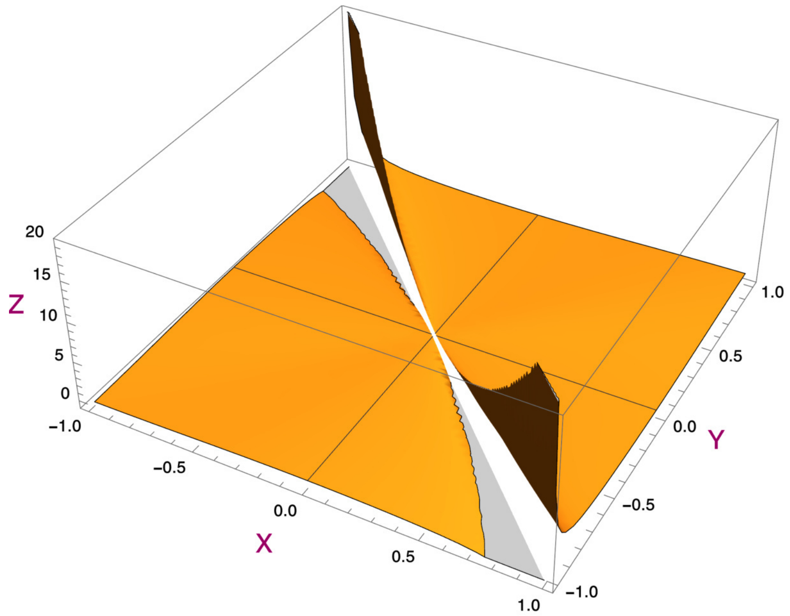


Figure 108. We see here a three dimensional figure of the graph of the function. The unbounded behaviour around $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

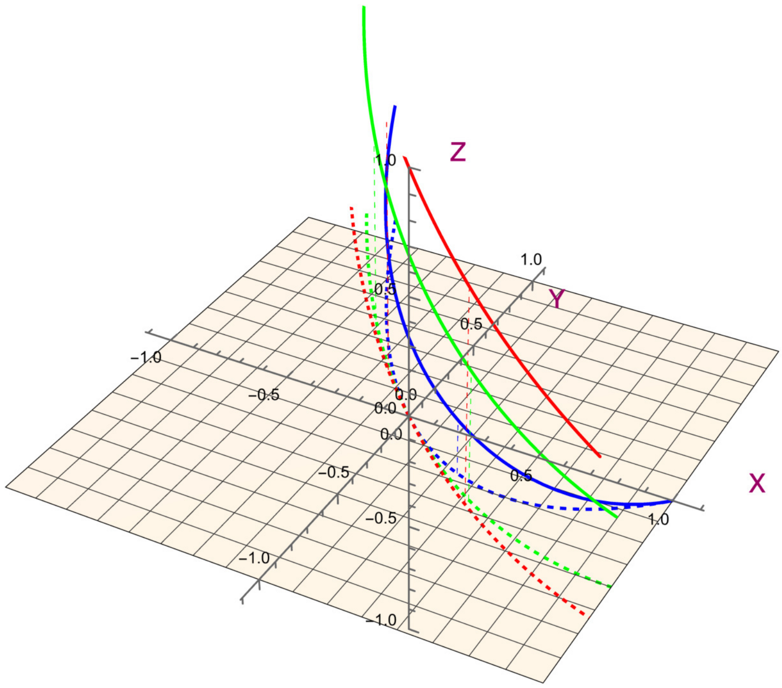


Figure 109. We have restricted the function here to $y = -x + 1/3\,x^2$ and $y = -x + 1/2\,x^2$ and $y = -x + x^2$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

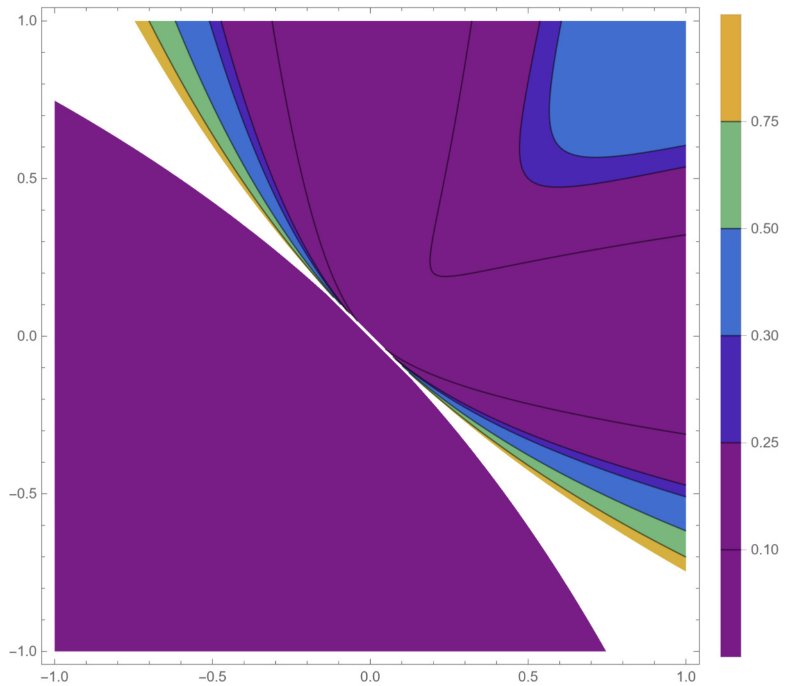


Figure 110. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 39.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x\,y\,(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ holds.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{x\,y\,(x^2 - y^2)}{x^2 + y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{x\,y\,(x^2 - y^2)}{x^2 + y^2} - 0 \right| &\leq \left| \frac{x\,y\,(x^2 - y^2)}{x^2 + y^2} \right| \\
&\leq \frac{|x|\,|y|\,(x^2 + y^2)}{x^2 + y^2} \\
&\leq |x|\,|y| \\
&\leq \sqrt{x^2 + y^2}\sqrt{x^2 + y^2} \\
&\leq \sqrt{x^2 + y^2}^{\,2}.
\end{aligned}$$

So it is sufficient to prove that $\sqrt{x^2 + y^2}^{\,2} < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2 + y^2} < \delta$. In order to do that it is sufficient to take $\delta = \sqrt{\epsilon}$. We can find a δ, so we conclude that the function is continuous.

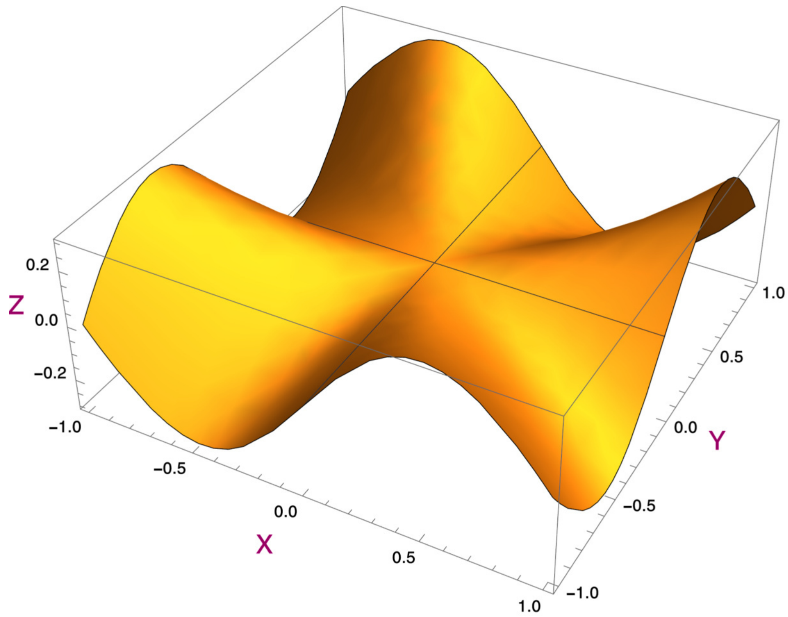


Figure 111. We see here a three dimensional figure of the graph of the function.

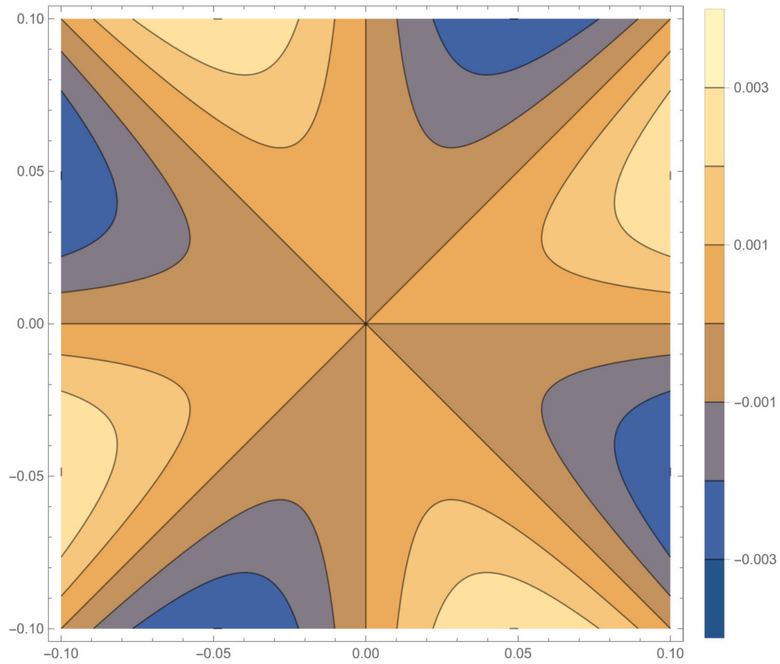


Figure 112. We see here a figure of the plot of the contour curves of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 40.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x^2+\sin^2(y)}{2x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We restrict the function to the continuous curves with equations $y = \lambda x$. We observe then that

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{\sin^2(\lambda x)+x^2}{\lambda^2 x^2 + 2x^2} = \dfrac{\sin^2(\lambda x)+x^2}{(\lambda^2+2)\, x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Let us calculate the limits.

$$\begin{aligned}\lim_{x\to 0} \frac{\sin^2(\lambda x) + x^2}{(\lambda^2+2)\, x^2} &= \lim_{x\to 0} \frac{\frac{\sin(\lambda x)}{\lambda x}\frac{\sin(\lambda x)}{\lambda x}\lambda^2 x^2 + x^2}{(\lambda^2+2)\, x^2} \\ &= \lim_{x\to 0} \frac{\frac{\sin(\lambda x)}{\lambda x}\frac{\sin(\lambda x)}{\lambda x}\lambda^2 + 1}{(\lambda^2+2)} \\ &= \frac{\lambda^2+1}{\lambda^2+2}.\end{aligned}$$

We have used the fact that $\lim_{\alpha\to 0} \sin(\alpha)/\alpha = 1$.

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

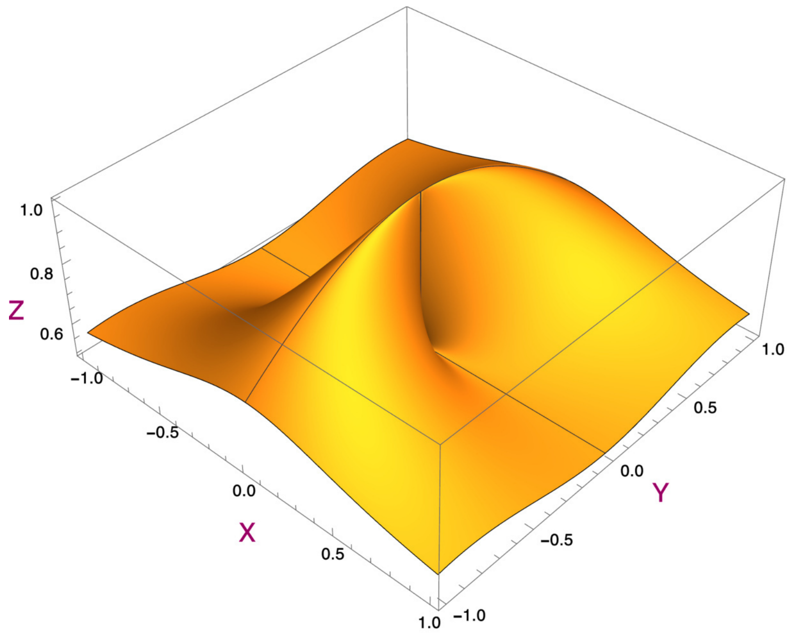


Figure 113. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

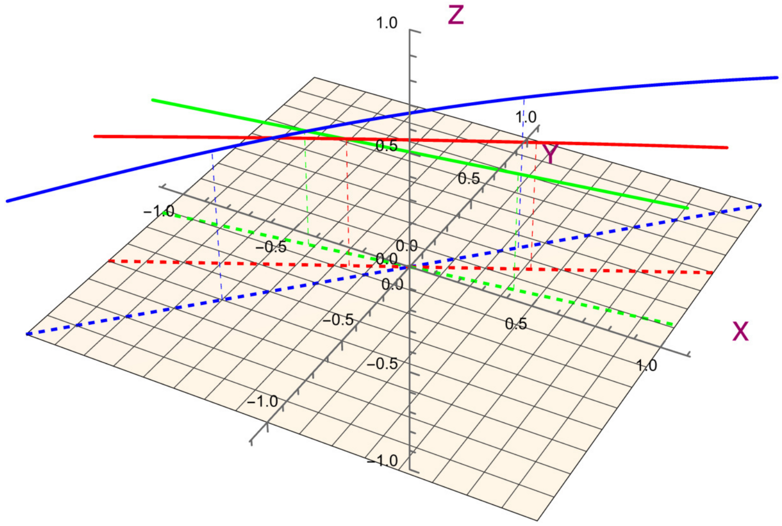


Figure 114. We have restricted the function here to $y = 1/7\,x$ and $y = 1/2\,x$ and $y = x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

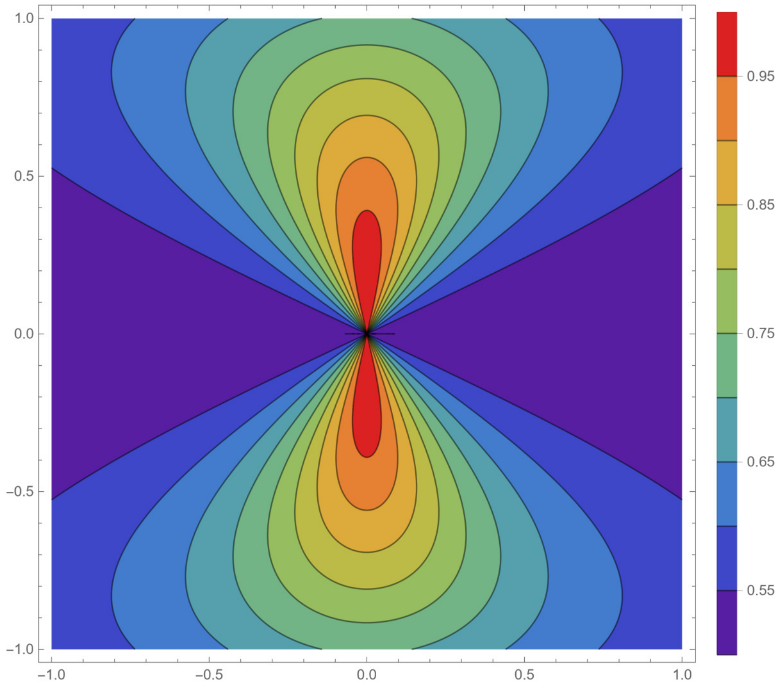


Figure 115. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 41.

Is the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. We try to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ holds.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{x\,y}{\sqrt{x^2 + y^2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{x\,y}{\sqrt{x^2 + y^2}} - 0 \right| &\leq \left| \frac{x\,y}{\sqrt{x^2 + y^2}} \right| \\
&\leq \frac{|x|\,|y|}{\sqrt{x^2 + y^2}} \\
&\leq \left| \frac{\sqrt{x^2 + y^2}\,\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} \right| \\
&\leq \sqrt{x^2 + y^2}.
\end{aligned}$$

So it is sufficient to prove that $\sqrt{x^2 + y^2} < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2 + y^2} < \delta$. In order to do that It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function is continuous.

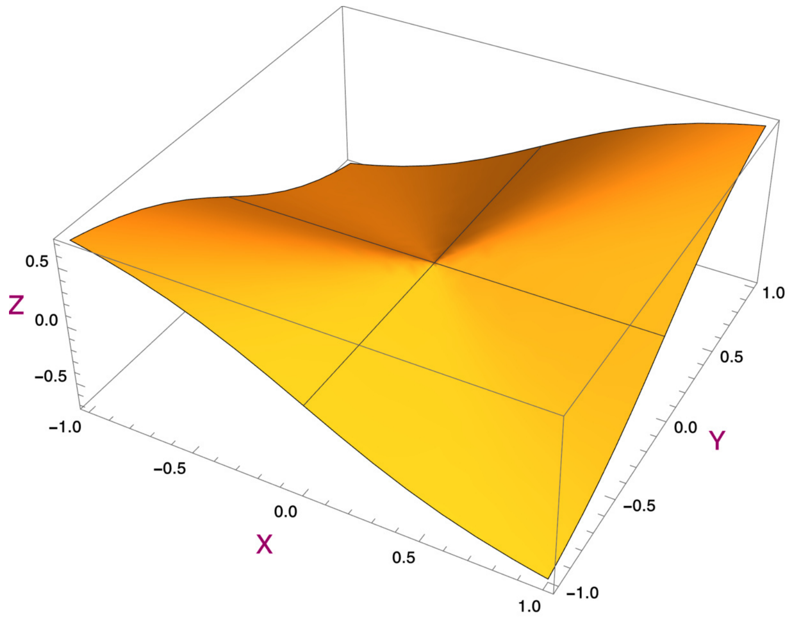


Figure 116. We see here a three dimensional figure of the graph of the function.

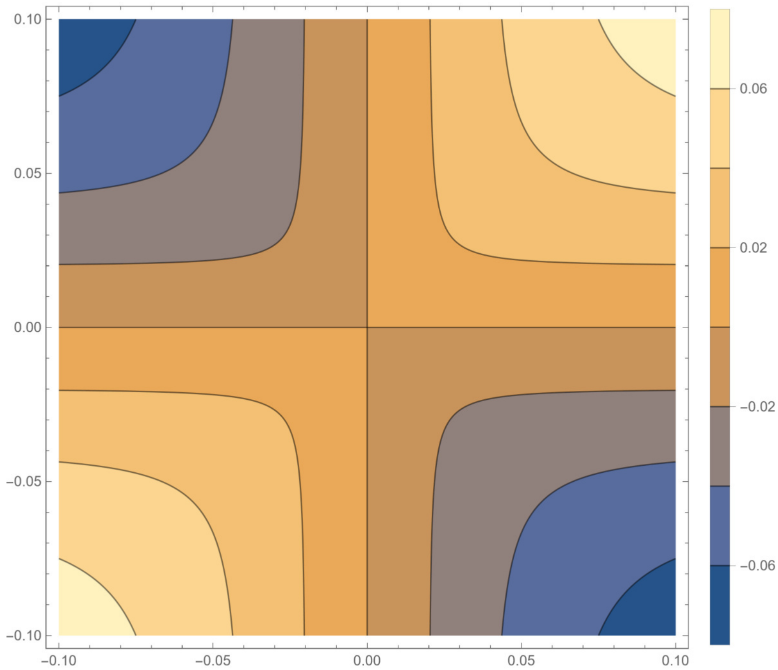


Figure 117. We see here a figure of the plot of the contour curves of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 42.

Is the function

$$f(x,y) = \begin{cases} \dfrac{\sin(x\,y)}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ holds.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left|\frac{\sin(xy)}{\sqrt{x^2+y^2}} - 0\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{\sin(xy)}{\sqrt{x^2+y^2}} - 0\right| &\leq \frac{|x|\,|y|}{\sqrt{x^2+y^2}} \\ &\leq \frac{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} \\ &\leq \sqrt{x^2+y^2}.\end{aligned}$$

We have used that $|\sin(x)| \leq |x|$. So it is sufficient to prove that $\sqrt{x^2+y^2} < \epsilon$ by manipulating the value of δ in the inequality $\sqrt{x^2+y^2} < \delta$. In order to do that it is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function is continuous.

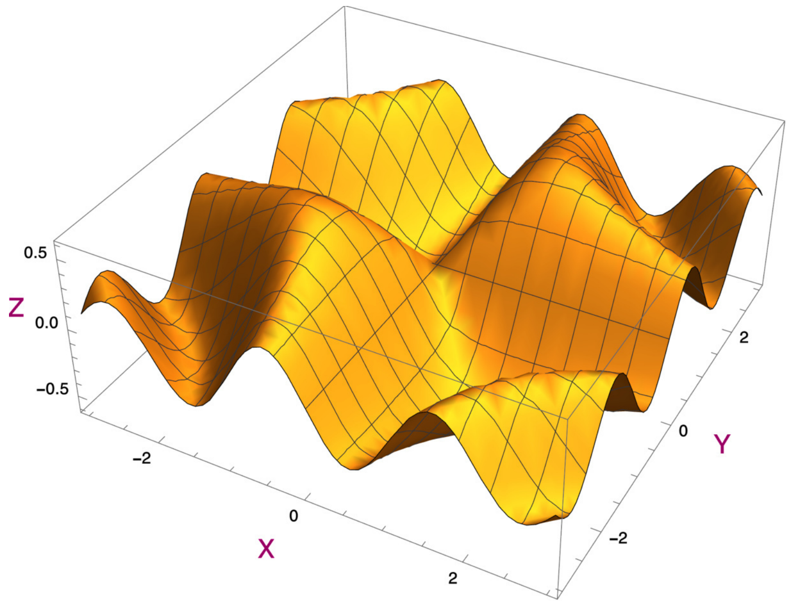


Figure 118. We see here a three dimensional figure of the graph of the function.

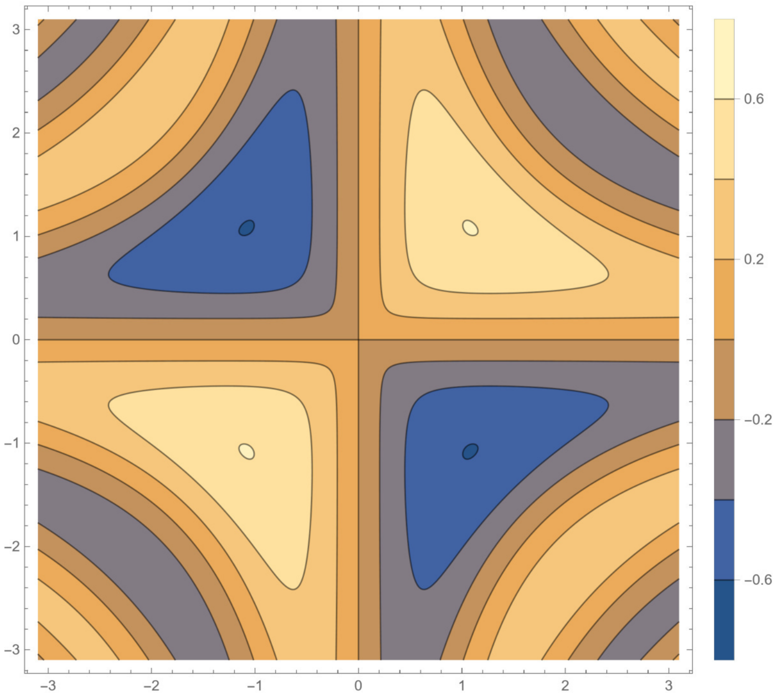


Figure 119. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0, 0)$.

Exercise 43.

Is the function

$$f(x,y) = \begin{cases} \dfrac{\arcsin(x/y)}{1+xy} & \text{if } xy \neq -1,\ y \neq 0 \text{ and } -1 \leq x/y \leq 1; \\ 0 & \text{if } xy = -1 \text{ or } y = 0 \text{ or } -1 > x/y \text{ or } x/y > 1 \end{cases}$$

continuous in $(0,0)$?

Solution.

We restrict the function to the continuous curves with equations $x = \lambda y$ with $|\lambda| < 1$. We observe then that in a small enough neighbourhood of x

$$f|_{x=\lambda y}(x,y) = \begin{cases} f(\lambda y, y) = \dfrac{\arcsin(\lambda)}{\lambda y^2 + 1} & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

Let us calculate the limits.

$$\lim_{y \to 0} = \frac{\arcsin(\lambda)}{\lambda y^2 + 1} = \arcsin(\lambda).$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

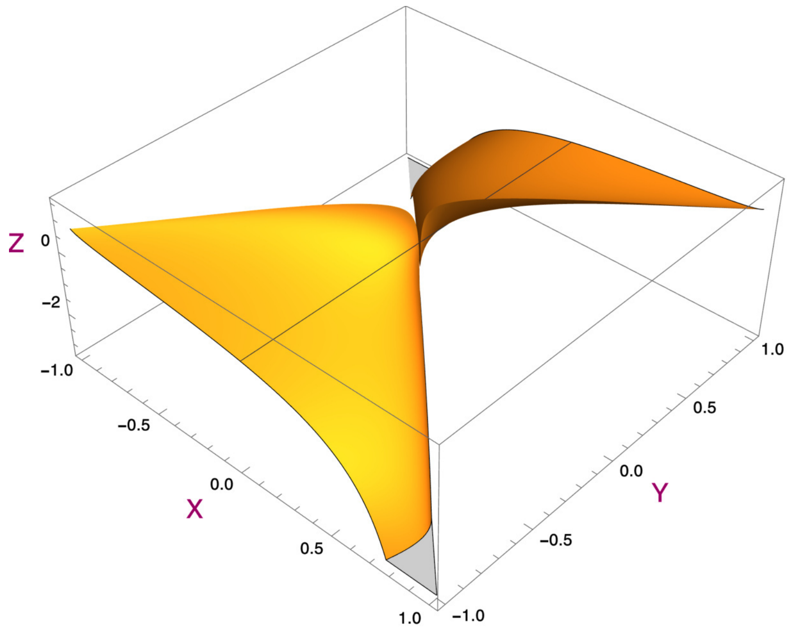


Figure 120. We see here a three dimensional figure of the graph of the function. The vertical line above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

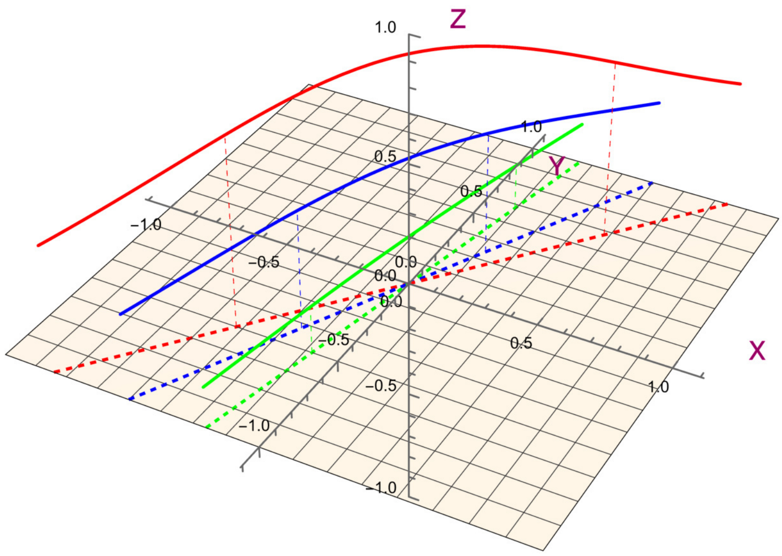


Figure 121. We have restricted the function here to $x = 4/5\,y$ and $x = 1/5\,y$ and $x = 1/2\,y$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

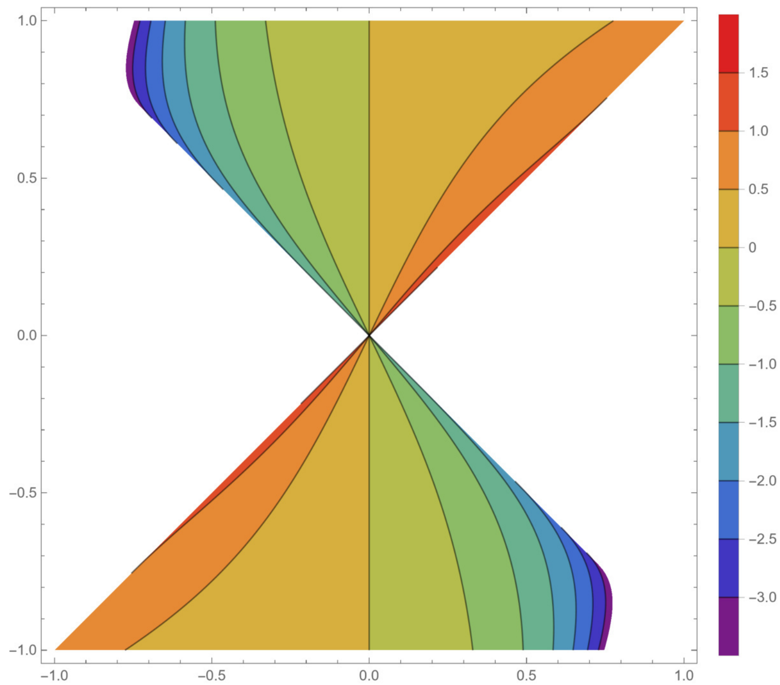


Figure 122. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 44.

Is the function

$$f(x,y) = \frac{x+y}{x^2+y^2+1}$$

continuous in $(0,0)$?

Solution.

This function is by the elementary properties of continuous functions evidently continuous where it exists. Observe that this function exists everywhere. All functions that are used in its definition like the power function, the addition, multiplication and division are continuous.
This exercise looks dull, but in fact this is the so called “general case” in the context of this type of exercises. Our functions have a very simple

function definition consisting of very old and classical functions that are continuous where they exist.

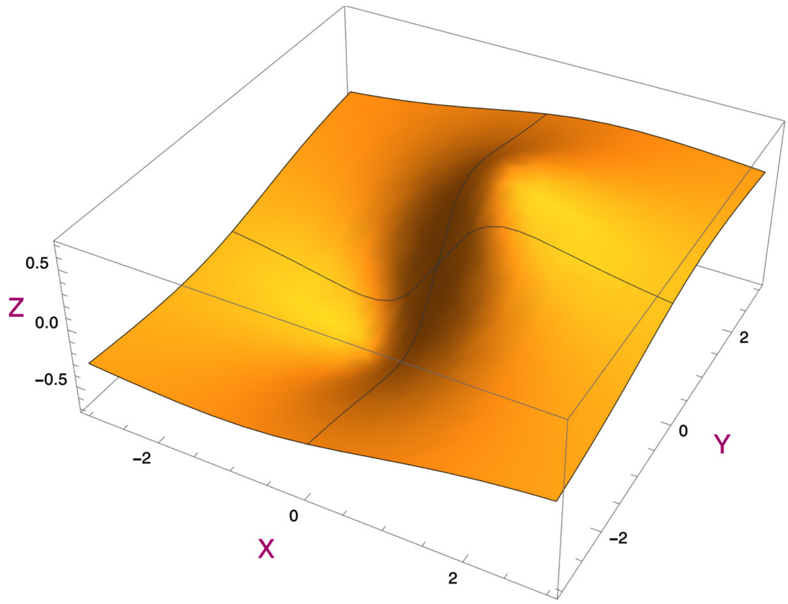


Figure 123. We see here a three dimensional figure of the graph of the function.

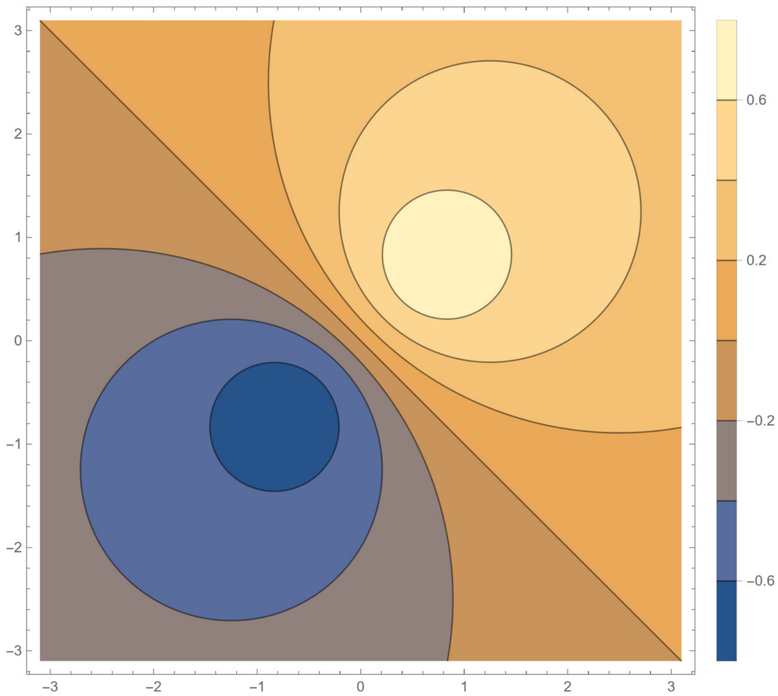


Figure 124. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 45.

Is the function

$$f(x,y) = \begin{cases} \dfrac{(x-1)^2 \ln(x)}{(x-1)^2 + y^2} & \text{if } (x,y) \neq (1,0) \text{ and } x > 0; \\ 0 & \text{elsewhere.} \end{cases}$$

continuous in $(1,0)$? We may use $\ln(1+\alpha) \leq \alpha$ for $\alpha > -1$. Consult the appendix for a proof that does not use differentiability.

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(1,0)| < \epsilon$. The problem is now to find a $\delta > 0$ that if $||(x,y) - (1,0)|| < \delta$ it follows that $|f(x,y) - f(1,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (1,0)|| = \sqrt{(x-1)^2 + y^2} < \delta$, we have that

$$\left| \frac{(x-1)^2 \ln(x)}{(x-1)^2 + y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

Before attacking the main inequality, let us try to get a grip on the behaviour of $\ln(x)$ in $x = 1$. We first recall the basic second inequality of the logarithmic function. See the appendix for a proof. This is $\ln(1+\alpha) \leq \alpha$ for $\alpha > -1$. By using the substitution $x = 1 + \alpha$, we have $\ln(x) \leq x - 1$ for $x > 0$. We use this inequality in our main inequality.

$$
\begin{aligned}
\left| \frac{(x-1)^2 \ln(x)}{(x-1)^2+y^2} - 0 \right| &\le \left| \frac{(x-1)^2 \ln(x)}{(x-1)^2+y^2} \right| \\
&\le \left| \frac{(x-1)^2 (x-1)}{(x-1)^2+y^2} \right| \\
&\le \frac{\sqrt{(x-1)^2+y^2}^{\,3}}{(x-1)^2+y^2} \\
&\le \sqrt{(x-1)^2+y^2}.
\end{aligned}
$$

So it is sufficient to prove that $\sqrt{(x-1)^2+y^2} < \epsilon$ by manipulating the value of $\sqrt{(x-1)^2+y^2}$. It is for example sufficient to take $\delta = \epsilon$. We can find a δ so we conclude that the function is continuous.

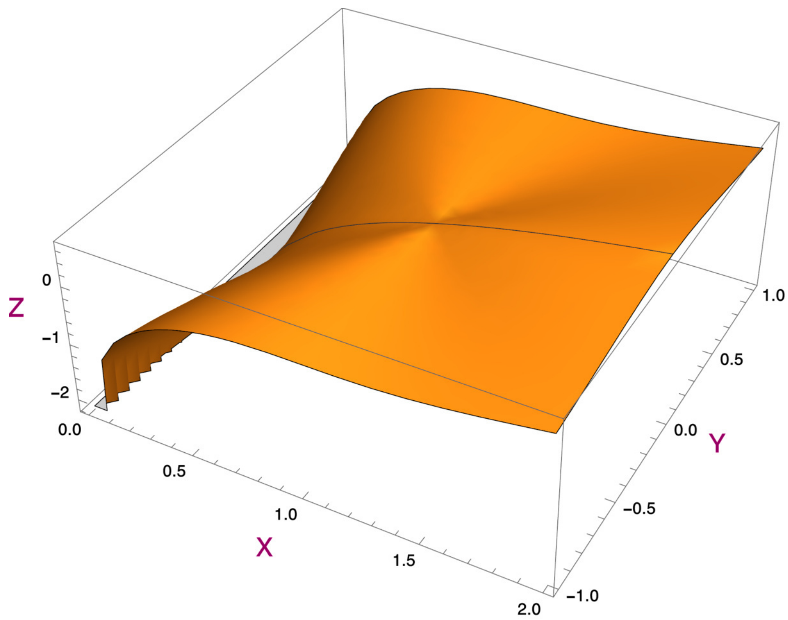


Figure 125. We see here a three dimensional figure of the graph of the function.

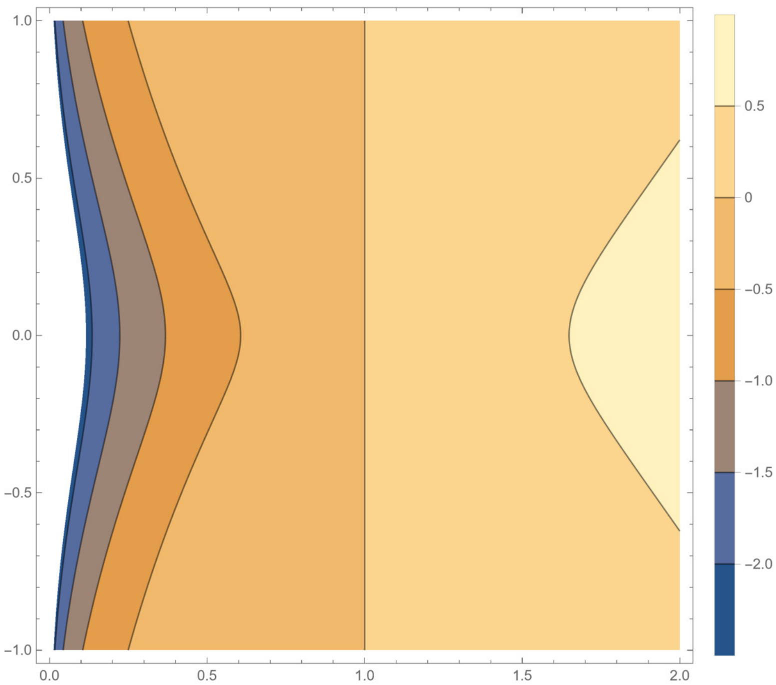


Figure 126. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 46.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x+y}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$?

Solution.

We restrict the function to the continuous curves with equations $y = -x + \lambda x^2$. We observe then that

$$f|_{y=-x+\lambda x^2}(x,y) = \begin{cases} \dfrac{\lambda x^2}{(\lambda x^2 - x)^2 + x^2} = \dfrac{\lambda}{\lambda^2 x^2 - 2\lambda x + 2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Let us calculate the limits.

$$\lim_{x\to 0} \frac{\lambda}{\lambda^2 x^2 - 2\lambda x + 2} = \frac{\lambda}{2}.$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

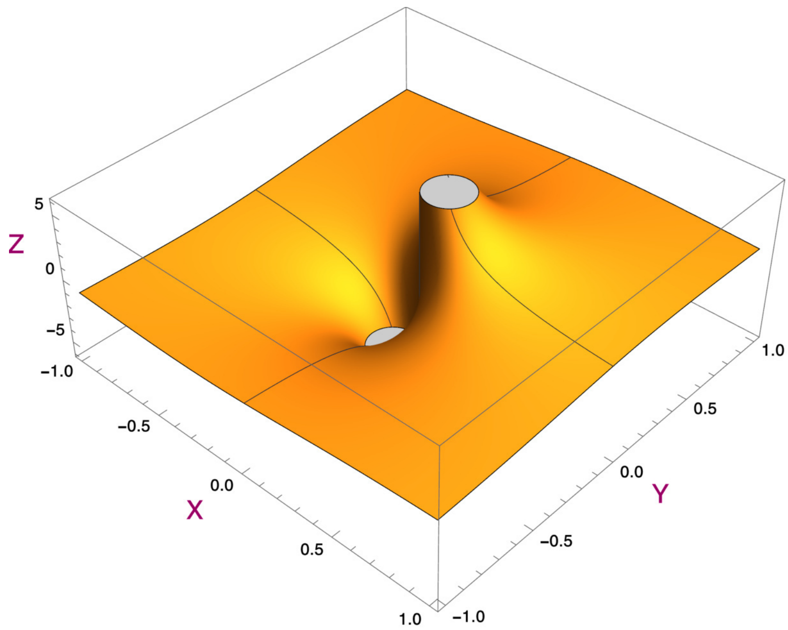


Figure 127. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ and the unboundedness in the neighbourhood of $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

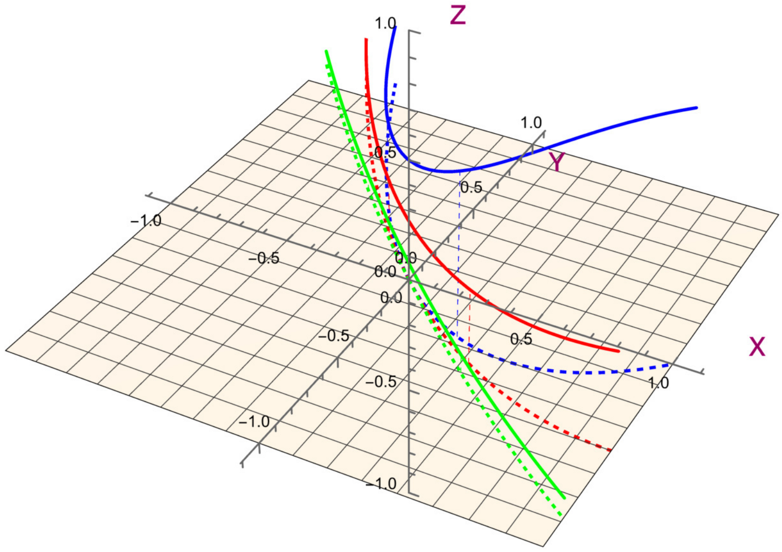


Figure 128. We have restricted the function here to $y = -x + 1/7\,x^2$, $y = -x + 1/2\,x^2$ and $y = -x + x^2$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

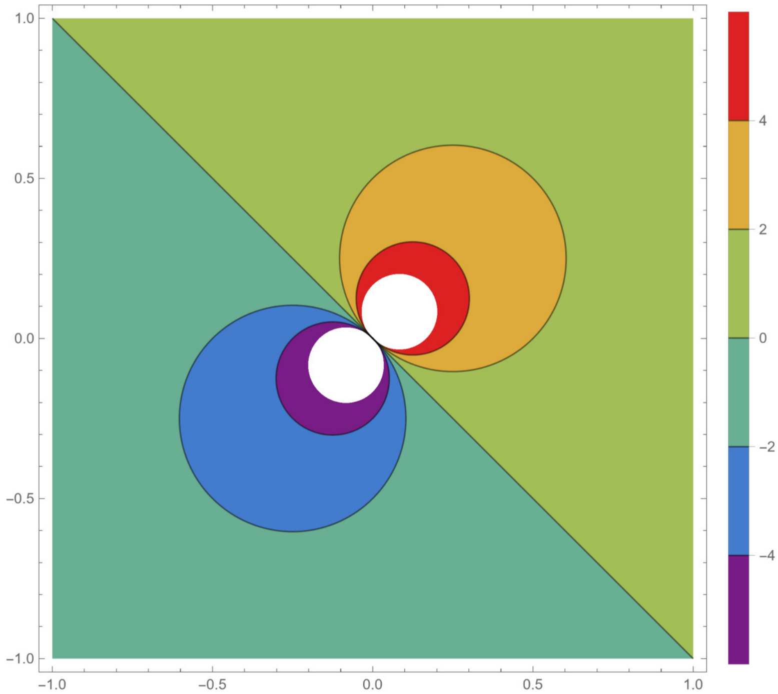


Figure 129. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 47.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^3 + y^3} & \text{if } x \neq -y; \\ 0 & \text{if } x = -y \end{cases}$$

continuous in $(0,0)$?

Solution.

We restrict the function to the continuous curves with equations $y = \lambda x$. We observe then that

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{x^3 - \lambda^3 x^3}{\lambda^3 x^3 + x^3} = \dfrac{1-\lambda^3}{\lambda^3+1} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

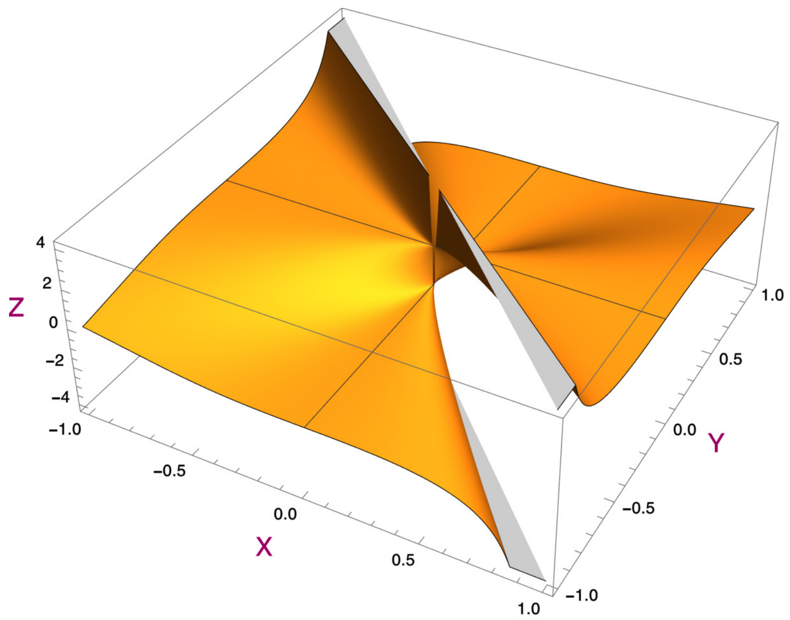


Figure 130. We see here a three dimensional figure of the graph of the function. The vertical line and the unboundedness above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

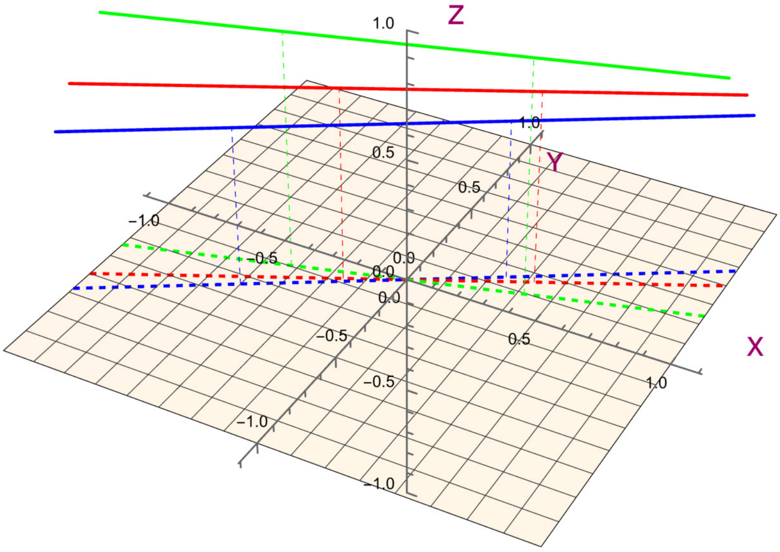


Figure 131. We have restricted the function here to $y = 1/2\,x$ and $y = 3/10\,x$ and $y = 3/5\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

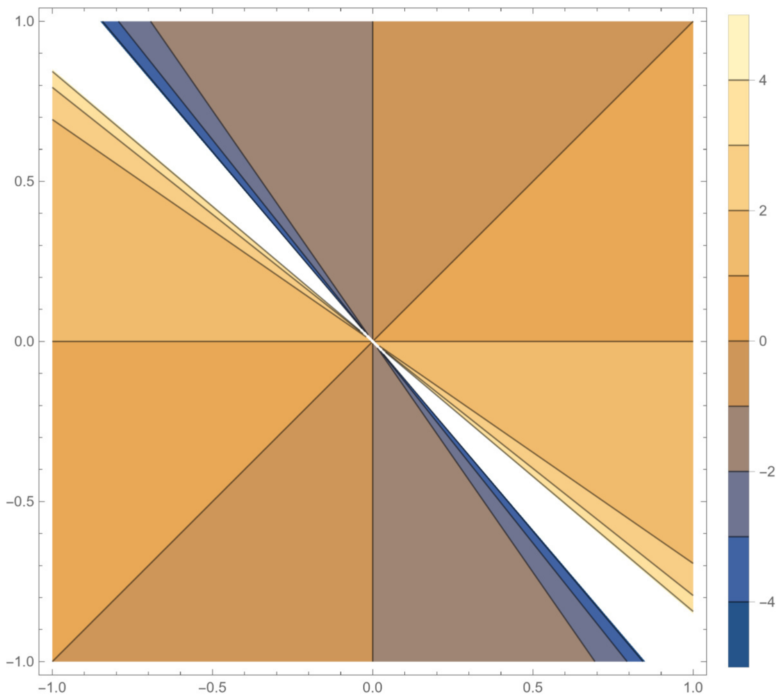


Figure 132. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 48.

Is the function

$$f(x,y) = \begin{cases} \arctan\left(\dfrac{|x|+|y|}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0); \\ \pi/2 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous in $(0,0)$? One may assume that we know that $\lim_{\alpha\to\infty} \arctan(\alpha) = \pi/2$.

Solution.

Let us prove it with a ϵ-δ approach.

We take an arbitrary $\epsilon > 0$. We know that $\lim_{\alpha\to\infty} \arctan(\alpha) = \pi/2$. So we know that there exists a number M so that if $\alpha > M$, then $|\arctan(\alpha) -$

$\pi/2| < \epsilon$. The problem is now to find a $\delta > 0$ such that if (x, y) is in the neighbourhood $||(x, y) - (0,0)|| < \delta$ punctured in $(0,0)$, so $(0,0)$ is excluded from this disk, it follows that $(|x| + |y|)/(x^2 + y^2) > M$ holds. So we are left to prove that if $(x, y) \neq (0,0)$ and $\sqrt{x^2 + y^2} < \delta$

$$\frac{|x| + |y|}{x^2 + y^2} > M \text{ or equivalently } |x| + |y| > M\,(x^2 + y^2).$$

by manipulating the value of δ in the inequality $\sqrt{x^2 + y^2} < \delta$.

If we look at the inequality $|x| + |y| > M\,(x^2 + y^2)$, we see that it is enough to have that both $|x| > M\,x^2$ and $|y| > M\,y^2$ or that $1 > M\,|x|$ and $1 > M\,|y|$. So we must have both $|x| < 1/M$ and $|y| < 1/M$. But if we take $\delta = 1/M$, then $|x| \leq \sqrt{x^2 + y^2} < 1/M$ and $|y| \leq \sqrt{x^2 + y^2} < 1/M$. So we can find a $\delta = 1/M$ and the function is continuous.

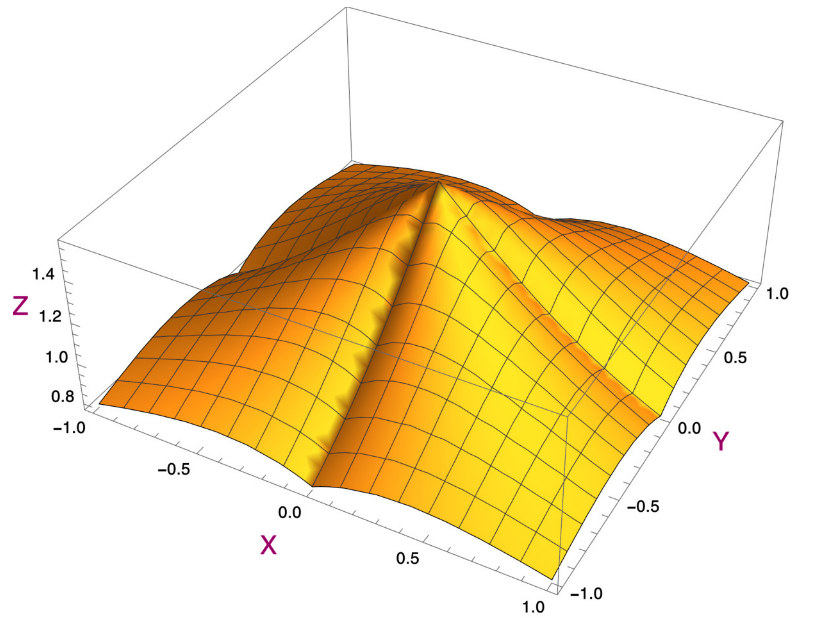


Figure 133. We see here a three dimensional figure of the graph of the function.

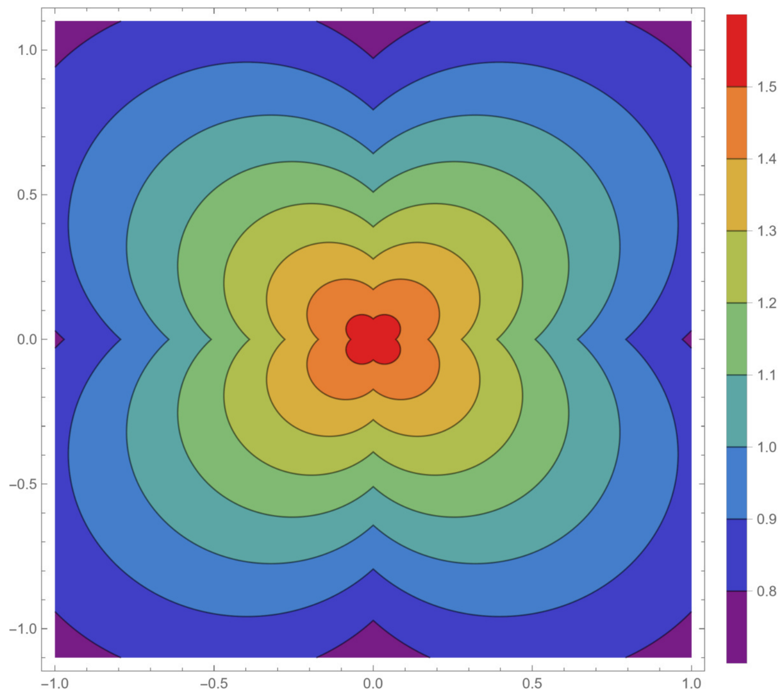


Figure 134. We see here a figure of the contour plot of the function. Only level curves of a larger level around $\pi/2$ come close to $(0,0)$.

Exercise 49.

Is the following function continuous in $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{e^{-x^2-y^2}-1}{x^2+y^2} + 1 & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

It is allowed to use the traditional inequality: $\alpha + 1 \leq e^{\alpha} \leq \dfrac{1}{1-\alpha}$ with $\alpha < 1$. See appendix for a proof.

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0, 0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{e^{-x^2-y^2} - 1}{x^2 + y^2} + 1 - 0 \right| < \epsilon.$$

We try to find something larger than the left side of the inequality and try then to keep that larger expression smaller than ϵ by manipulating the value of δ in the inequality $\sqrt{x^2 + y^2} < \delta$. If we could do that, then we have proven our inequality.
We will use the classical inequality of the exponential function which one can prove without using differentiation theory as explained in the appendix.

$$\alpha + 1 \leq e^{\alpha} \leq \frac{1}{1 - \alpha}$$

which is valid for $\alpha < 1$.
We restrict further and demand that $\alpha > 0$. We invert the three terms of the inequality and multiply them with -1. We add 1 to the three terms and divide by α. We subtract -1 of the three terms and find again with $0 < \alpha < 1$

$$\frac{-\alpha}{1 + \alpha} \leq \frac{1 - e^{-\alpha}}{\alpha} - 1 \leq 0.$$

When $\alpha = x^2 + y^2$ we have then by using this inequality

$$\begin{aligned}
\left| \frac{e^{-x^2-y^2} - 1}{x^2 + y^2} + 1 - 0 \right| &\leq \left| \frac{1 - e^{-x^2-y^2}}{x^2 + y^2} - 1 \right| \\
&\leq \frac{x^2 + y^2}{x^2 + y^2 + 1} \\
&\leq x^2 + y^2 \\
&\leq \sqrt{x^2 + y^2}^{\,2}.
\end{aligned}$$

It is sufficient to take $\delta = \min\{1, \sqrt{\epsilon}\}$. We can find a δ, so we conclude that the function is continuous.

Remark. The inequalities are easier if one is allowed to use the theory of differentiation in one variable. We can use e.g. the theorem of McLaurin.

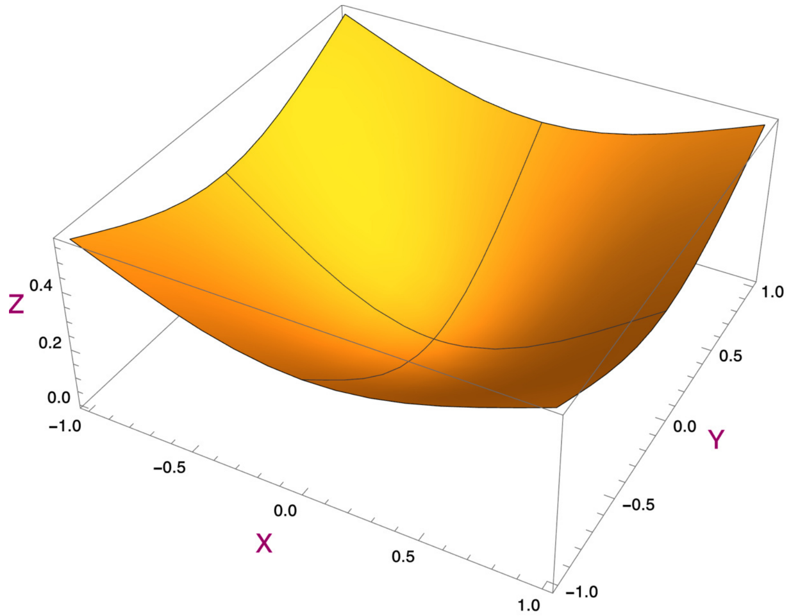


Figure 135. We see here a three dimensional figure of the graph of the function.

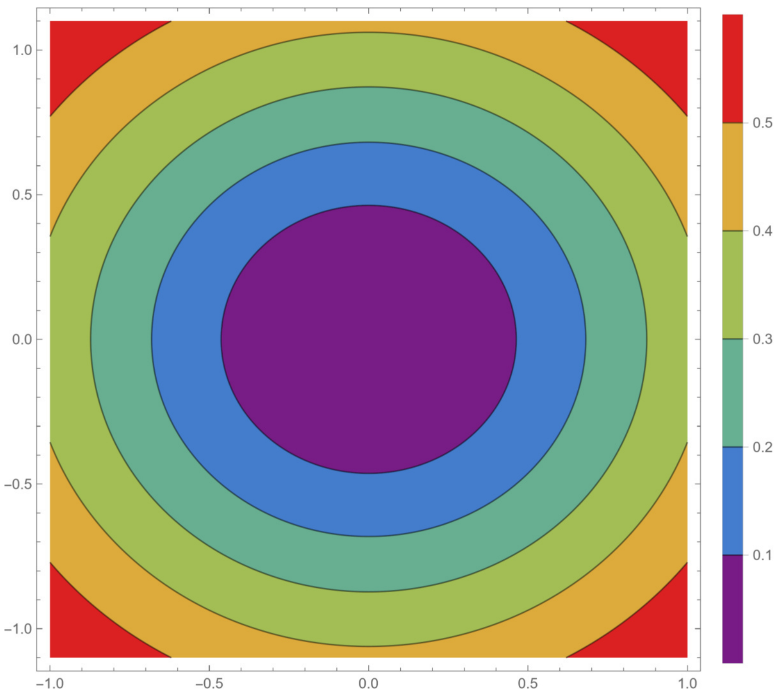


Figure 136. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 50.

Is the following function continuous in $(0,0)$?

$$f(x,y) = \begin{cases} 0 & \text{if } x\,y \neq 0; \\ 1 & \text{if } x\,y = 0. \end{cases}$$

Solution.

Let us restrict this function to the curve $y = x$. We observe then that

$$f|_{y=x}(x,y) = \begin{cases} f(x,x) = 0 & \text{if } x \neq 0; \\ 1 & \text{if } x = 0. \end{cases}$$

We see that this restricted function is discontinuous in $x = 0$. But if $f(x,y)$ is continuous, then it should be continuous when restricted to this continuous curve. We conclude that $f(x,y)$ is discontinuous.

Exercise 51.
Is the following function continuous in $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{xy}{|x|+|y|} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Solution.
We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{xy}{|x|+|y|} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned} \left| \frac{xy}{|x|+|y|} - 0 \right| &\leq \left| \frac{xy}{|x|+|y|} \right| \\ &\leq \frac{|x|\,|y|}{|x|+|y|} \\ &\leq \frac{(|x|+|y|)\,(|x|+|y|)}{|x|+|y|} \\ &\leq |x|+|y| \\ &\leq 2\sqrt{x^2+y^2}. \end{aligned}$$

It is sufficient to take $\delta = \epsilon/2$. We can find a δ, so we conclude that the function is continuous.

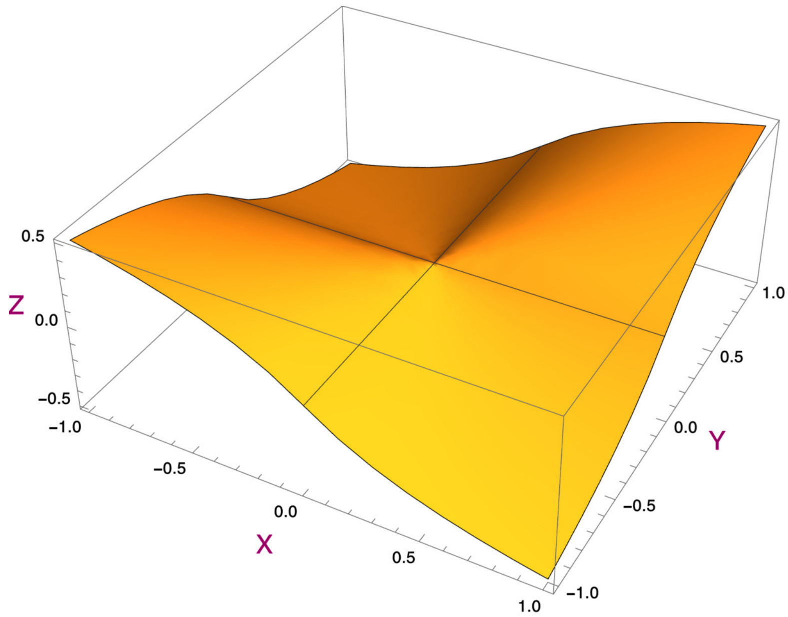


Figure 137. We see here a three dimensional figure of the graph of the function.

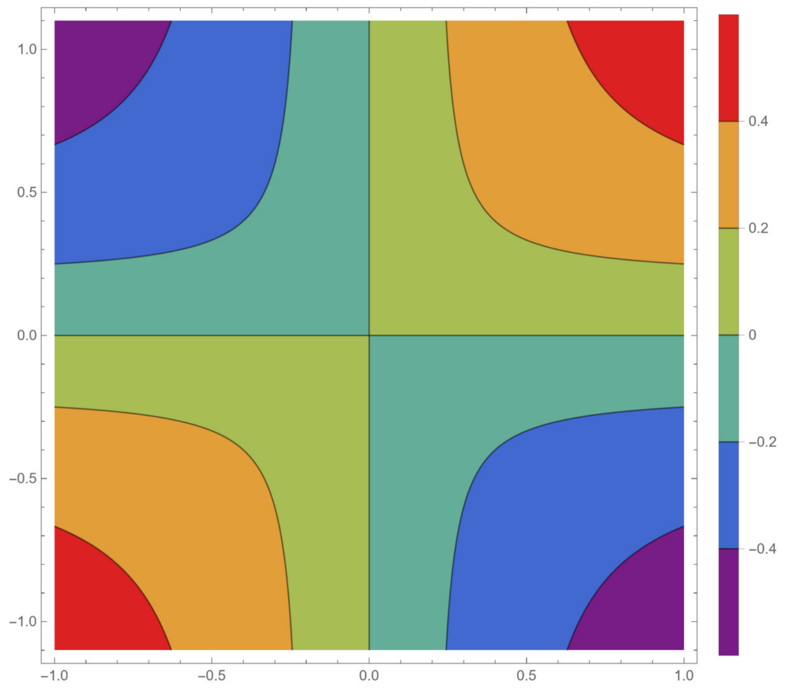


Figure 138. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 52.

Is the following function continuous in $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{x^2-2y}{y^2+2x} & \text{if } y^2+2x \neq 0; \\ 0 & \text{if } y^2+2x = 0. \end{cases}$$

Solution.

We restrict the function to the continuous curves with equations $y = \lambda x$. We observe then that

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{x-2\lambda}{\lambda^2 x+2} & \text{if } x \neq 0 \text{ and } x \neq -2/\lambda^2; \\ 0 & \text{if } x = 0 \text{ or } x = -2/\lambda^2. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

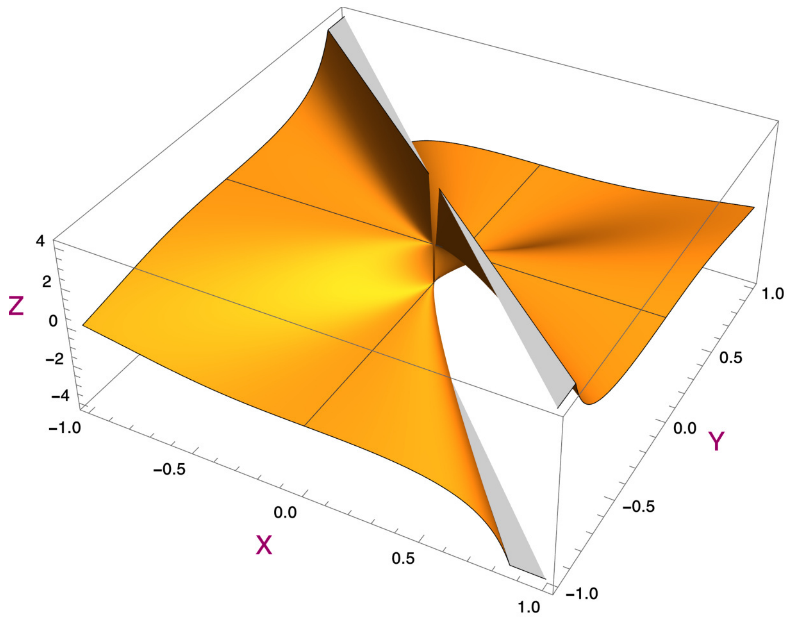


Figure 139. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

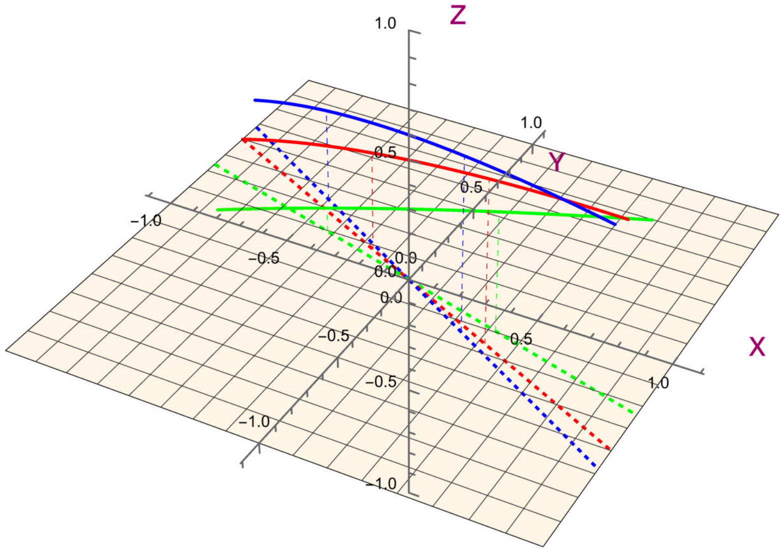


Figure 140. We have restricted the function here to $y = 1/2\,x$ and $y = 3/10\,x$ and $y = 3/5\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

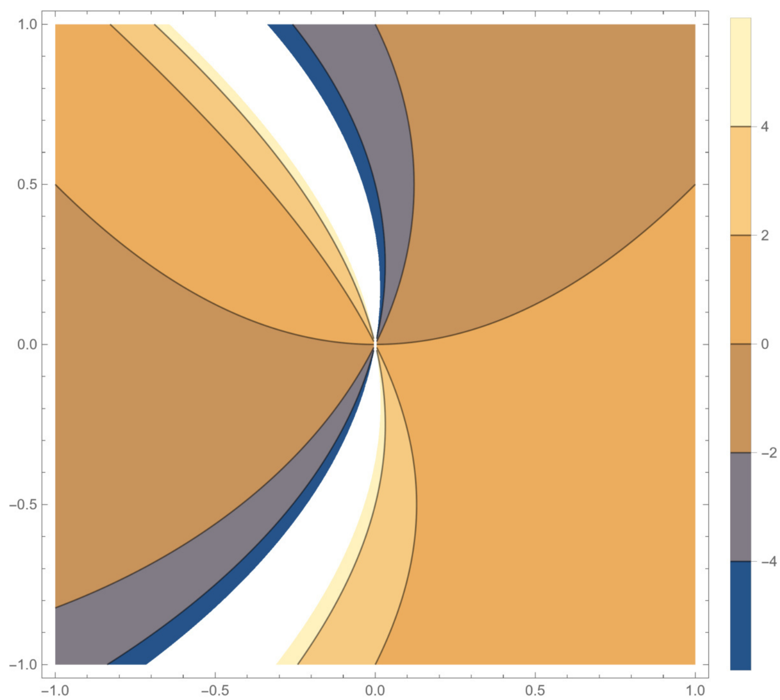


Figure 141. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 53.

Is the function

$$f(x,y) = \begin{cases} \dfrac{x^2 y^2}{x^4 + y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous in $(0,0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{x^2 y^2}{x^4 + y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$
\begin{aligned}
\left|\frac{x^2y^2}{x^4+y^2}-0\right| &\leq \frac{x^2y^2}{x^4+y^2}\\
&\leq \frac{\sqrt{x^2+y^2}^2\sqrt{x^2+y^2}^2}{\sqrt{x^2+y^2}^4+\sqrt{x^2+y^2}^2}\\
&\leq \frac{\sqrt{x^2+y^2}^4}{\sqrt{x^2+y^2}^2\left(\sqrt{x^2+y^2}^2+1\right)}\\
&\leq \frac{\sqrt{x^2+y^2}^2}{\sqrt{x^2+y^2}^2+1}\\
&\leq \sqrt{x^2+y^2}^2.
\end{aligned}
$$

It is sufficient to take $\delta = \sqrt{\epsilon}$. We can find a δ, so we conclude that the function is continuous.

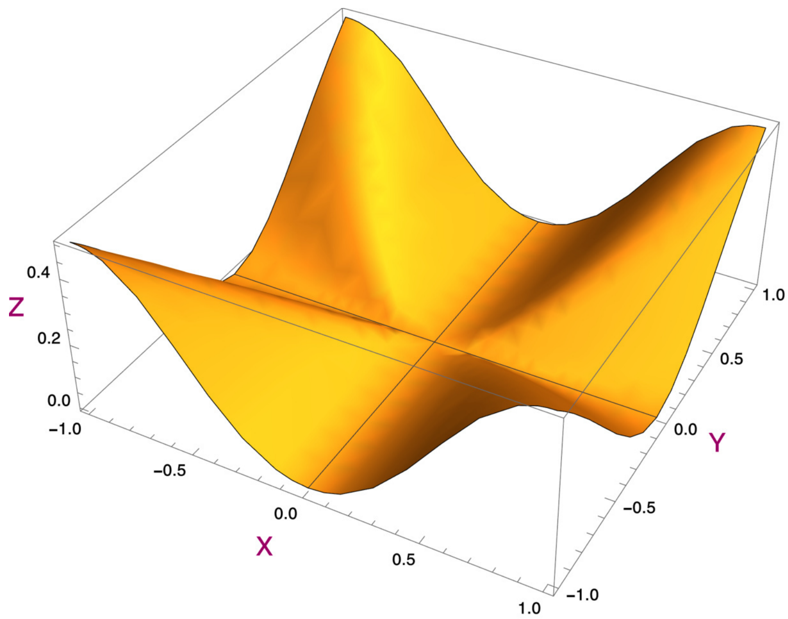


Figure 142. We see here a three dimensional figure of the graph of the function.

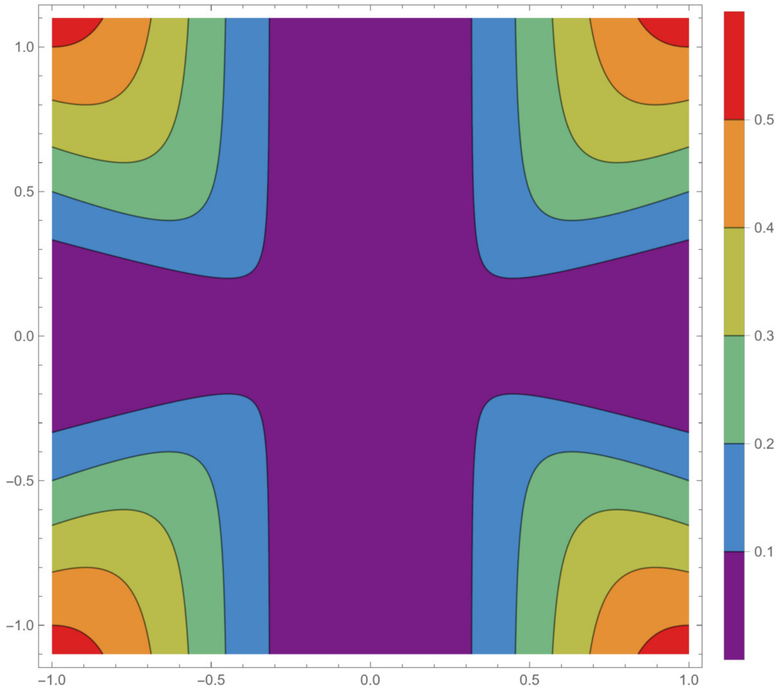


Figure 143. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 54.

Is the following function continuous in $(0,0)$?

$$f(x,y)=\begin{cases}\dfrac{\sin\left(x^2+4y^2\right)}{x^2+y^2} & \text{if } (x,y)\neq(0,0);\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

We assume that we know that $\lim_{\alpha\to 0}\sin(\alpha)/\alpha=1$.

Solution.

We can rewrite the function definition as

$$\frac{\sin\left(x^2+4y^2\right)}{x^2+y^2}=\frac{\sin\left(x^2+4y^2\right)}{x^2+4y^2}\,\frac{x^2+4y^2}{x^2+y^2}.$$

We know that $\frac{\sin\left(x^2+4y^2\right)}{x^2+4y^2}$ is continuous in $(0,0)$, so it is enough to prove or disprove that $\frac{x^2+4y^2}{x^2+y^2}$ is continuous.

We restrict this last function to the continuous curves with equations $y = \lambda x$. We observe then that

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{4\lambda^2 x^2 + x^2}{\lambda^2 x^2 + x^2} = \dfrac{4\lambda^2+1}{\lambda^2+1} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

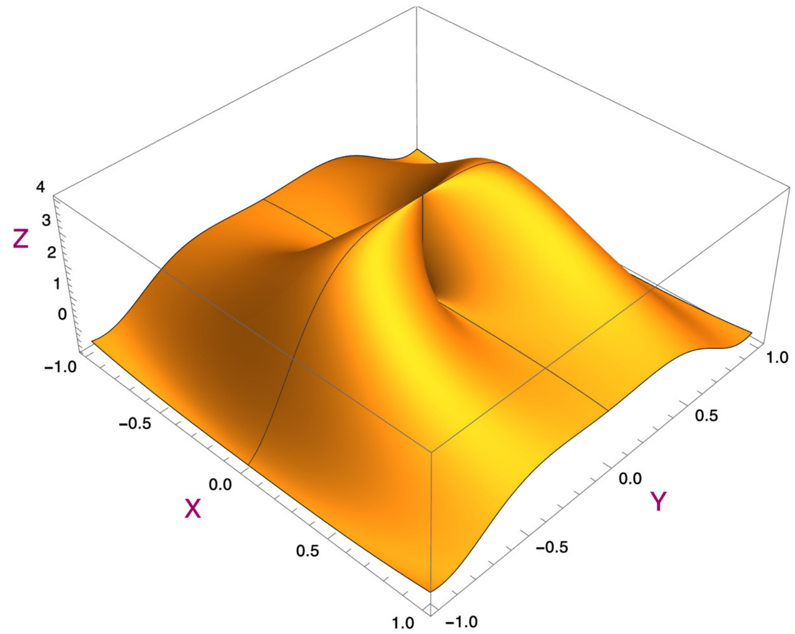


Figure 144. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

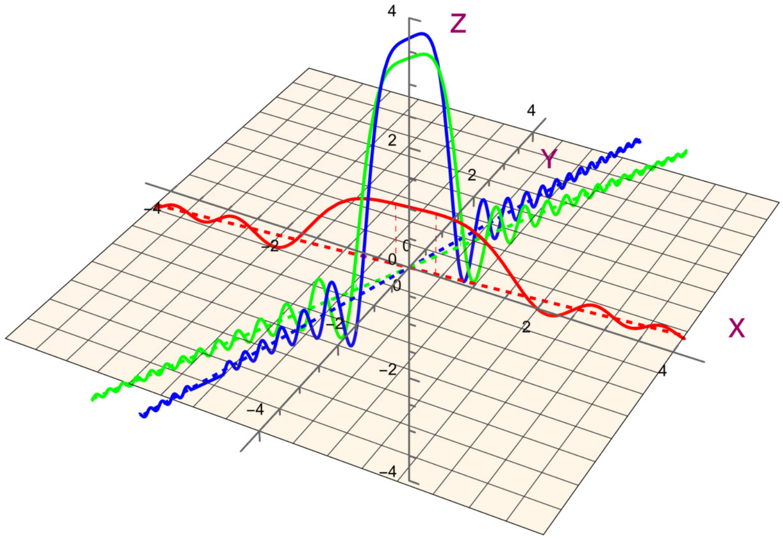


Figure 145. We have restricted the function here to $y = 1/10\,x$, $y = 2\,x$ and $y = 3\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

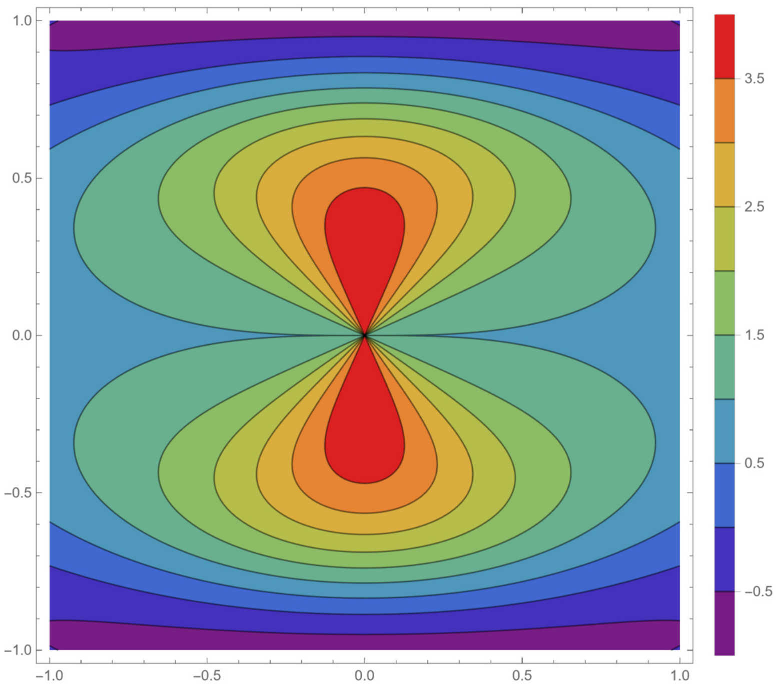


Figure 146. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0, 0)$. This looks discontinuous indeed.

Exercise 55.

Is the following function continuous in $(0, 0)$?

$$f(x, y) = \begin{cases} |x|^y - 1 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Solution.

We restrict this function to the continuous curves with equations $y = \frac{\lambda}{\ln |x|}$. These curve are continuous in $x = 0$ with limit value $y = 0$. We do look to these curves only in a small neighbourhood of $(0, 0)$ in order to avoid difficulties with $x = -1$ and $x = 1$. We observe then that

$$f\,\Big|_{y=\frac{\lambda}{\ln |x|}}(x, y) = \begin{cases} f\left(x, \frac{\lambda}{\ln |x|}\right) = e^{\lambda} - 1. & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x, y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x, y)$ is not continuous in $(0,0)$.

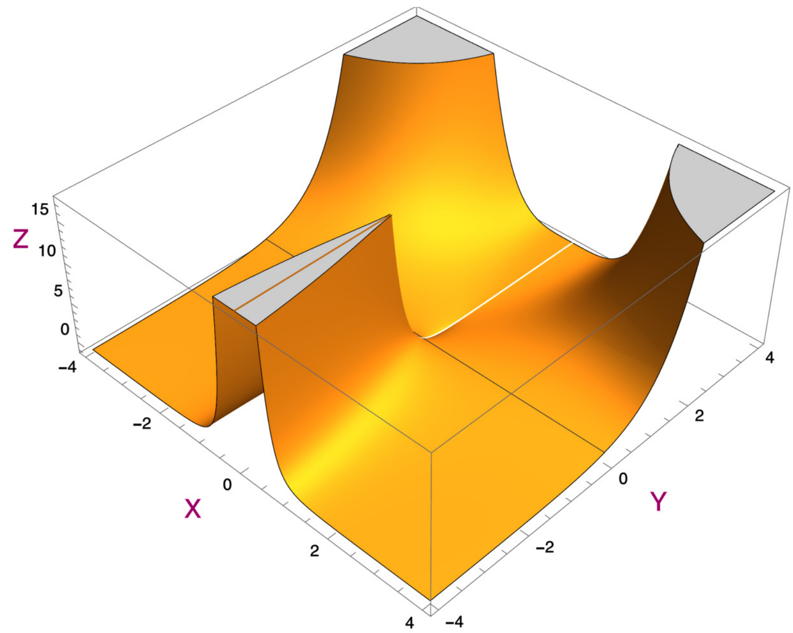


Figure 147. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

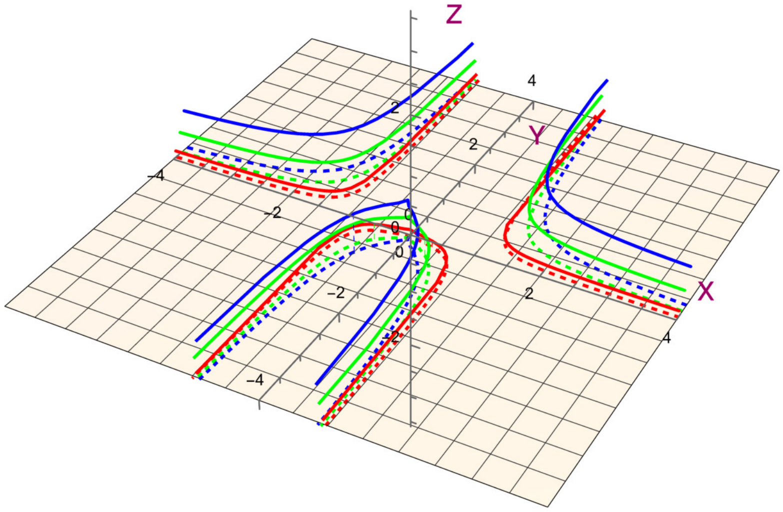


Figure 148. We have restricted the function here to $y = \frac{1/10}{\ln(|x|)}$, $y = \frac{2/5}{\ln(|x|)}$ and $y = \frac{7/10}{\ln(|x|)}$. We see in this figure clearly that the restrictions of the function to these curves are functions that have different limits in 0.

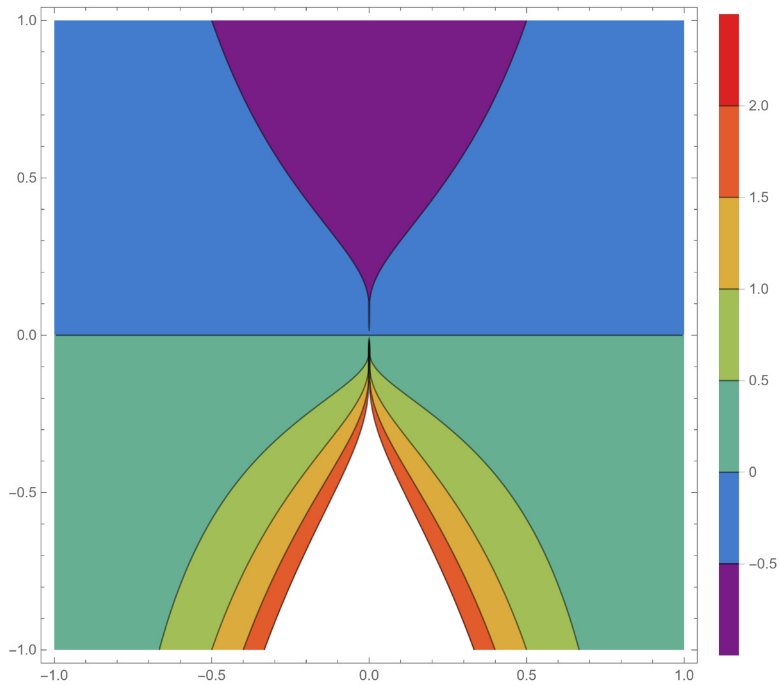


Figure 149. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0, 0)$. This looks discontinuous indeed.

Exercise 56.

Is the following function continuous in $(0, 0)$?

$$f(x, y) = \begin{cases} \dfrac{e^{-x/y}}{y} & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

Solution.

We restrict this function to the continuous curves with equations $y = \lambda x$ with $\lambda \neq 0$. We observe then that

$$f|_{y=\lambda x}(x, y) = \begin{cases} f(x, \lambda x) = \dfrac{e^{-1/\lambda}}{\lambda x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions are unbounded in any neighbourhood of $x = 0$. But if $f(x, y)$ is continuous in $(0, 0)$, then it is bounded in a small enough neighbourhood of $(0, 0)$. So this function $f(x, y)$ is not continuous.

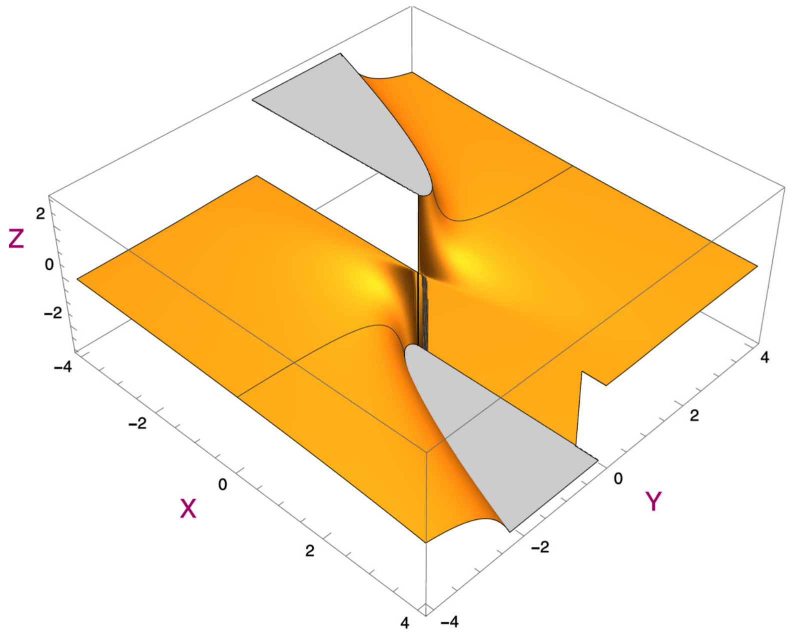


Figure 150. We see here a three dimensional figure of the graph of the function. The vertical line above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

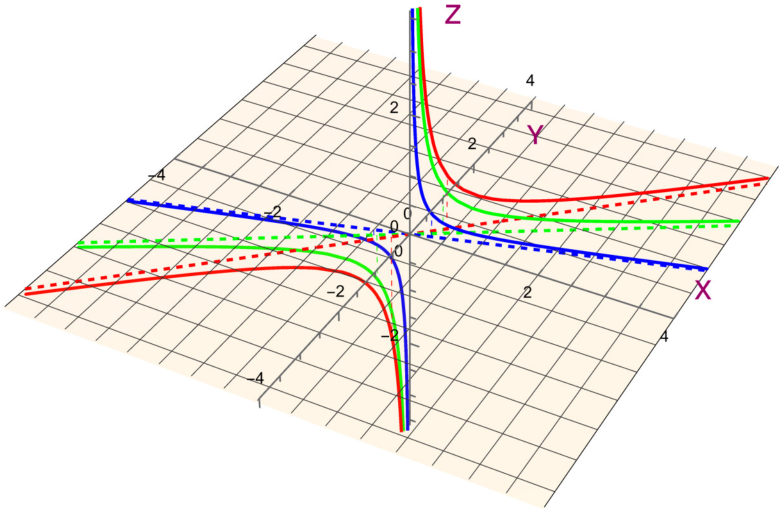


Figure 151. We have restricted the function here to $y = 3/10\,x$ and $y = 6/10\,x$ and $y = 9/10\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have unbounded behaviour in any neighbourhood of 0.

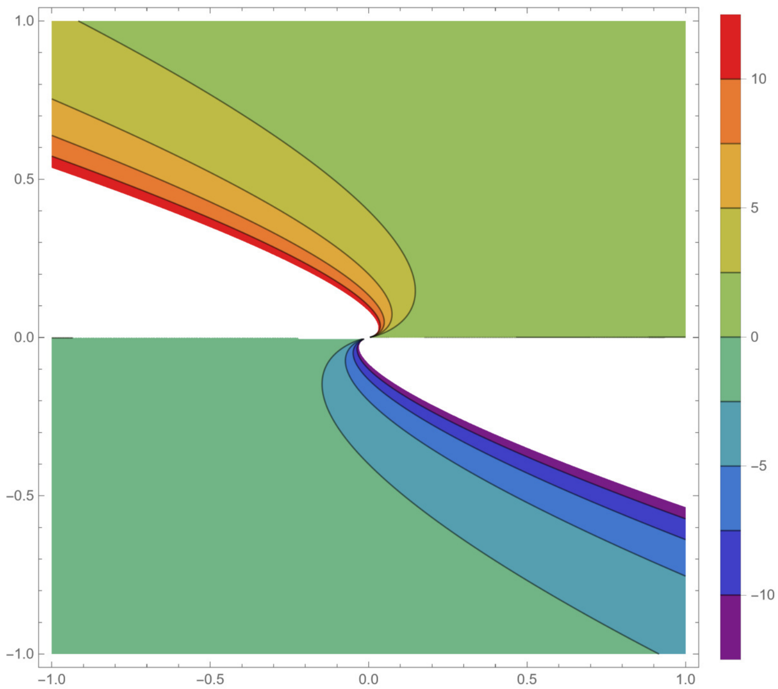


Figure 152. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 57.

Is the following function continuous in $(0,0)$?

$$f(x,y)=\begin{cases}\dfrac{\sin\left(x\,y^3\right)}{x^2} & \text{if } x\neq 0;\\ 0 & \text{if } x=0.\end{cases}$$

Solution.

We can approach this problem by writing the function as follows.

$$\frac{\sin\left(x\,y^3\right)}{x^2}=\frac{\sin\left(x\,y^3\right)}{x\,y^3}\,\frac{x\,y^3}{x^2}.$$

We know that $\lim_{\alpha\to 0}\frac{\sin(\alpha)}{\alpha}=1$. So we are left with the investigation of the discontinuity of the second factor $\frac{xy^3}{x^2} = \frac{y^3}{x}$. We have done this before elsewhere in this text. Let us try out another strategy.

We restrict this function to the continuous curves with equations $y = \sqrt[3]{\arcsin(\lambda x^2)}/\sqrt[3]{x}$. We take care here that we work in a neighbourhood of $x = 0$ small enough so that the arcsin function is defined and note also that these curves are continuous in $x = 0$. We observe then that

$$f\big|_{y=\sqrt[3]{\arcsin(\lambda x^2)}/\sqrt[3]{x}}(x,y) = \begin{cases} f(x, \sqrt[3]{\arcsin(\lambda x^2)}/\sqrt[3]{x}) = \lambda & \text{if } x \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

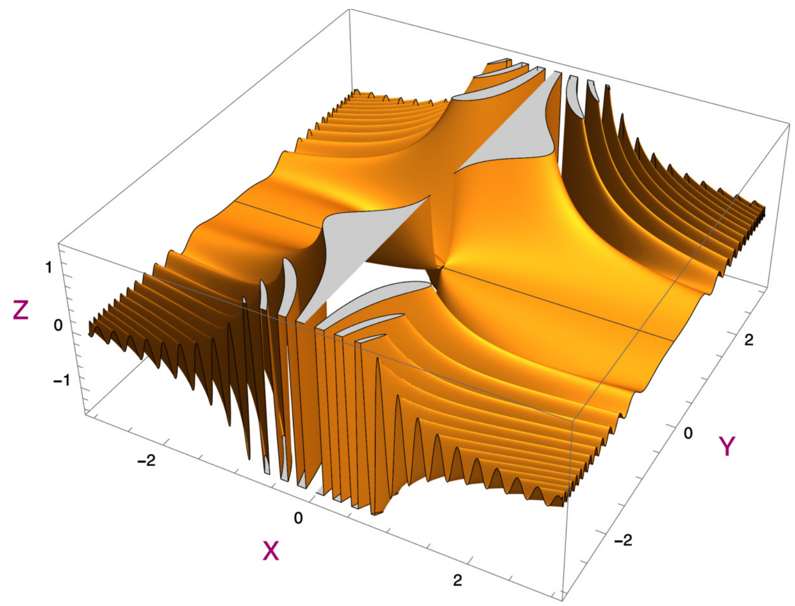


Figure 153. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

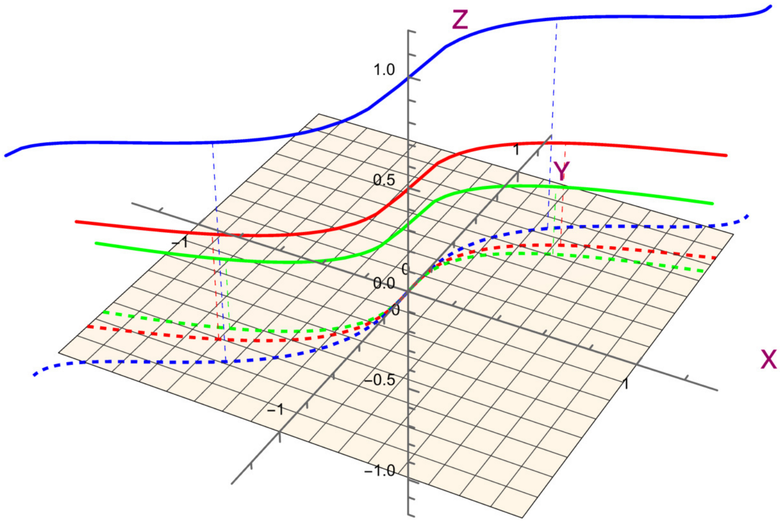


Figure 154. We have restricted the function here to $y = \frac{\sqrt[3]{\arcsin(1/2\,x^2)}}{\sqrt[3]{x}}$ and $y = \frac{\sqrt[3]{\arcsin(1/3\,x^2)}}{\sqrt[3]{x}}$ and $y = \frac{\sqrt[3]{\arcsin(x^2)}}{\sqrt[3]{x}}$. We see in this figure clearly that the restrictions of the function to these curves are functions that have different limits in 0.

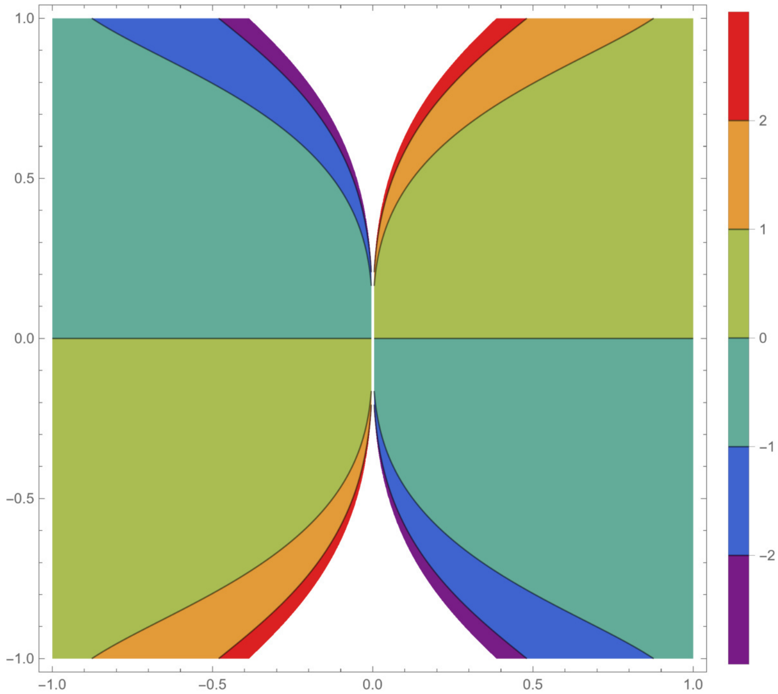


Figure 155. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 58.

Is the following function continuous in $(0,0)$?

$$f(x,y)=\begin{cases}\dfrac{x^3+y^5}{x^2+2y^2} & \text{if } (x,y)\neq(0,0);\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

Solution.

Let us prove it with an ϵ-δ approach.

We take an arbitrary $\epsilon>0$ and we have to prove that $|f(x,y)-f(0,0)|<\epsilon$. The problem is now to find a $\delta>0$ such that if $||(x,y)-(0,0)||<\delta$ it follows that $|f(x,y)-f(0,0)|<\epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{x^3+y^5}{x^2+2\,y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

We assume that $\sqrt{x^2+y^2} < 1$.

$$\begin{aligned}
\left| \frac{x^3+y^5}{x^2+2\,y^2} - 0 \right| &\leq \left| \frac{x^3+y^5}{x^2+y^2} \right| \\
&\leq \frac{\sqrt{x^2+y^2}^{\,3} + \sqrt{x^2+y^2}^{\,5}}{x^2+y^2} \\
&\leq \frac{\sqrt{x^2+y^2}^{\,3} + \sqrt{x^2+y^2}^{\,3}}{x^2+y^2} \\
&\leq 2\,\frac{\sqrt{x^2+y^2}^{\,3}}{x^2+y^2} \\
&\leq 2\sqrt{x^2+y^2}.
\end{aligned}$$

It is sufficient to take $\sqrt{x^2+y^2} < \min\{\epsilon/2, 1\}$. We can find a δ, so we conclude that the function is continuous.

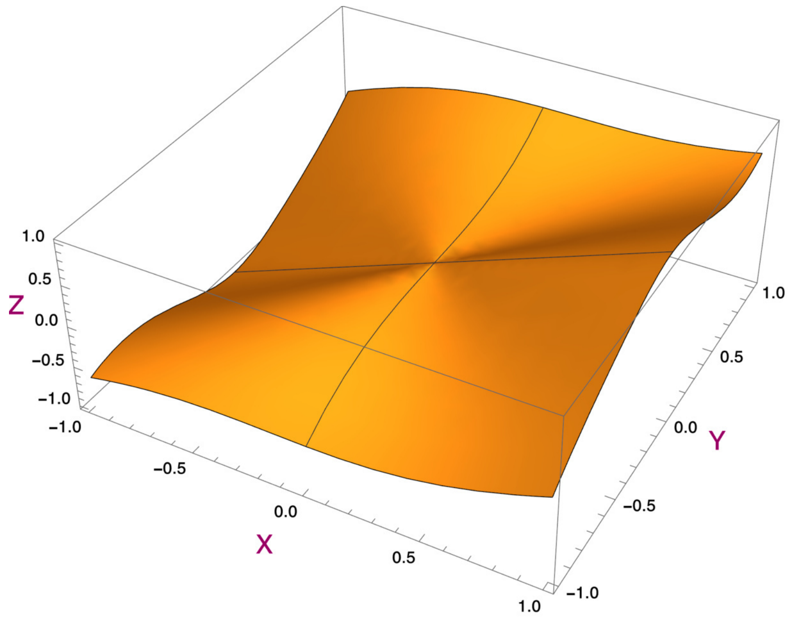


Figure 156. We see here a three dimensional figure of the graph of the function.

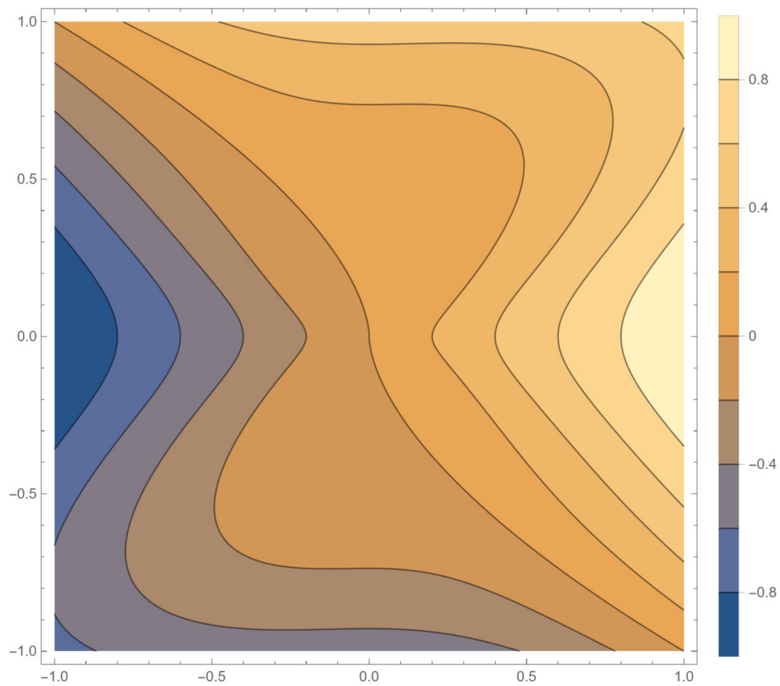


Figure 157. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 59.

Is the following function continuous in $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{x^2+\sin^2(y)}{x^2+2\,y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) \neq (0,0). \end{cases}$$

Solution.

We restrict this function to the continuous curves with equations $x = \lambda\, y$. We observe then that

$$f\,|_{x=\lambda\, y}(x,y) = \begin{cases} f(\lambda\, y, y) = \dfrac{\lambda^2\, y^2 + \sin^2(y)}{\lambda^2\, y^2 + 2\, y^2} & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

We have also the following limit by dividing denominator en numerator by y^2.

$$\lim_{y\to 0} \frac{\lambda^2\, y^2 + \sin^2(y)}{\lambda^2\, y^2 + 2y^2} = \lim_{y\to 0} \frac{y^2\,(\lambda^2 + \sin^2(y)/y^2)}{y^2\,(\lambda^2+2)} = \frac{\lambda^2+1}{\lambda^2+2}.$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

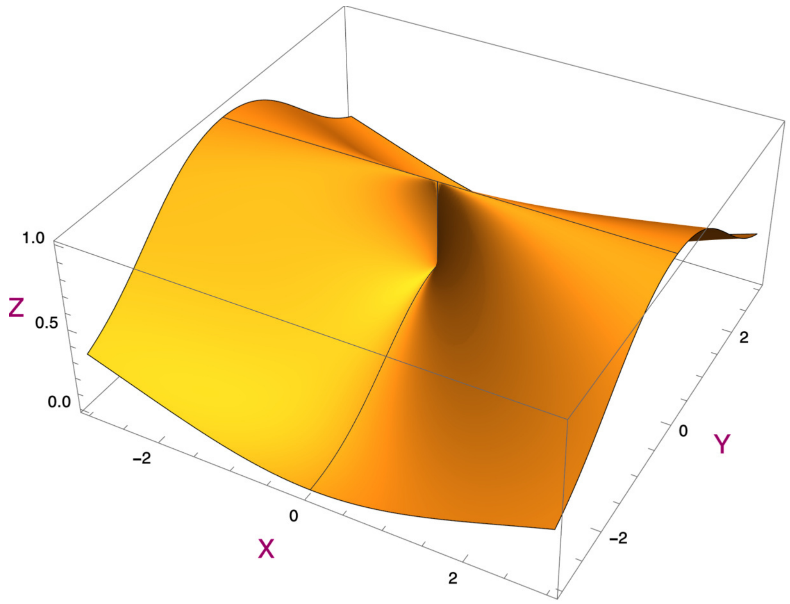


Figure 158. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

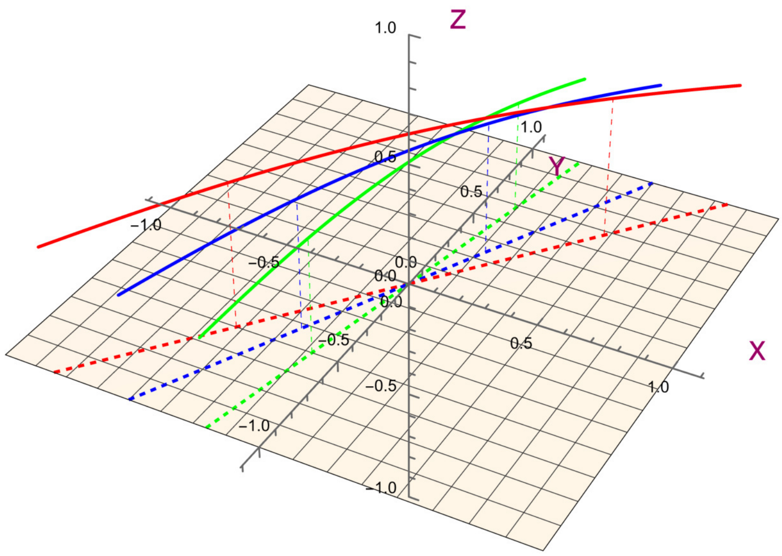


Figure 159. We have restricted the function here to $x = 4/5\,y$ and $x = 1/5\,y$ and $x = 1/2\,y$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

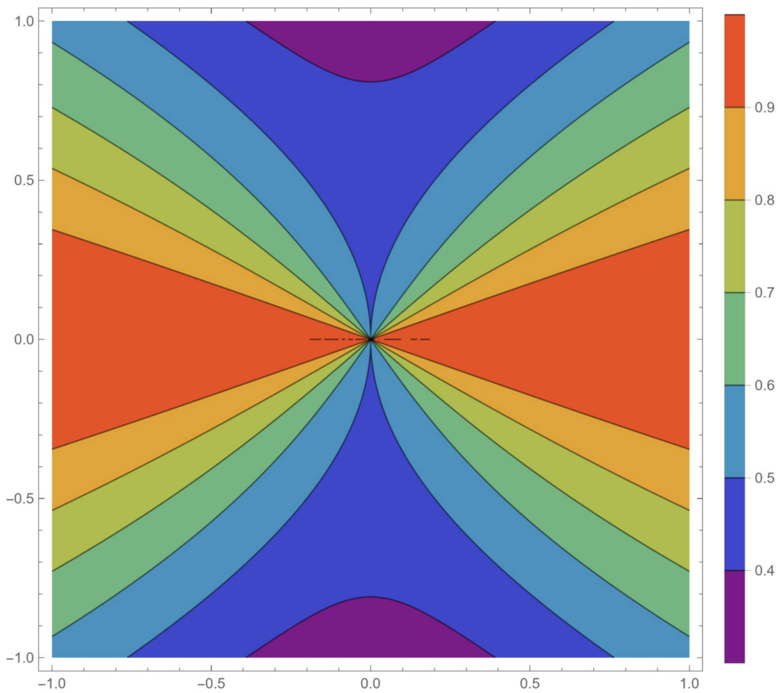


Figure 160. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 60.

Is the following function continuous in $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{\tan\left(\frac{x\,y}{x\,y+1}\right)}{x\,y} & \text{if } x\,y \neq 0; \\ 1 & \text{if } x\,y = 0. \end{cases}$$

We assume here also that the function takes its values only in a neighbourhood small enough around $(0,0)$ in order to avoid difficulties with the tangent function and the hyperbola $x\,y + 1 = 0$. It is sufficient to take the neighbourhood $D = \{(x,y) \mid x < 1/2, y < 1/2\}$. We assume also that we know that $\lim_{\alpha \to 0} \tan(\alpha)/\alpha = 1$.

Solution.

We observe that

$$f(x,y) = \frac{1}{x\,y+1}\,\frac{\tan\left(\frac{x\,y}{x\,y+1}\right)}{\frac{x\,y}{x\,y+1}}.$$

The second factor has a limit going to the value 1. So we are left with investigating the continuity of $\frac{1}{x\,y+1}$. This function is obviously continuous in $(0,0)$. So we conclude that the function $f(x,y)$ is continuous in $(0,0)$.

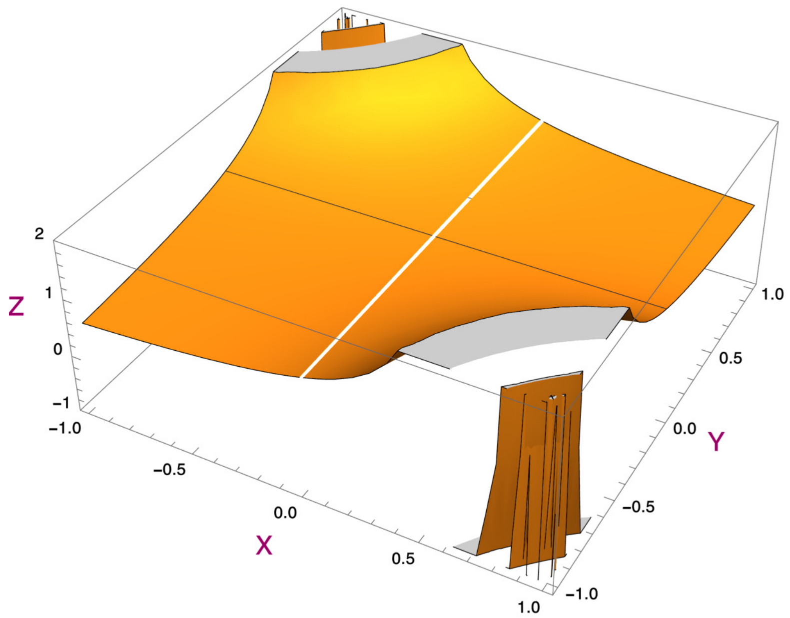


Figure 161. We see here a three dimensional figure of the graph of the function.

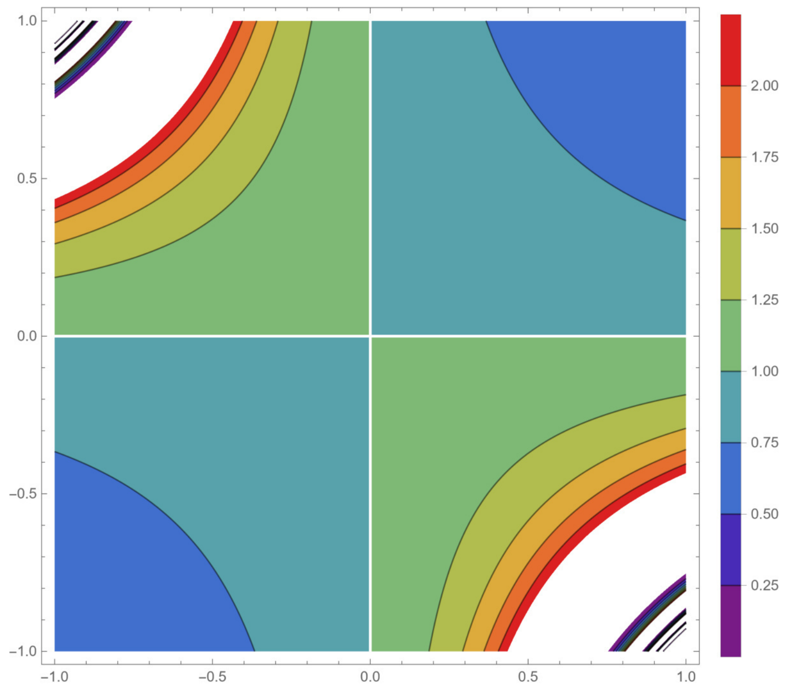


Figure 162. We see here a figure of the contour plot of the function.

Exercise 61.

Is the function

$$f(x,y) = \begin{cases} (x^2+y^2)^{x^2y^2} & \text{if } (x,y) \neq (0,0); \\ 1 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous in $(0,0)$? We may use that $t/(1+t) < \ln(1+t) < t$ with $t > -1$.

Solution.

We remark first that

$$(x^2+y^2)^{x^2y^2} = e^{x^2y^2\ln(x^2+y^2)}.$$

We want to prove now that the function

$$g(x,y) = \begin{cases} x^2y^2\ln(x^2+y^2) & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

is continuous. By the continuity of the exponential function, we have then immediately our assertion.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|g(x,y)-g(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|g(x,y)-g(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| x^2 y^2 \ln(x^2+y^2) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

Before we do that, we want to proof the fact that the one variable function $x\ \ln(x)$ is bounded in a neighbourhood of $x = 0$. We start from the inequality

$$t/(1+t) \leq \ln(1+t) \leq t \quad \text{with} \quad t > -1.$$

We have proven this inequality in the appendix without using derivatives.
By substituting $x = t+1$ with x where $x > 0$ we have that this inequality is equivalent with the following inequality

$$\frac{x-1}{x} \leq \ln(x) \leq x-1 \quad \text{with} \quad x > 0.$$

By multiplying with the positive x, we have $x-1 \leq x\ \ln(x) \leq x\ (x-1)$. So if $0 < x < 1$, then $-1 \leq x\ \ln(x) \leq 0$ and $|x\ \ln(x)| \leq M = 1$. In order to use the inequality $|x\ \ln(x)| \leq 1$ for $0 < x < 1$,
We will need later that $\sqrt{x^2+y^2} < 1$, and to take care of that we take $\delta < 1$.

$$
\begin{aligned}
\left|x^2 y^2 \ln(x^2+y^2) - 0\right| &\le \left|\frac{x^2 y^2}{x^2+y^2}\,(x^2+y^2)\ln(x^2+y^2)\right| \\
&\le \left|M\,\frac{x^2 y^2}{x^2+y^2}\right| \\
&\le M\,\frac{\sqrt{x^2+y^2}^{\,2}\,\sqrt{x^2+y^2}^{\,2}}{x^2+y^2} \\
&\le M\,\sqrt{x^2+y^2}^{\,2}.
\end{aligned}
$$

If this last term must be smaller than ϵ, then it is sufficient to take $\sqrt{x^2+y^2} < \delta = \min\{\sqrt{\epsilon/M}, 1\}$. Because we can choose a δ, we have continuity of $g(x,y)$ in $(0,0)$. And this is enough to prove the assertion that $f(x,y)$ is continuous.

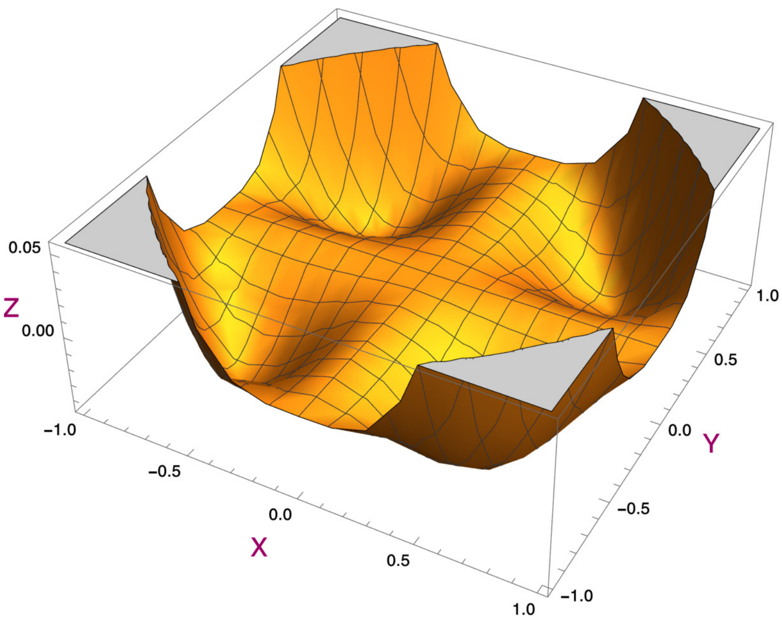


Figure 163. We see here a three dimensional figure of the graph of the function $g(x,y) = x^2 y^2 \ln(x^2+y^2)$.

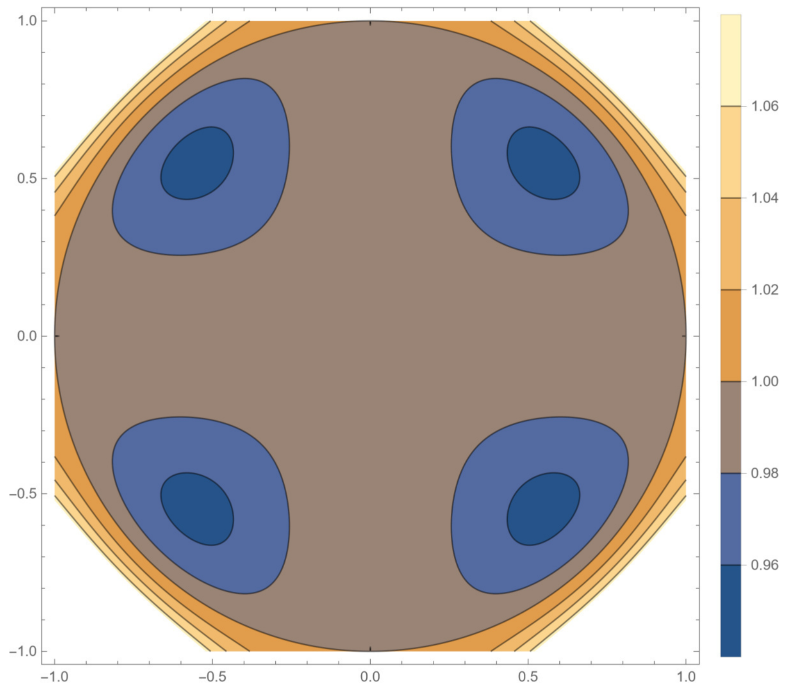


Figure 164. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0, 0)$. This plot is drawn for the function $f(x, y) = (x^2 + y^2)^{x^2 y^2}$.

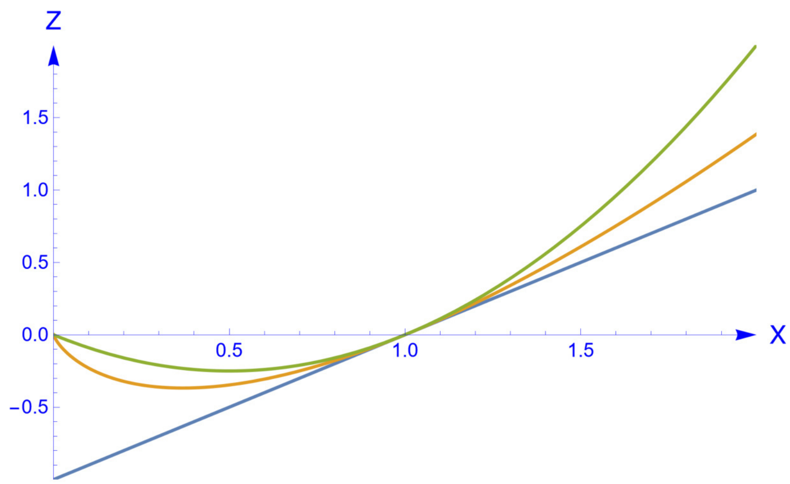


Figure 165. We see here a figure showing the inequality $x - 1 \leq x \ln(x) \leq x(x-1)$. The graph of $x - 1$ is in blue, that of $x \ln(x)$ is in ochre, and that of $x(x-1)$ is in green.

Exercise 62.

Is the following function continuous in $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{(x+1)(y-x)}{(y+1)(x+y)} & \text{if } x \neq -y \text{ and } y \neq -1; \\ 0 & \text{if } x = -y \text{ or } y = -1. \end{cases}$$

Solution.

We restrict this function to the continuous curves with equations $y = \lambda x$, where $\lambda \neq -1$. We observe then that

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{(\lambda-1)(x+1)}{(\lambda+1)(\lambda x+1)} & \text{if } x \neq 0 \text{ and } x \neq -1/\lambda; \\ 0 & \text{if } y = 0 \text{ or } x = -1/\lambda. \end{cases}$$

We have also the following limit.

$$\lim_{x\to 0} \frac{(\lambda-1)(x+1)}{(\lambda+1)(\lambda x+1)} = \frac{\lambda-1}{\lambda+1}.$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

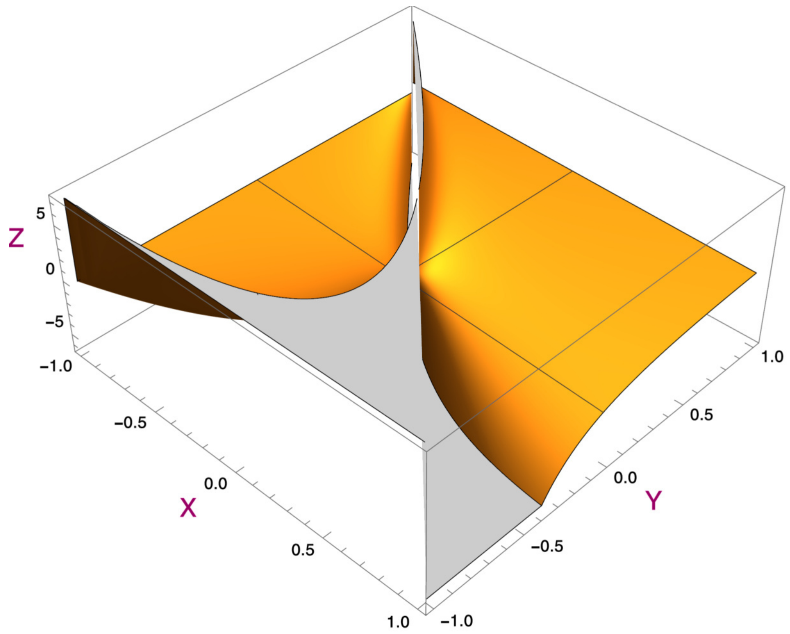


Figure 166. We see here a three dimensional figure of the graph of the function. The vertical line above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

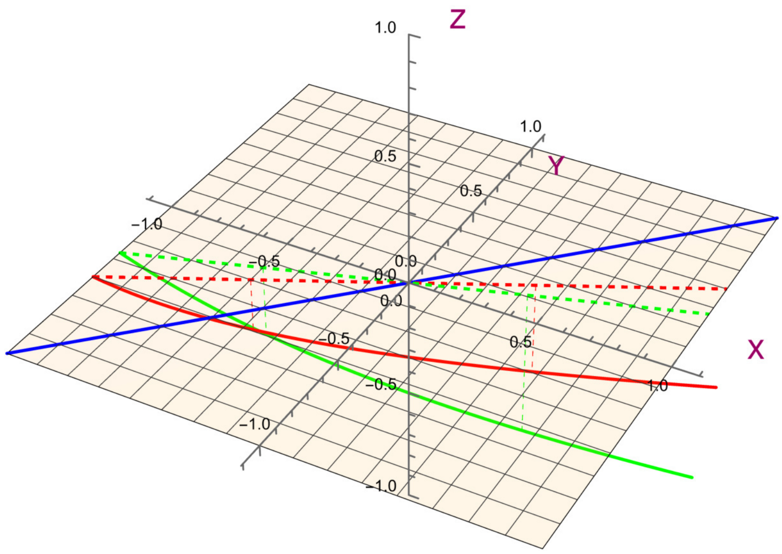


Figure 167. We have restricted the function here to $y = 1/2\,x$ and $y = 1/3\,x$ and $y = x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

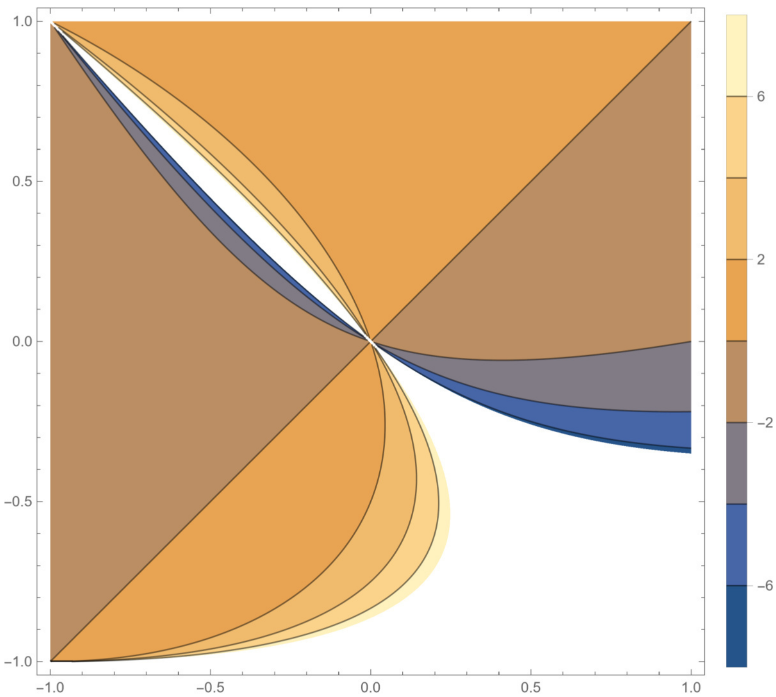


Figure 168. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 63.

Is the following function with $a > 0$ and $b > 0$ continuous in $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{|y|^a\,|x|^b}{|x|^a + |y|^b} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Solution.

Let us investigate this with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{|y|^a\,|x|^b}{|x|^a+|y|^b} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

We will make use on the second line of the main inequality the following inequalities

$$|x|^{a/2} \le \sqrt{\left(|x|^{a/2}\right)^2+\left(|y|^{b/2}\right)^2},$$
$$|y|^{b/2} \le \sqrt{\left(|x|^{a/2}\right)^2+\left(|y|^{b/2}\right)^2}.$$

Here is the main inequality.

$$\begin{aligned}
\frac{|x|^b\,|y|^a}{|x|^a+|y|^b} &\le \frac{\left(|x|^{a/2}\right)^{2\,b/a}\,\left(|y|^{b/2}\right)^{2\,a/b}}{\left(|x|^{a/2}\right)^2+\left(|y|^{b/2}\right)^2}\\
&\le \frac{\sqrt{\left(|x|^{a/2}\right)^2+\left(|y|^{b/2}\right)^2}^{2\,b/a}\,\sqrt{\left(|x|^{a/2}\right)^2+\left(|y|^{b/2}\right)^2}^{2\,a/b}}{\sqrt{\left(|x|^{a/2}\right)^2+\left(|y|^{b/2}\right)^2}^{2}}\\
&\le \sqrt{\left(|x|^{a/2}\right)^2+\left(|y|^{b/2}\right)^2}^{2\,b/a+2\,a/b-2}\\
&\le \sqrt{\left(|x|^{a/2}\right)^2+\left(|y|^{b/2}\right)^2}^{2\,((a-b)^2+a\,b)}.
\end{aligned}$$

We must have

$$\sqrt{\left(|x|^{a/2}\right)^2+\left(|y|^{b/2}\right)^2}^{2\,((a-b)^2+a\,b)} < \epsilon$$

or

$$\sqrt{|x|^a + |y|^b} < \epsilon^{\frac{1}{2((a-b)^2+ab)}}.$$

Let us define a shorthand for the right hand side of the inequality

$$\epsilon_0 = \epsilon^{\frac{1}{2((a-b)^2+ab)}}.$$

To satisfy the inequality, it is sufficient to have

$$|x|^a < \epsilon_0^2/2 \quad \text{and} \quad |y|^b < \epsilon_0^2/2.$$

Or

$$|x| < \sqrt[a]{\epsilon_0^2/2} \quad \text{and} \quad |y| < \sqrt[b]{\epsilon_0^2/2}.$$

So it is sufficient to take the following δ in $\sqrt{x^2+y^2} < \delta$ stated at the start of the solution: $\delta = \min\left\{\sqrt[a]{\epsilon_0^2/2}, \sqrt[b]{\epsilon_0^2/2}\right\}$.

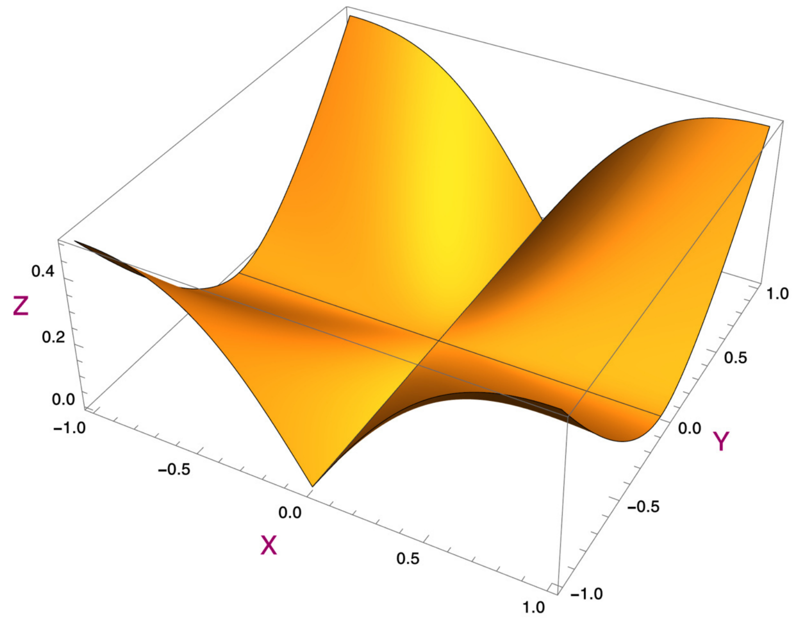


Figure 169. We see here a three dimensional figure of the graph of the function. The picture is made with parameters $a = 2$ and $b = 3$.

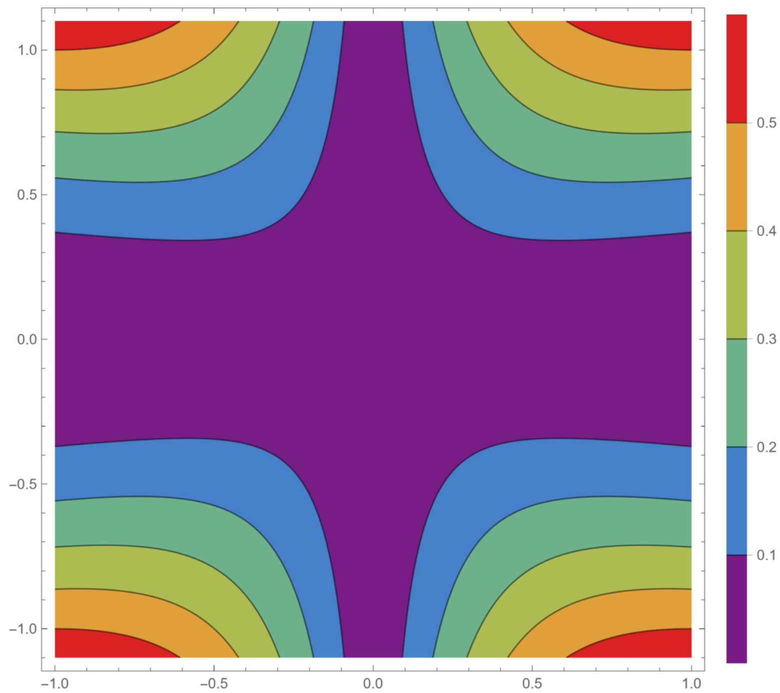


Figure 170. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

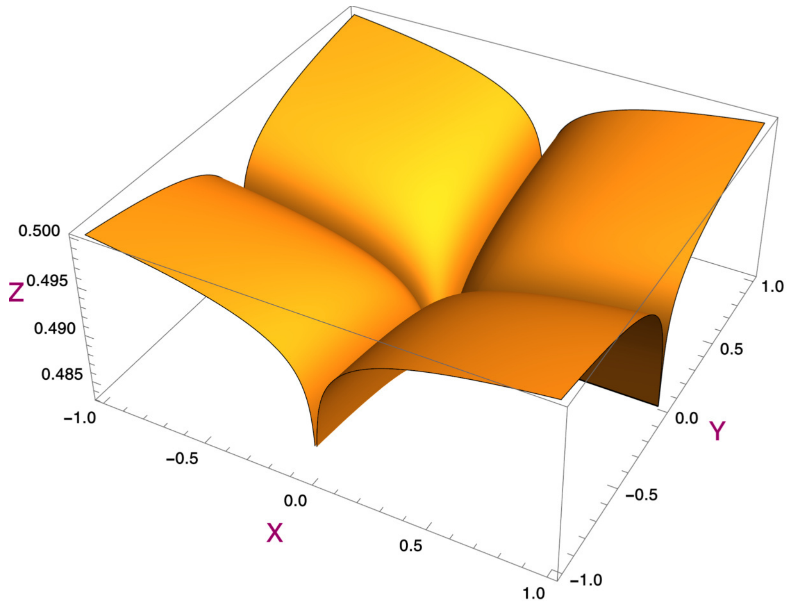


Figure 171. We see here a three dimensional figure of the graph of the function. The picture is made with parameters $a = 0.01$ and $b = 0.01$.

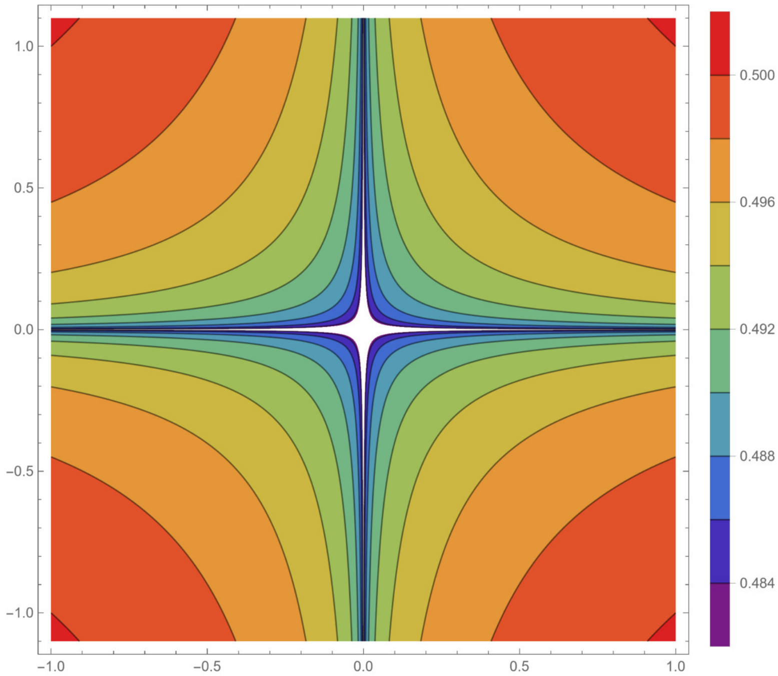


Figure 172. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$. We have to take extreme care when examining this picture. When (x,y) approaches zero, the function hovers a time at level around 1 and dives extremely fast to the level zero. The smaller the parameters a and b, the stronger is this phenomenon.

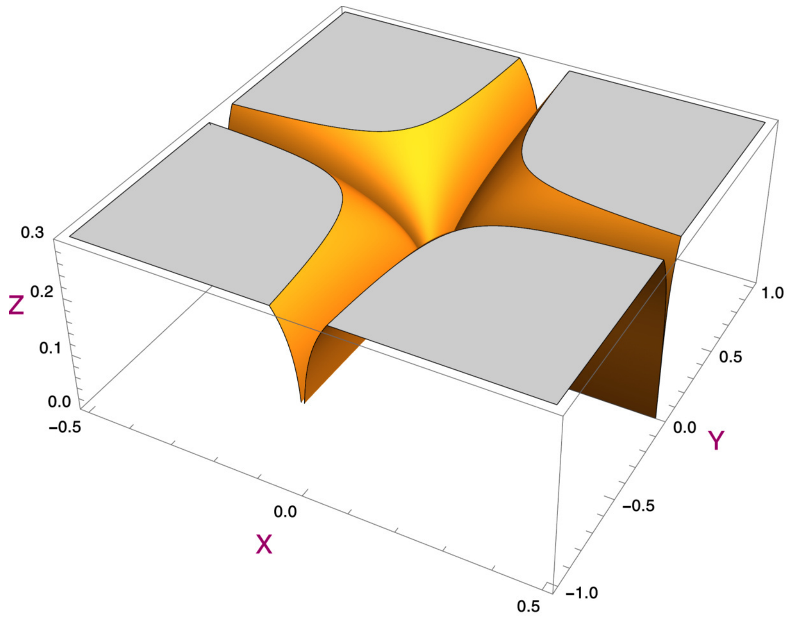


Figure 173. We see here a figure of the graph of the function with parameters $a = 0.3$ and $b = 0.3$. We have to take extreme care when examining this picture. When (x, y) approaches zero, then the function dives extremely fast to the level zero. The smaller the parameters a and b, the stronger is this phenomenon.

Let us examine for example the behaviour above $y = x$. We have then the function $|x|^{a+b}/(x^a + x^b)$. Suppose also that $a < b$. Then we have the function $|x|^{a+b}/(|x|^a\,(1 + |x|^{b-a})) \approx |x|^b$ in a neighbourhood of $x = 0$.

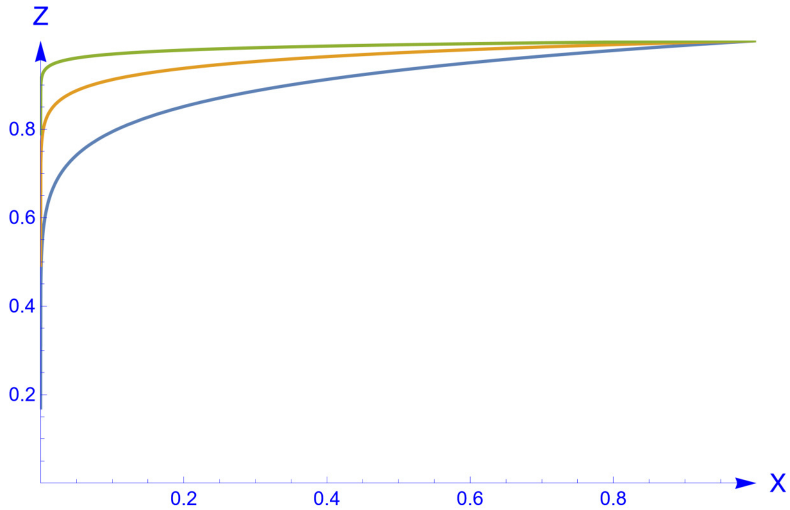


Figure 174. We see here a figure of the graph of the three functions with function definitions $g_{10}(x) = x^{1/10}$ pictured in blue and $g_{25}(x) = x^{1/25}$ pictured in ochre and $g_{75}(x) = x^{1/75}$ pictured in green. When x approaches zero, then the function tends to wait longer at level around 1 before diving extremely fast to the level zero for small exponents. The smaller the exponents, the stronger is this phenomenon. For the function depicted in the colour green, the phenomenon of fast diving is even barely observable because of colour overlapping.

Exercise 64.

Is the following function continuous in $(0,0)$?

$$f(x,y) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Solution.

We restrict this function to the continuous curves with equations $y = \lambda x$. We observe then that

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \arctan(\lambda) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$

is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x, y)$ is not continuous.

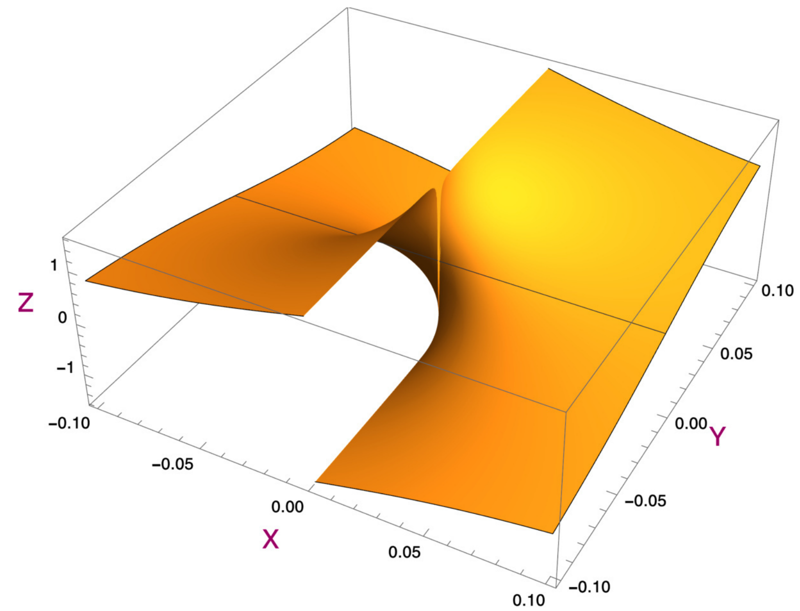


Figure 175. We see here a three dimensional figure of the graph of the function. The vertical line above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

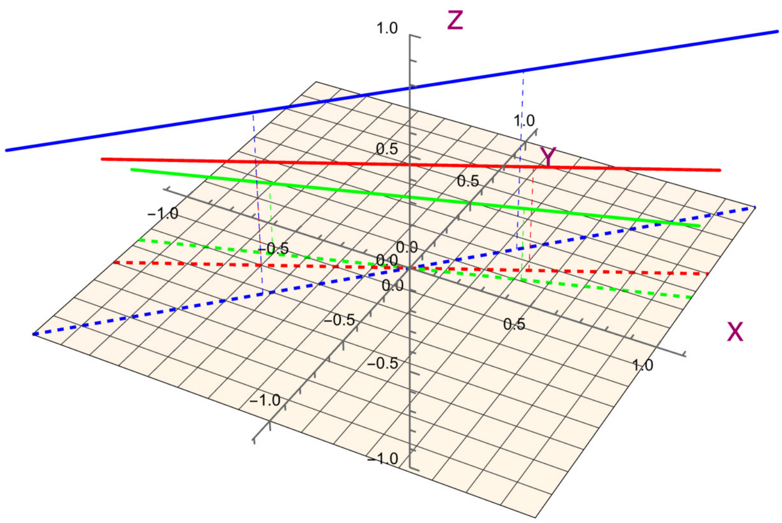


Figure 176. We have restricted the function here to $y = x$ and $y = 1/2\,x$ and $y = 1/3\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

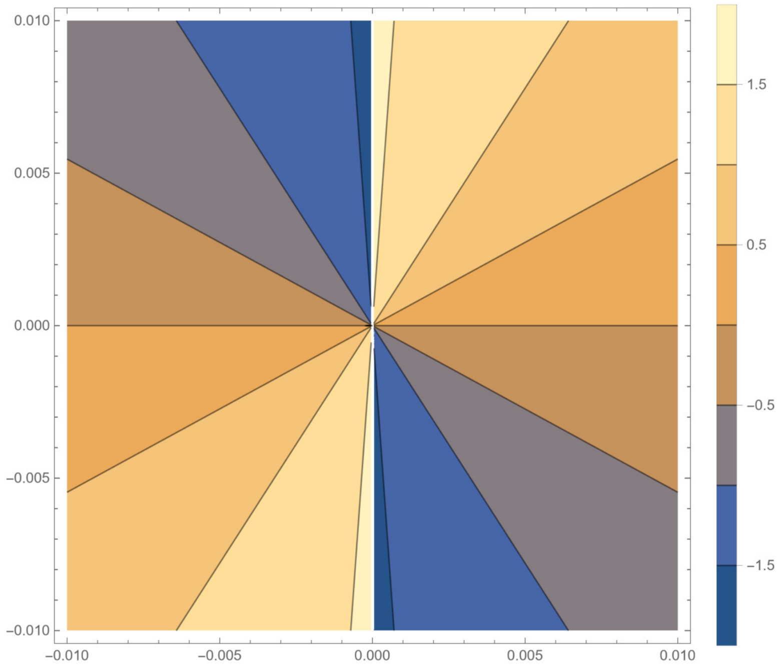


Figure 177. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0, 0)$. This looks discontinuous indeed.

Exercise 65.

Is the following function continuous in $(0, 0)$?

$$f(x, y) = \begin{cases} \dfrac{x^3 y^3}{x^6 + y^6} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Solution.

We restrict this function to the continuous curves with equations $y = \lambda x$. We observe then that

$$f|_{x,\lambda x}(x, y) = \begin{cases} f(x, \lambda x) = \dfrac{\lambda^3 x^6}{\lambda^6 x^6 + x^6} = \dfrac{\lambda^3}{\lambda^6 + 1} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0)=0$. So this function $f(x,y)$ is not continuous.

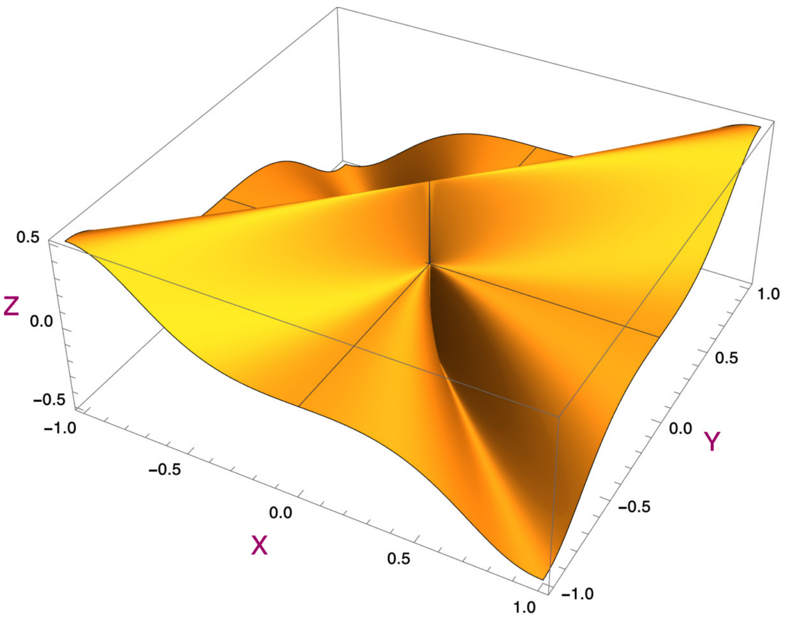


Figure 178. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

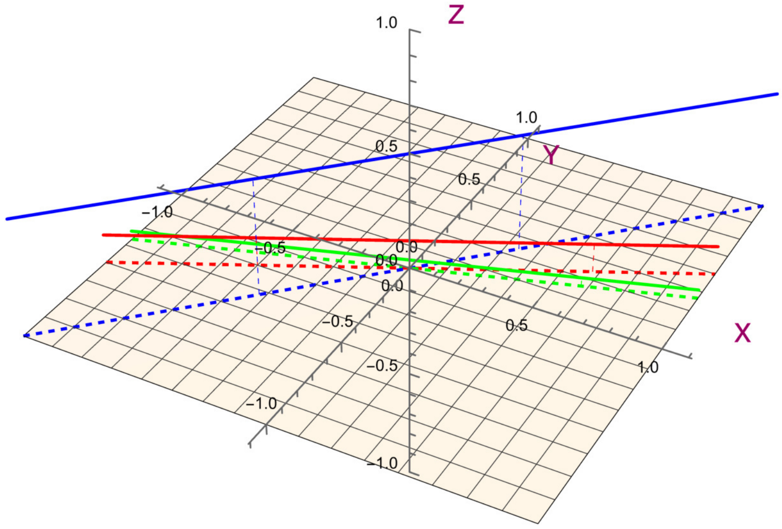


Figure 179. We have restricted the function here to $y = x$ and $y = 1/2\,x$ and $y = 1/3\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

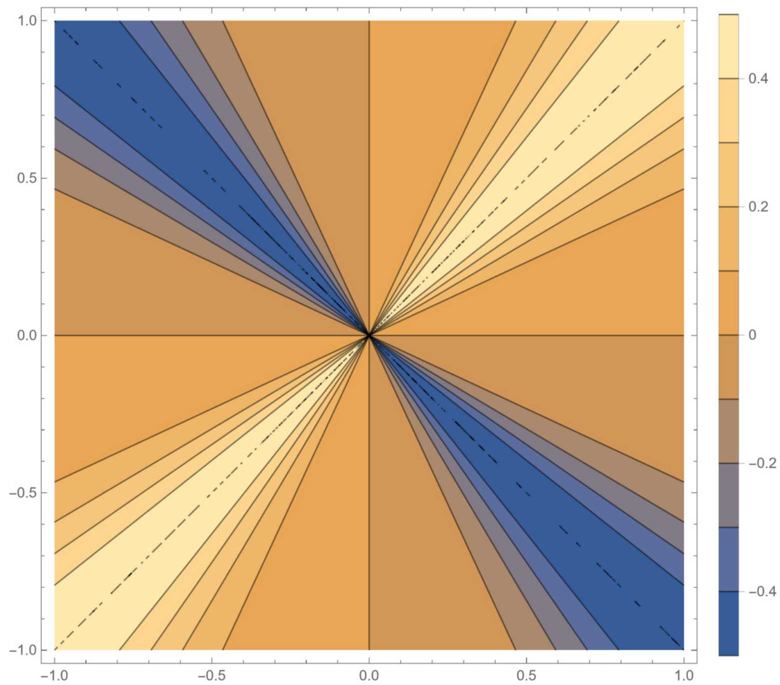


Figure 180. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 66.

Is the following function continuous in $(0,0)$?

$$f(x,y)=\begin{cases}\dfrac{x^3y}{x^6+y^2} & \text{if } (x,y)\neq(0,0);\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

Solution.

We restrict this function to the continuous curves with equations $y=\lambda x^3$. We observe then that

$$f|_{y=\lambda x^3}(x,y)=\begin{cases}f(x,\lambda x^3)=\dfrac{\lambda x^6}{x^6+\lambda^2x^6}=\dfrac{\lambda}{\lambda^2+1} & \text{if } x\neq 0;\\ 0 & \text{if } x=0.\end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

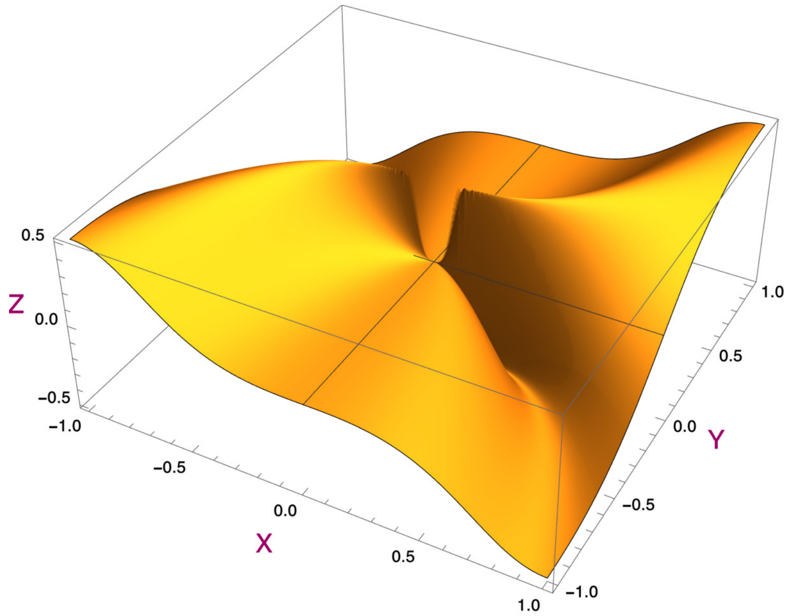


Figure 181. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

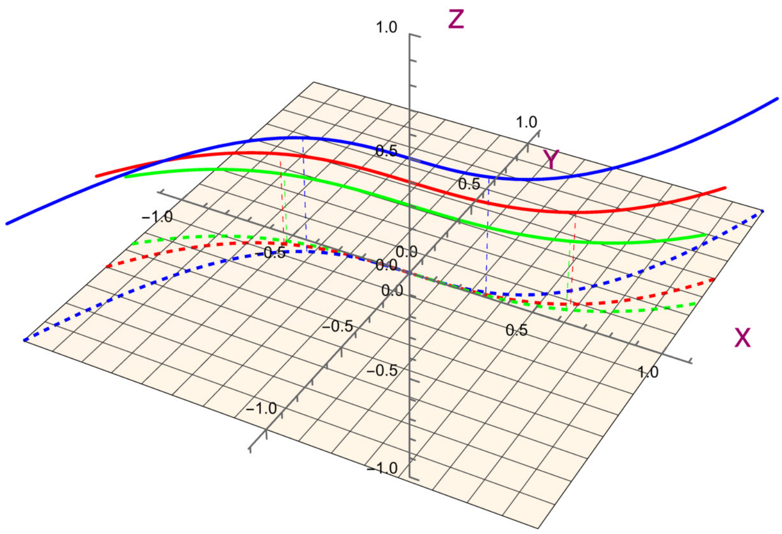


Figure 182. We have restricted the function here to $y = x^3$ and $y = 1/2\,x^3$ and $y = 1/3\,x^3$. We see in this figure clearly that the restrictions of the function to these curves are functions that have different limits in 0.

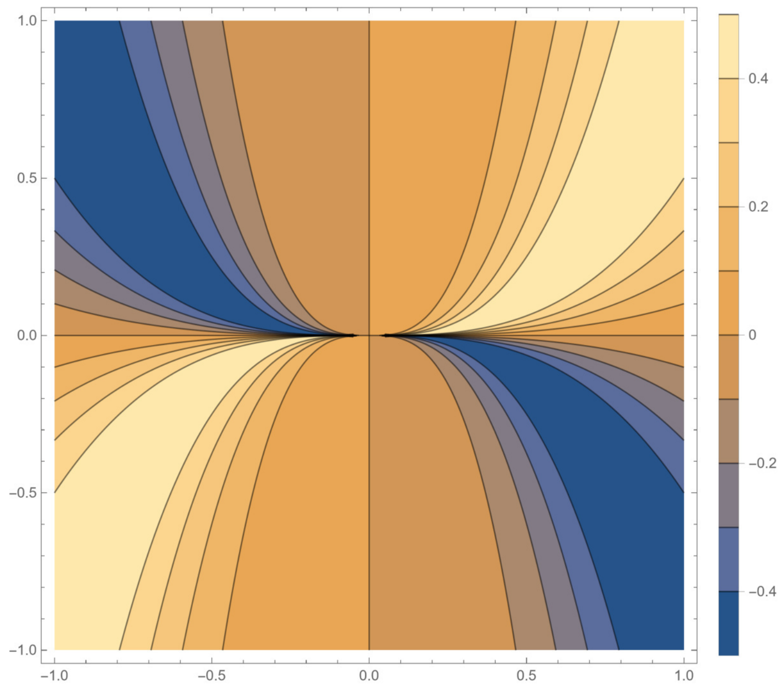


Figure 183. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 67.

Is the following function continuous in $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{(x-1)\,y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Solution.

We restrict this function to the continuous curves with equations $y = \lambda x$. We observe then that

$$f\,|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{\lambda^2\,(x-1)\,x^2}{\lambda^2\,x^2+x^2} = \dfrac{\lambda^2\,(x-1)}{\lambda^2+1} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We take the limit to 0.

$$\lim_{x\to 0} \frac{\lambda^2 (x-1)}{\lambda^2+1} = -\frac{\lambda^2}{\lambda^2+1}.$$

We see that these restricted functions have different limits. But if $f(x, y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x, y)$ is not continuous.

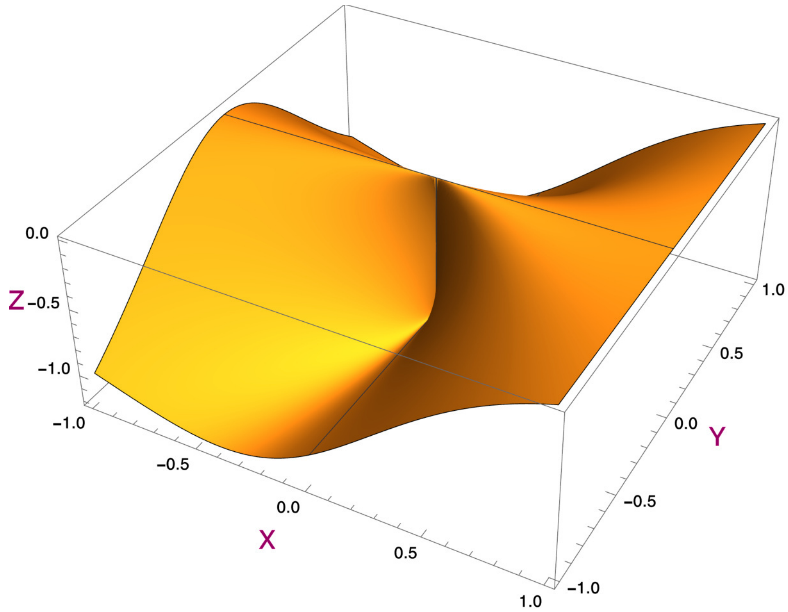


Figure 184. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

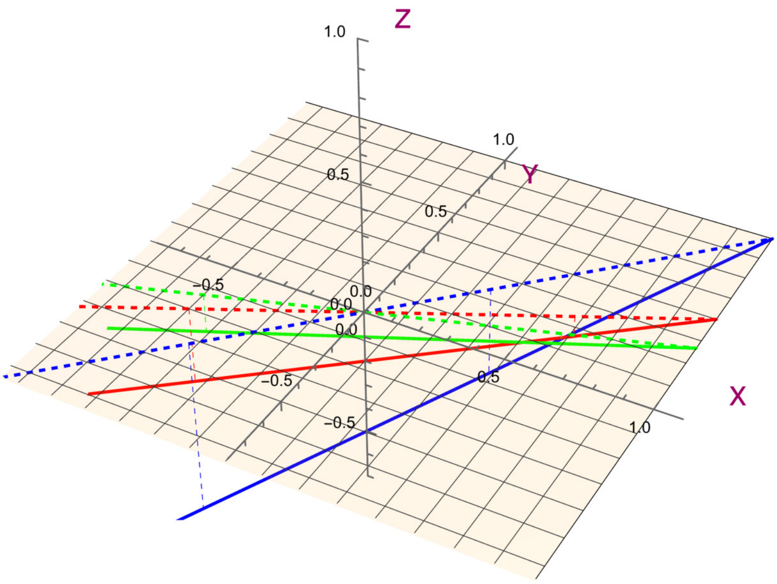


Figure 185. We have restricted the function here to $y = x$ and $y = 1/2\,x$ and $y = 1/3\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

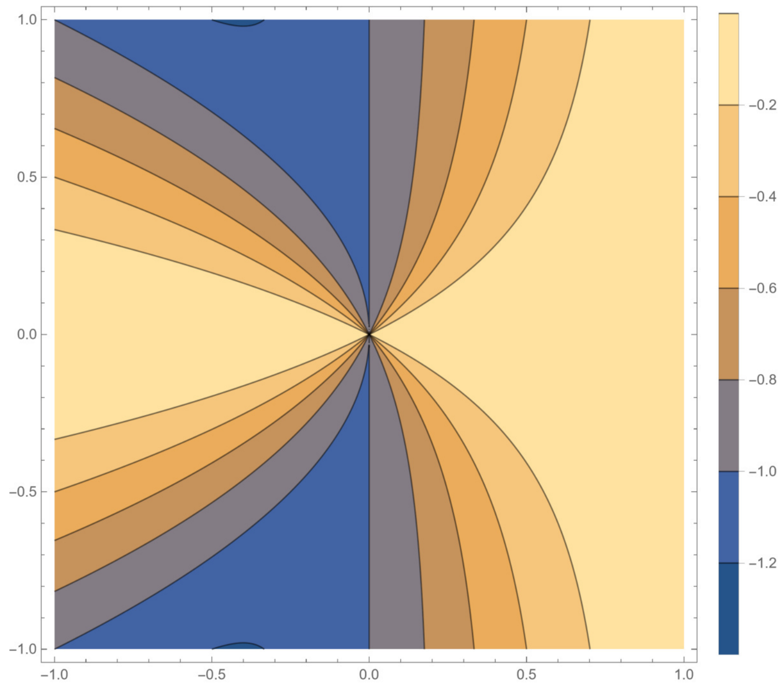


Figure 186. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0, 0)$. This looks discontinuous indeed.

Exercise 68.

Is the following function continuous in $(0, 0)$?

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^6}{x y^3} & \text{if } x \neq 0 \text{ and } y \neq 0; \\ 0 & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

Solution.

We restrict this function to the continuous curves with equations $x = \lambda y^3$. We observe then that

$$f(x, y) = f(\lambda y^3, y) = \frac{\lambda^2 y^6 - y^6}{\lambda y^6} = \lambda - \frac{1}{\lambda}.$$

We see that these restricted functions have different limits. But if $f(x, y)$ is continuous, all these limit values should be $f(0, 0) = 0$. So this function $f(x, y)$ is not continuous.

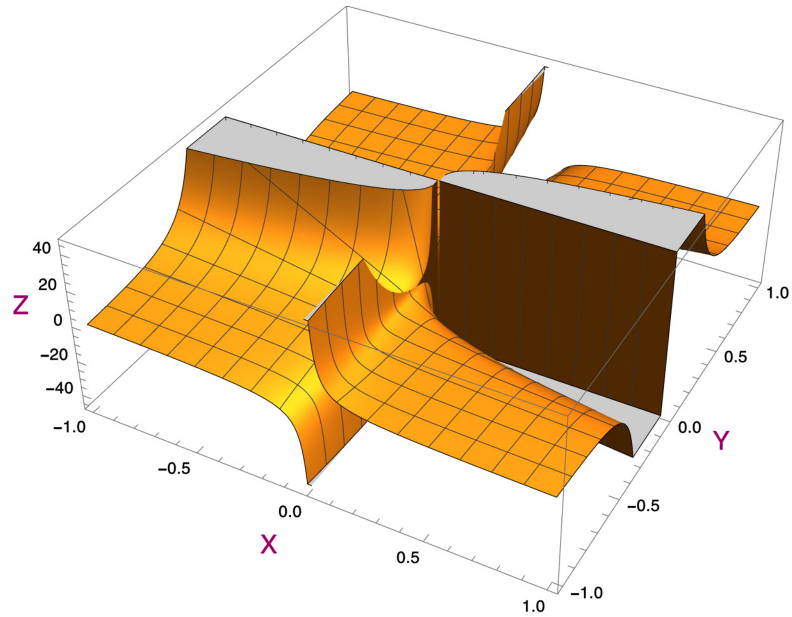
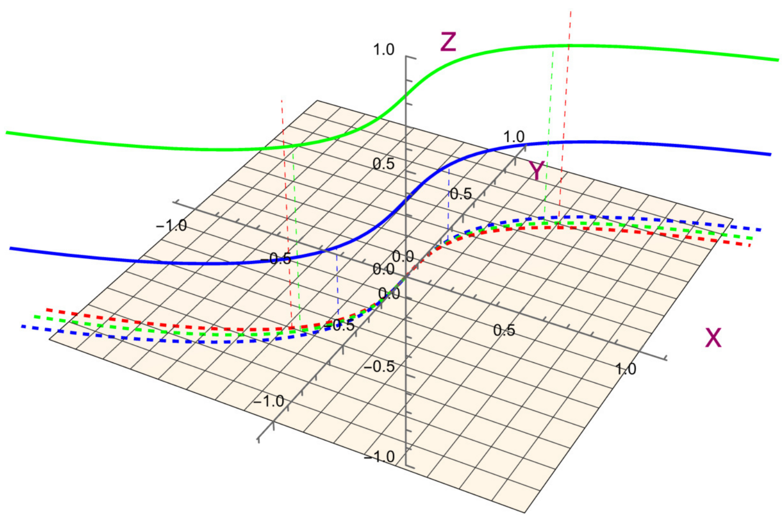


Figure 187. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

Figure 188. We have restricted the function here to $x = 6/5\,y^3$ and $x = 3/2\,y^3$ and $x = 0/5\,y^3$. We see in this figure clearly that the restrictions of the function to these curves are functions that have different limits in 0.



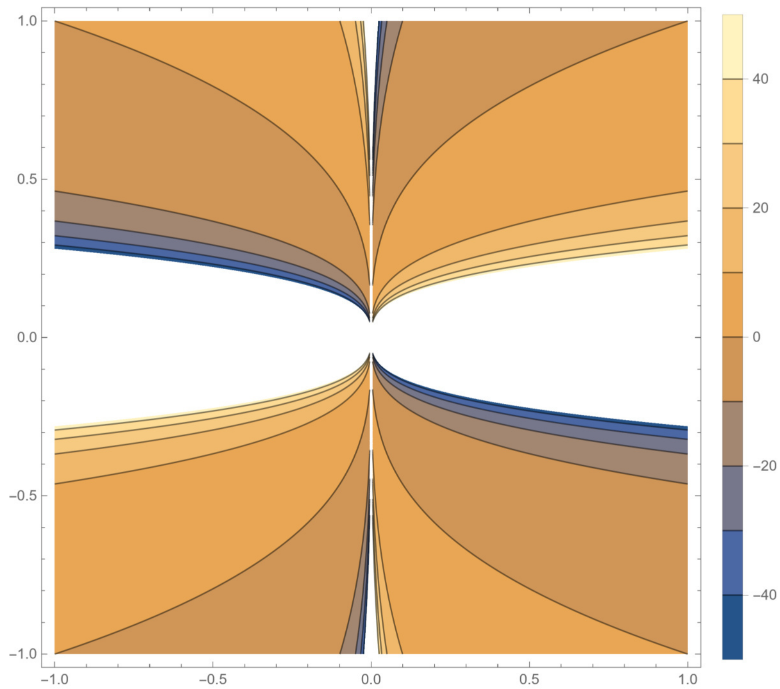


Figure 189. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 69.

Is the following function continuous in $(0,0)$?

$$f(x,y) = \begin{cases} \arctan\left(\dfrac{2\,y}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Solution.

We restrict this function to the continuous curves with equations $x = \sqrt{2\,\lambda\,y - y^2}$. These are parts of circles with centre $(0,\lambda)$ and radius λ. If $\lambda > 0$, then $0 \leq y \leq 2\,\lambda$ and if $\lambda < 0$, then $2\,\lambda \leq y \leq 0$. We observe then that with the same restrictions as mentioned

$$f|_{x=\sqrt{2\lambda y-y^2}}(x,y) = \begin{cases} f\left(\sqrt{2\lambda y - y^2}, y\right) = \arctan\left(\frac{1}{\lambda}\right) & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

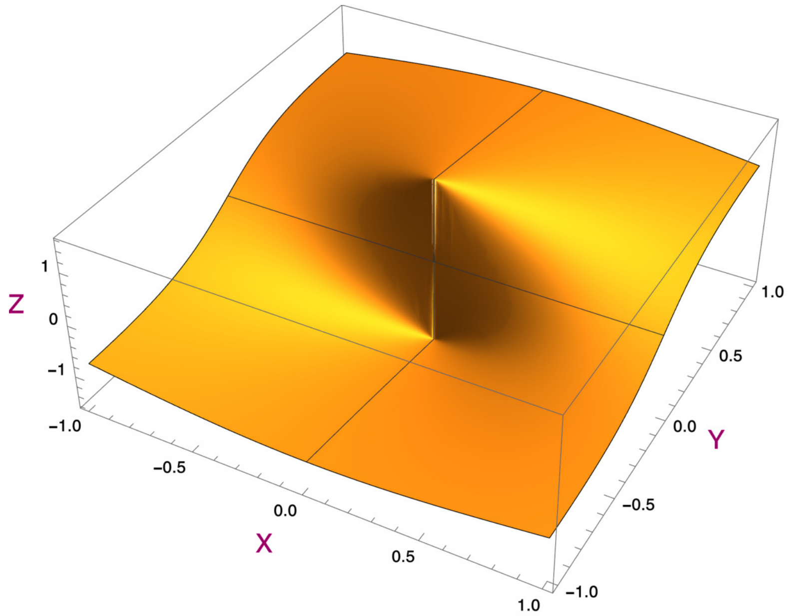


Figure 190. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

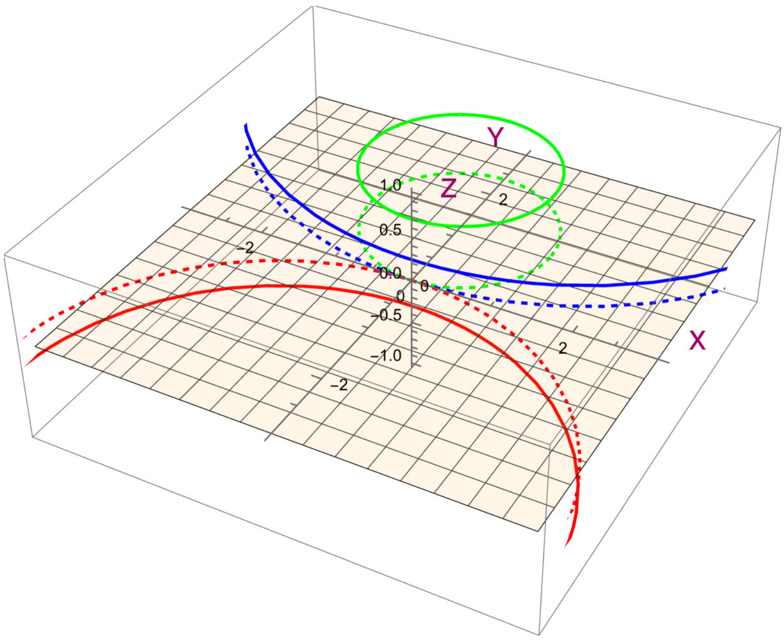


Figure 191. We have restricted the function here to circles with parametric equation $\{x = \lambda \cos(\theta), y = \lambda \sin(\theta) + \lambda\}$, $0 \leq \theta < 2\pi$ and λ-values $\lambda = \cot(-3/10)$, $\lambda = \cot(7/10)$ and $\lambda = \cot(12/5)$. We see in this figure clearly that the restrictions of the function to these circles are functions that have different limits in 0.

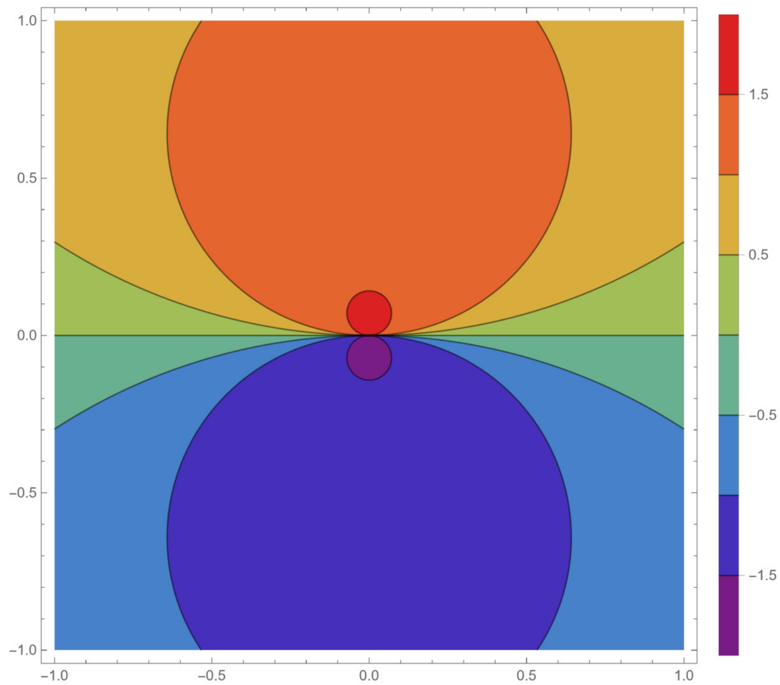


Figure 192. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 70.

Is the function

$$f(x,y) = \begin{cases} |y| \sin\left(\frac{x}{\sqrt{|y|}}\right) & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

continuous in $(0,0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| |y| \sin\left(\frac{x}{\sqrt{|y|}}\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left| |y| \sin\left(\frac{x}{\sqrt{|y|}}\right) - 0 \right| &\leq |y| \left| \sin\left(\frac{x}{\sqrt{|y|}}\right) \right| \\ &\leq |y| \\ &\leq \sqrt{x^2+y^2}.\end{aligned}$$

If this last term must be smaller than ϵ, then it is enough to take $\sqrt{x^2+y^2} < \delta = \epsilon$. Because we can choose a δ, we have continuity of $f(x,y)$ in $(0,0)$.

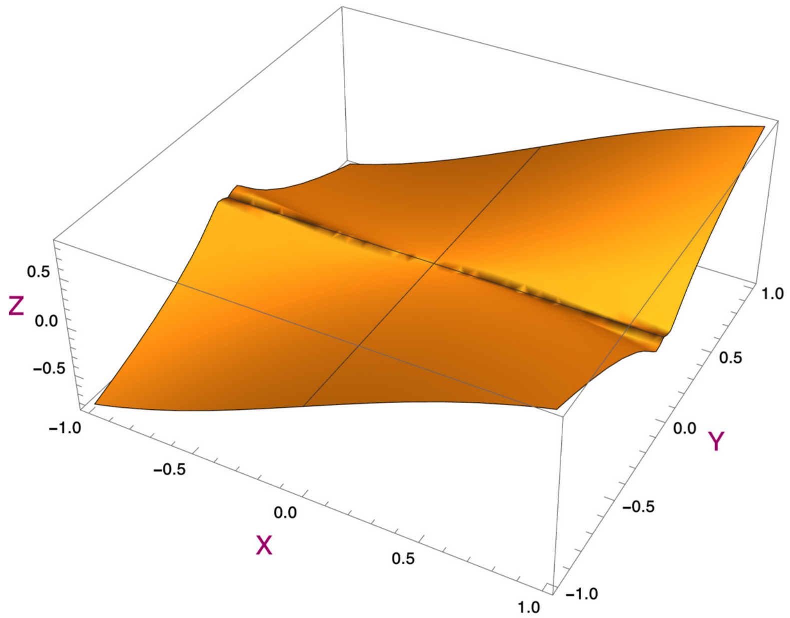


Figure 193. We see here a three dimensional figure of the graph of the function.

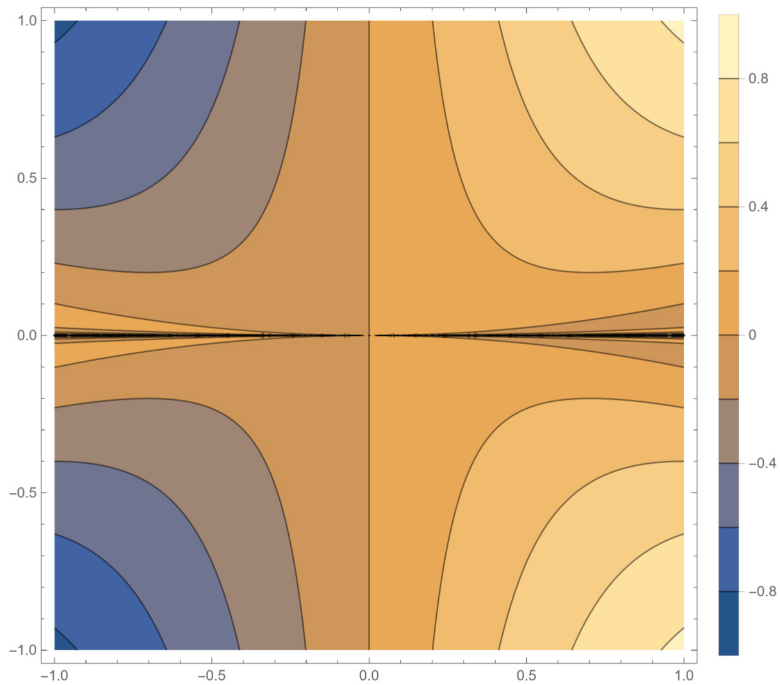


Figure 194. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$. The black area around the X-axis is caused by high frequency of the oscillations caused by the sinus function.

Exercise 71.

Is the function

$$f(x,y)=\begin{cases} x^2\left(1-\cos\left(\frac{y}{x}\right)\right) & \text{if } x\neq 0;\\ 0 & \text{if } x=0.\end{cases}$$

continuous in $(0,0)$?

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon>0$ and we have to prove that $|f(x,y)-f(0,0)|<\epsilon$. The problem is now to find a $\delta>0$ such that if $||(x,y)-(0,0)||<\delta$ it follows that $|f(x,y)-f(0,0)|<\epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0, 0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| x^2 \left(1 - \cos\left(\frac{y}{x}\right)\right) - 0 \right| < \epsilon.$$

We try to find something larger than the left side of the inequality. Then we try to keep that larger expression smaller than ϵ by manipulating the value of δ in the inequality $\sqrt{x^2 + y^2} < \delta$. If we could do that, then we have proven our inequality.

$$\begin{aligned}
\left| x^2 \left(1 - \cos\left(\frac{y}{x}\right)\right) - 0 \right| &\le \left| x^2 \left(1 - \cos\left(\frac{y}{x}\right)\right) \right| \\
&\le 2\,|x^2| \\
&\le 2\sqrt{x^2 + y^2}^{\,2}.
\end{aligned}$$

If this last term must be smaller than ϵ, then it is enough to take $\sqrt{x^2 + y^2} < \delta = \sqrt{\epsilon/2}$. Because we can choose a δ, we have continuity of $f(x, y)$ in $(0, 0)$.

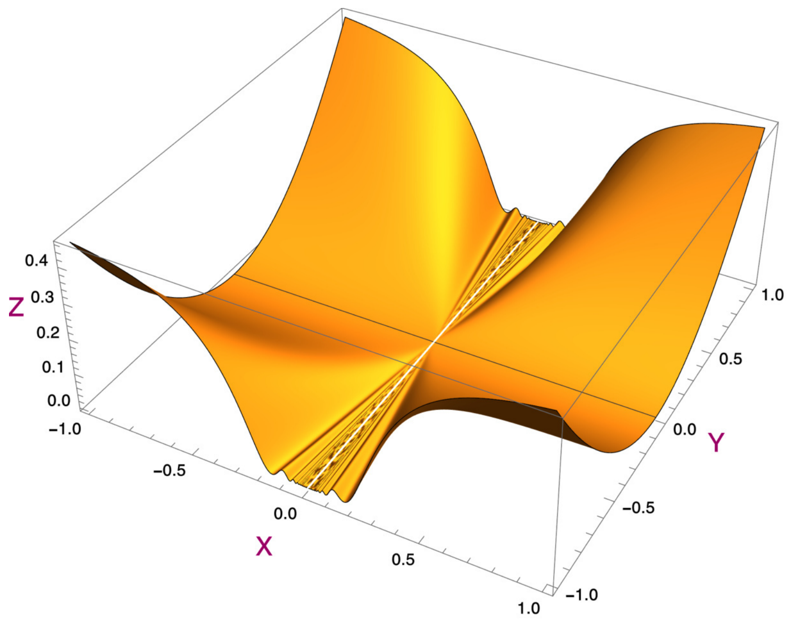


Figure 195. We see here a three dimensional figure of the graph of the function.

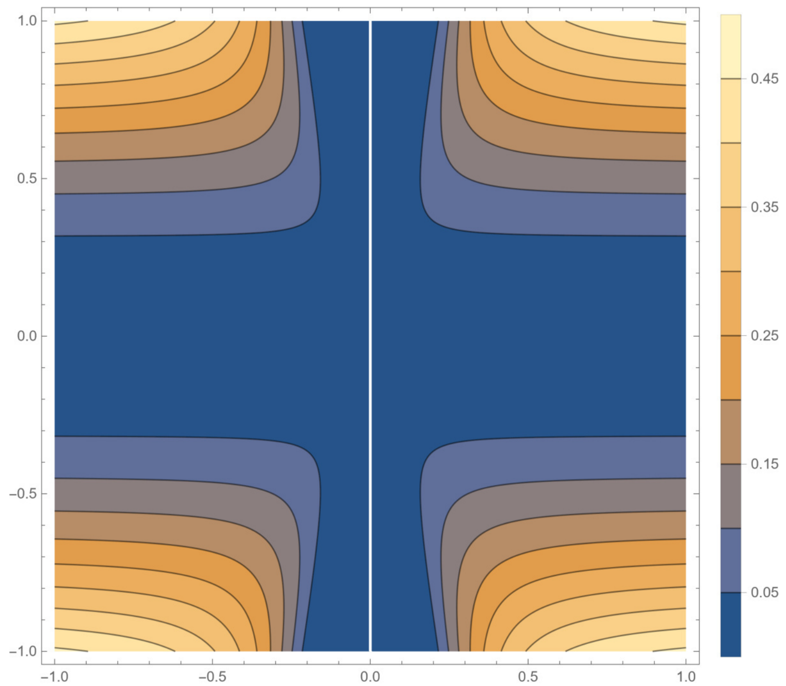


Figure 196. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 72.

Is the function

$$f(x,y)=\begin{cases}\sin\left(\dfrac{1}{x\,y}\right) & \text{if } x\neq 0 \text{ and } y\neq 0;\\ 0 & \text{if } x=0 \text{ or } y=0.\end{cases}$$

continuous in $(0,0)$?

Solution.

We will give a proof for discontinuity. When we studied the theory of continuity in one variable, we surely were acquainted with the topologist's sine curve defined by $f(x)=\sin(1/x)$ for $x\neq 0$. This is a curve with heavy and frequent oscillations becoming more frequent in a neighbourhood of $x=0$. We have an analogue phenomenon here.

We take $\epsilon = 1/2$ and it is enough to prove that $|f(x,y) - f(0,0)| \geq 1/2$ for any $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ there is at least one (x,y) such that $|f(x,y) - f(0,0)| \geq 1/2$ holds.

We see that curves of level $\sin(\mu)$ are of the form $1/(x\,y) = \mu$ and have consequently the equation $x\,y = 1/\mu$ and are hyperbolas. If $\mu = 1/(\pi/2 + 2\,k\,\pi)$, $k \in \mathbf{Z}$, then there will always be infinitely many points lying on hyperbolas that are closer to $(0,0)$ then any δ mapped onto level $y = 1$. So no δ can be chosen.

Because we cannot choose a δ for $\epsilon = 1/2$, we have discontinuity of $f(x,y)$ in $(0,0)$.

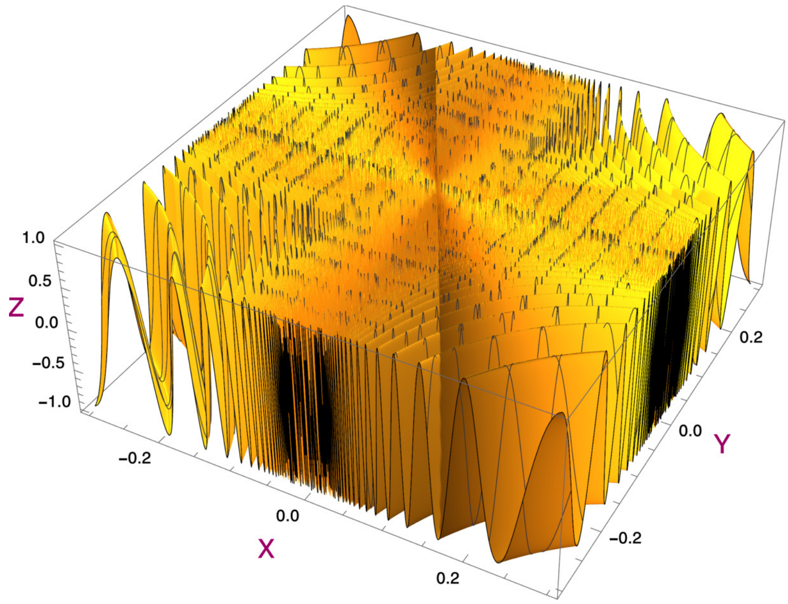


Figure 197. We see here a three dimensional figure of the graph of the function. This does not seem to be a graph of a continuous function. We strain the graphics software to the maximum. This function has a graph that is essentially almost impossible to draw. But the graph gives an idea about what is going on.

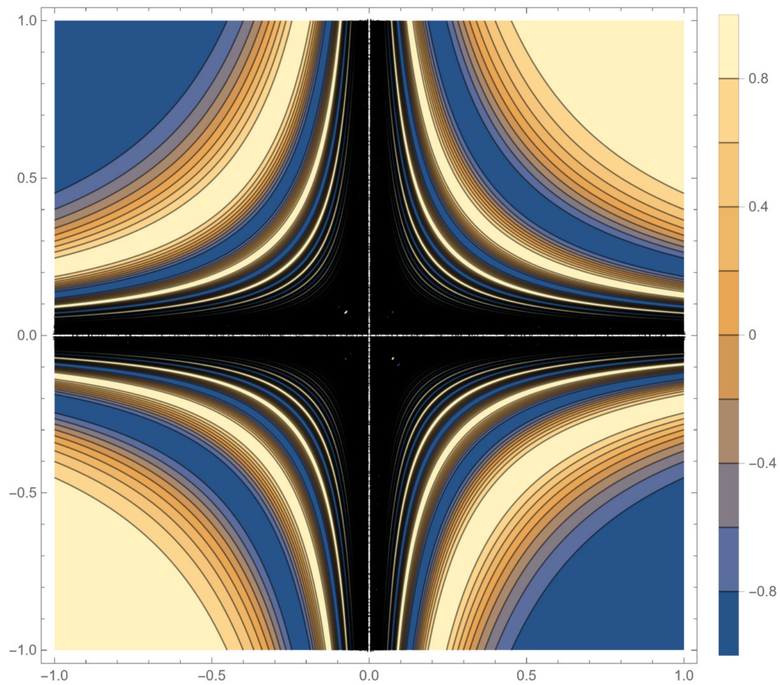


Figure 198. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed. Remark that when the function starts to oscillate very heavily, many colours start to mix giving a black colour. We remark also that the heavy colours and light colours approach $(0,0)$ in a recurring fashion representing the infinitely many oscillations.

Exercise 73.

Is the function

$$f(x,y) = \begin{cases} \sin\left(\dfrac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous in $(0,0)$?

Solution.

We will give a proof for discontinuity. When we studied the theory of continuity in one variable, we surely were acquainted with the topologist's sine curve defined by $f(x) = \sin(1/x)$ for $x \neq 0$. This is a curve with heavy and fast oscillations becoming more frequent in a neighbourhood of $x = 0$. We have an analogue phenomenon here.

We take $\epsilon = 1/2$ and it is enough to prove that $|f(x,y) - f(0,0)| \geq 1/2$ for any $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ there is at least one (x,y) such that $|f(x,y) - f(0,0)| \geq 1/2$ holds.

We see that curves of level μ are of the form $1/(x^2 + y^2) = \mu$ and have consequently the equation $x^2 + y^2 = 1/\mu$ and are circles. If $\mu = 1/(\pi/2 + 2k\pi)$, $k \in \mathbf{Z}$, then there will always be infinitely many circles closer to $(0,0)$ then any δ mapped onto level $y = 1$. So no δ can be chosen.

Because we cannot choose a δ for $\epsilon = 1/2$, we have discontinuity of $f(x,y)$ in $(0,0)$.

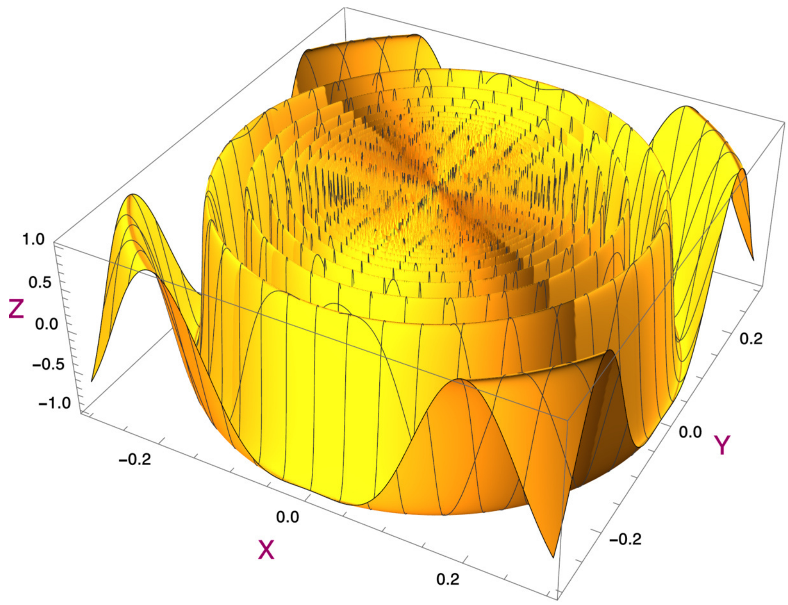


Figure 199. We see here a three dimensional figure of the graph of the function. The heavy and frequent oscillations always having the same amplitude, going from -1 to 1, cause the discontinuity.

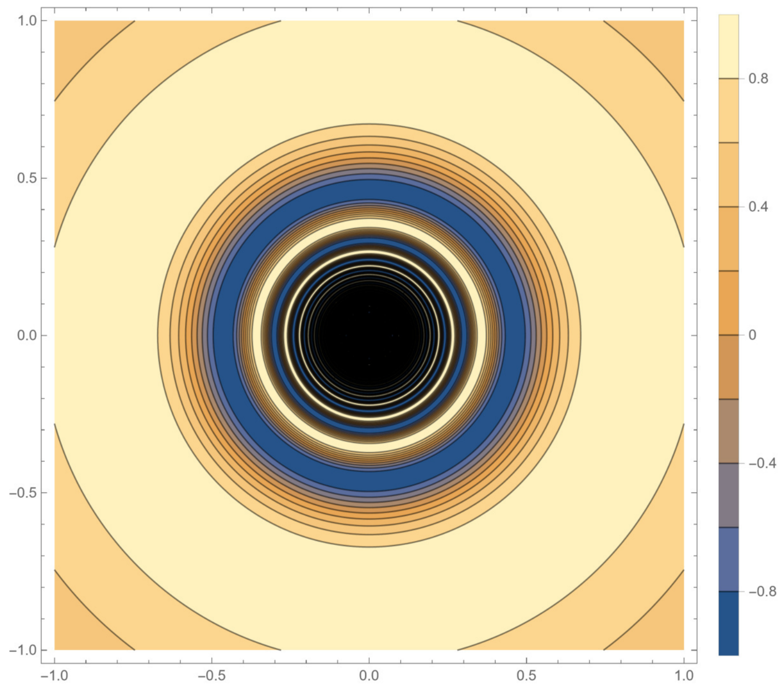


Figure 200. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

We see in this plot a one dimensional analogue of what is going on.

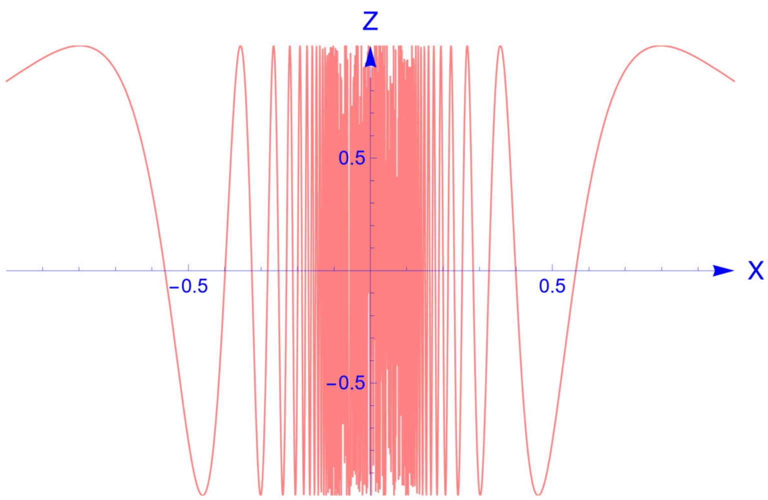


Figure 201. We see here a figure of the graph of $\sin(1/x^2)$.

Exercise 74.

Is the following function with $\mu > 0$ continuous in $(0,0)$?

$$f(x,y) = \begin{cases} (x^2+y^2)^\mu \sin\left(\dfrac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Solution.

We saw before in the theory of one variable calculus the function $f(x) = \sin(1/x)$ if $x \neq 0$. This function is not continuous in $x = 0$, but we saw also that $f(x) = x^\mu \sin(1/x)$ with $\mu > 0$ forces this function to be continuous in $x = 0$. This is true because essentially $\sin(1/x)$ is bounded in a neighbourhood of $x = 0$.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left|(x^2+y^2)^\mu \sin\left(\frac{1}{x^2+y^2}\right) - 0\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned} \left|(x^2+y^2)^\mu \sin\left(\frac{1}{x^2+y^2}\right) - 0\right| &\leq (x^2+y^2)^\mu \left|\sin\left(\frac{1}{x^2+y^2}\right)\right| \\ &\leq (x^2+y^2)^\mu \\ &\leq \sqrt{x^2+y^2}^{\,2\mu}. \end{aligned}$$

If this last term must be smaller than ϵ, then it is enough to take $\sqrt{x^2+y^2} < \delta = \epsilon^{1/(2\mu)}$. Because we can choose a δ, we have continuity of $f(x,y)$ in $(0,0)$.

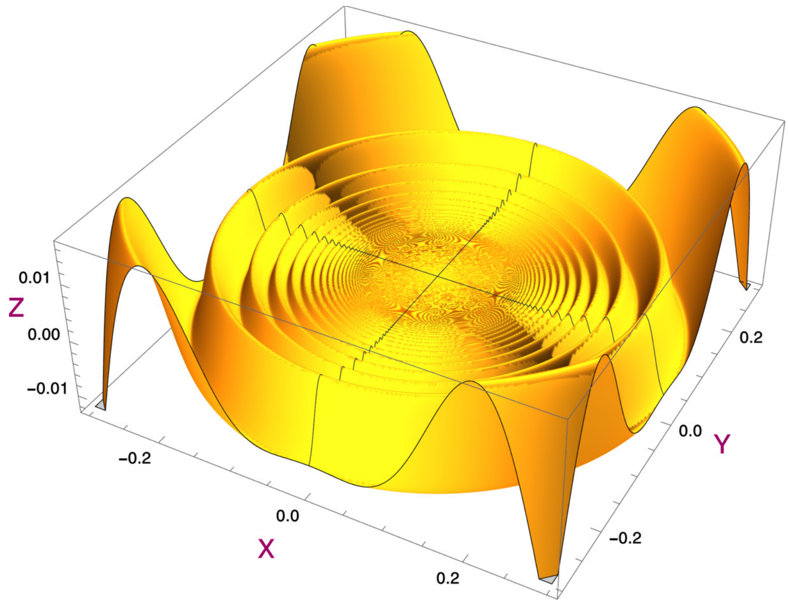


Figure 202. We see here a three dimensional figure of the graph of the function. The amplitudes of the sine are dampened by the factor $(x^2 + y^2)^\mu$ in a neighbourhood of $(0, 0)$. This causes continuity. The picture is made with $\mu = 2$.

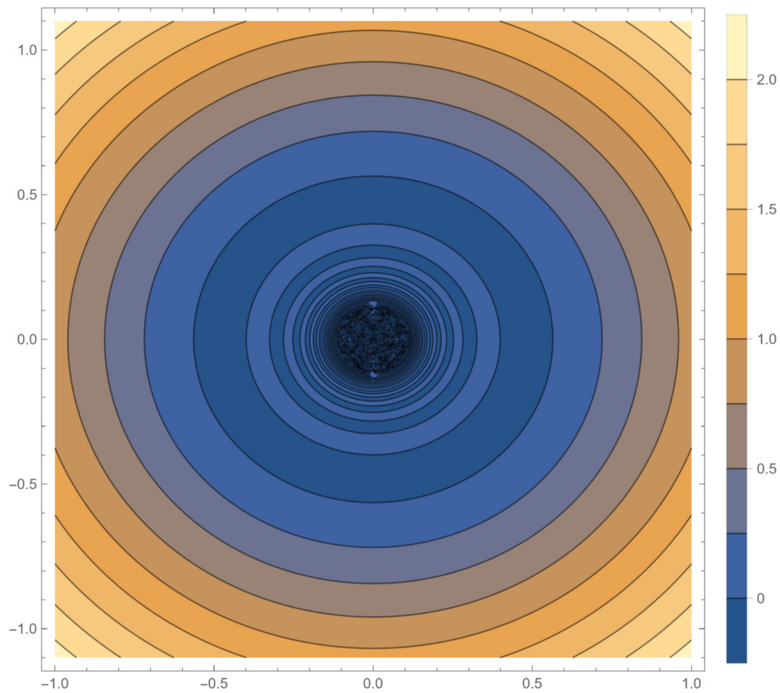


Figure 203. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$. The distortion in the centre is caused by numerical noise due to the high frequency of the oscillations in a neighbourhood of $(0,0)$. The picture is made with $\mu = 2$.

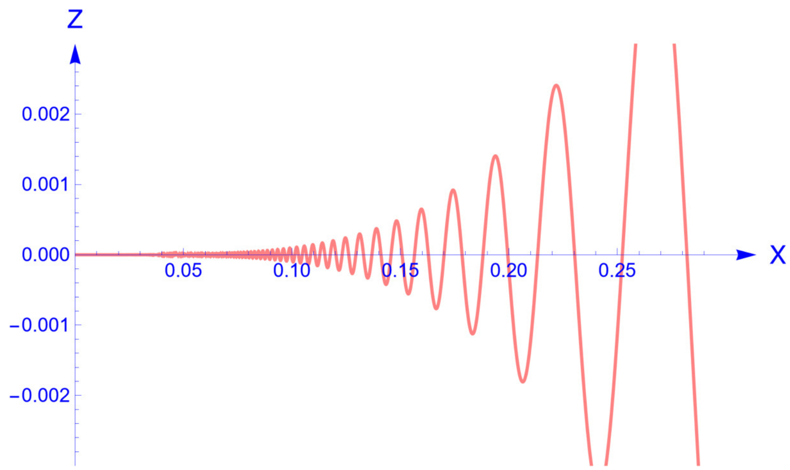


Figure 204. We see here a figure of the of the function $(x^2)^\mu \sin\left(\frac{1}{x^2}\right)$. The amplitudes of the sine curve are dampened by the factor $(x^2)^\mu$. The picture is made with $\mu = 2$.

Exercise 75.

Is the following function continuous in $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{\cos(x\,y)\ \sin\left(4\,x\sqrt{|y|}\right)}{\sqrt{|x\,y|}} & \text{if } x\,y \neq 0; \\ 0 & \text{if } x\,y = 0. \end{cases}$$

Solution.

Before starting the calculations, we will analyse this situation. We see that the factor $\cos(x\,y)$ is continuous and takes the value 1 in $(0,0)$. The continuity depends essentially on the continuity of the other factor. So we drop this factor and work further with the function

$$g(x,y) = \frac{\sin\left(4\,x\sqrt{|y|}\right)}{\sqrt{|x\,y|}}.$$

We rewrite this expression as

$$\frac{\sin\left(4\,x\sqrt{|y|}\right)}{4\,x\sqrt{|y|}}\,\frac{4\,x\sqrt{|y|}}{\sqrt{|x\,y|}}.$$

By remarking that $\lim_{\alpha\to 0} \frac{\sin(\alpha)}{\alpha} = 1$, we are by reasoning as above, left with investigating the function

$$h(x,y) = \frac{4\,x\sqrt{|y|}}{\sqrt{|x\,y|}} = \frac{4\,x}{\sqrt{|x|}}.$$

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|h(x,y) - h(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|h(x,y) - h(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{4x}{\sqrt{|x|}} - 0 \right| < \epsilon.$$

We try to find something larger than the left side of the inequality and try then to keep that larger expression smaller than ϵ by manipulating the value of δ in the inequality $\sqrt{x^2 + y^2} < \delta$. If we could do that, then we have proven our inequality.

$$\begin{aligned} \left| \frac{4x}{\sqrt{|x|}} - 0 \right| &\leq 4\sqrt{|x|} \\ &\leq 4\sqrt{\sqrt{x^2 + y^2}} \\ &\leq 4\sqrt{x^2 + y^2}^{\,1/2}. \end{aligned}$$

If this last term must be smaller than ϵ, then it is enough to take $\sqrt{x^2 + y^2} < \delta = \epsilon^2/4$. Because we can choose a δ, we have continuity of $h(x, y)$ in $(0, 0)$ and consequently also for $f(x, y)$ in $(0, 0)$.

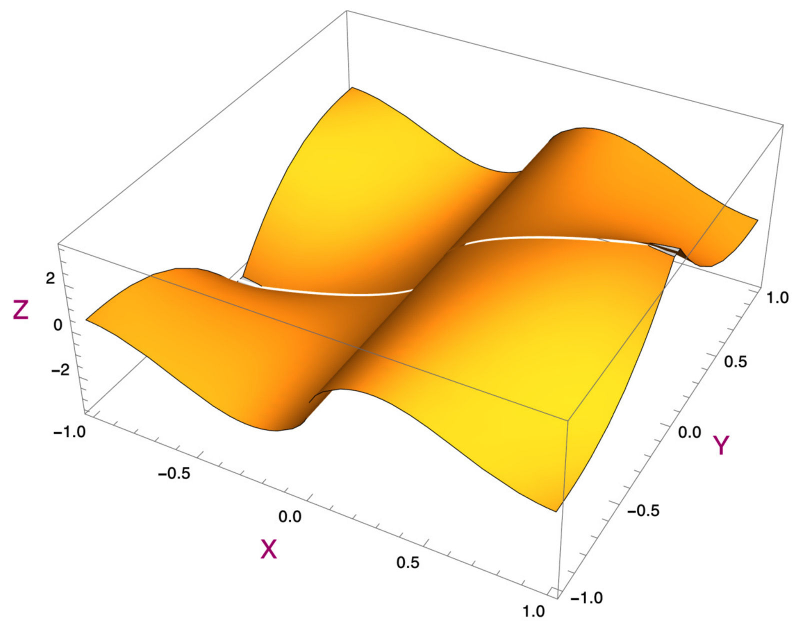


Figure 205. We see here a three dimensional figure of the graph of the function.

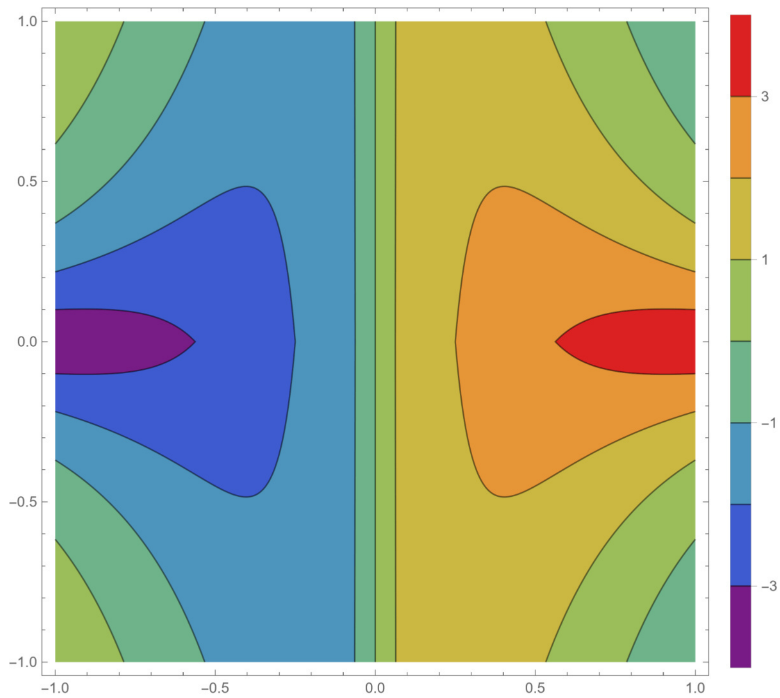


Figure 206. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 76.

Let $m > 0$ and $n > 0$. Prove that $f(x,y)$ defined by

$$f(x,y) = \begin{cases} x^n y^m \sin\left(\dfrac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

is continuous in $(0,0)$.

Solution.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| |x|^n\,|y|^m \sin\left(\frac{1}{x^2+y^2}\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| |x|^n\,|y|^m \sin\left(\frac{1}{x^2+y^2}\right) - 0 \right| &\le \left| |x|^n\,|y|^m \sin\left(\frac{1}{x^2+y^2}\right) \right| \\
&\le |x|^n\,|y|^m \\
&\le \sqrt{x^2+y^2}^{\,n}\,\sqrt{x^2+y^2}^{\,m}. \\
&\le \sqrt{x^2+y^2}^{\,n+m}.
\end{aligned}$$

If this last term must be smaller than ϵ, then it is enough to take $\sqrt{x^2+y^2} < \delta = \epsilon^{1/(n+m)}$. Because we can choose a δ, we have continuity of $f(x,y)$ in $(0,0)$.

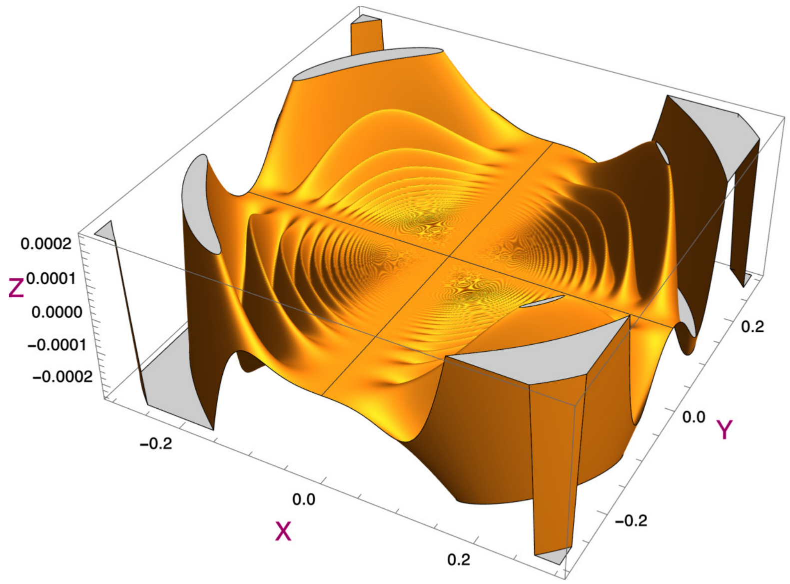


Figure 207. We see here a three dimensional figure of the graph of the function. We observe the strong dampening of the amplitudes of the sine at the centre of the picture. We made the picture using $m = 2$ and $n = 3$.

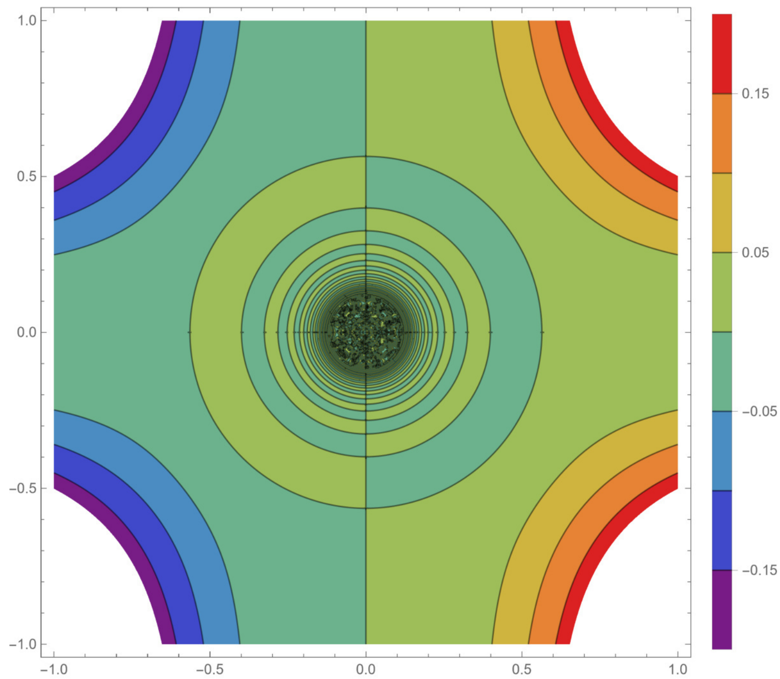


Figure 208. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$. We made the picture using $m = 2$ and $n = 3$. We can observe numerical noise caused by the extreme oscillations in the neighbourhood of $(0,0)$.

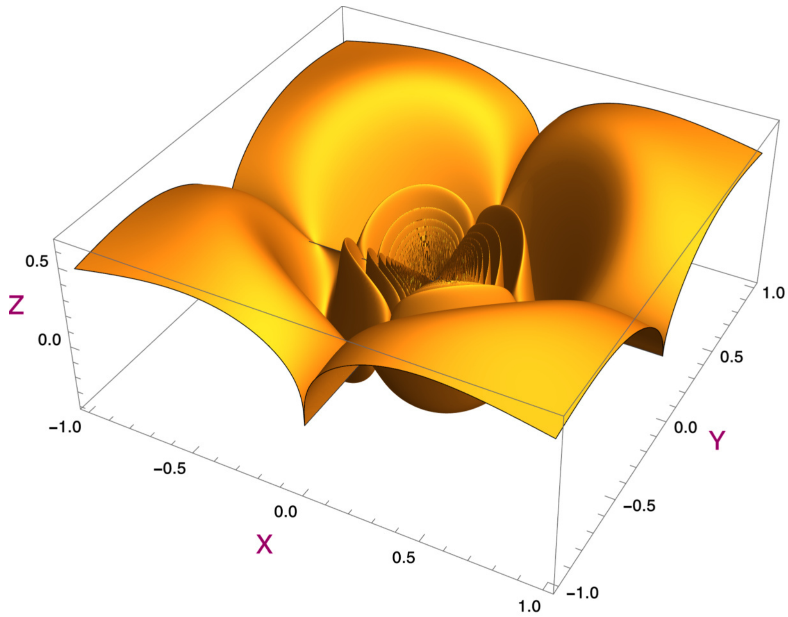


Figure 209. We see here a three dimensional figure of the graph of the function. We observe the strong dampening of the sine at the centre of the picture. We made the figure using $m = 1/2$ and $n = 1/3$.

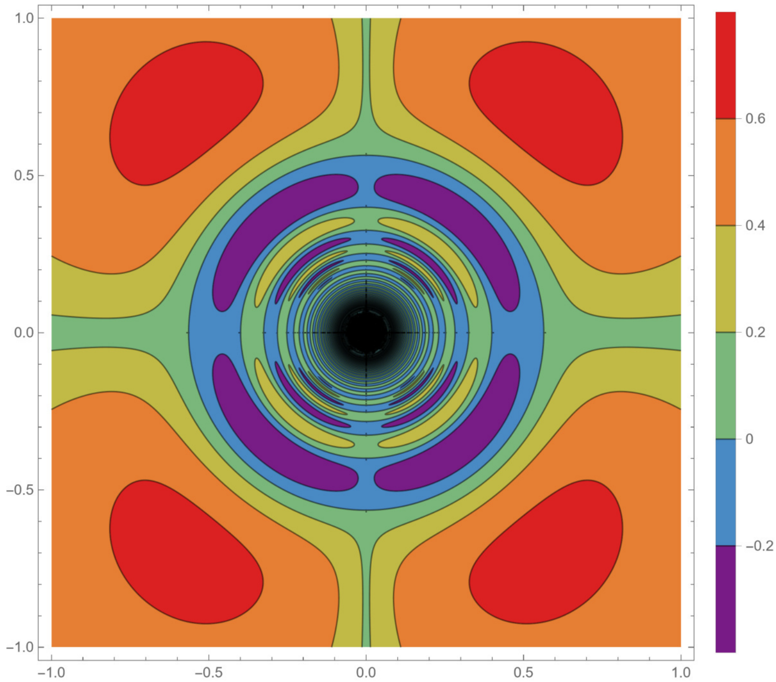


Figure 210. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0, 0)$. The picture is made with $m = 1/2$ and $n = 1/3$. We can observe numerical noise caused by the extreme oscillations in the neighbourhood of $(0, 0)$.

Exercise 77.

Is the following function continuous in $(0, 0)$?

$$f(x, y) = \begin{cases} \dfrac{1 - \cos(x^2 y)}{x^2 + 2y^2} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Solution.

Let us first try to write the function differently. We observe that the numerator can be written as follows.

$$1 - \cos\left(x^2 y\right) = 1 - \left(1 - 2\sin^2\left(\frac{x^2 y}{2}\right)\right) = 2\sin^2\left(\frac{x^2 y}{2}\right).$$

So the fraction can be written as

$$\frac{1-\cos\left(x^2\,y\right)}{x^2+2\,y^2}=\frac{2\,\sin^2\left(\frac{x^2\,y}{2}\right)}{x^2+2\,y^2}=2\,\frac{\left(\frac{x^2\,y}{2}\right)^2}{\left(x^2+2\,y^2\right)}\;\frac{\sin^2\left(\dfrac{x^2\,y}{2}\right)}{\left(\frac{x^2\,y}{2}\right)^2}.$$

We apply now the formula $\lim_{\alpha->0}\frac{\sin(\alpha)}{\alpha}=1$. So we are left with investigating the continuity in $(0,0)$ of the function

$$\frac{x^4\,y^2}{2\left(x^2+2\,y^2\right)}.$$

Let us prove it with an ϵ-δ approach.

We take an arbitrary $\epsilon>0$ and we have to prove that $|f(x,y)-f(0,0)|<\epsilon$. The problem is now to find a $\delta>0$ such that if $||(x,y)-(0,0)||<\delta$ it follows that $|f(x,y)-f(0,0)|<\epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)||=\sqrt{x^2+y^2}<\delta$, we have that

$$\left|\frac{x^4\,y^2}{2\left(x^2+2\,y^2\right)}-0\right|<\epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{x^4\,y^2}{2\left(x^2+2\,y^2\right)}-0\right| &\le \frac{1}{2}\left|\frac{x^4\,y^2}{x^2+2\,y^2}\right|\\ &\le \frac{1}{2}\left|\frac{x^4\,y^2}{x^2+y^2}\right|\\ &\le \frac{1}{2}\,\frac{\sqrt{x^2+y^2}^{\,4}\,\sqrt{x^2+y^2}^{\,2}}{\sqrt{x^2+y^2}^{\,2}}\\ &\le \frac{1}{2}\sqrt{x^2+y^2}^{\,4}.\end{aligned}$$

It is sufficient to take $\sqrt{x^2 + y^2} < 2\,\epsilon^{1/4}$. We can find a δ, so we conclude that the function is continuous.

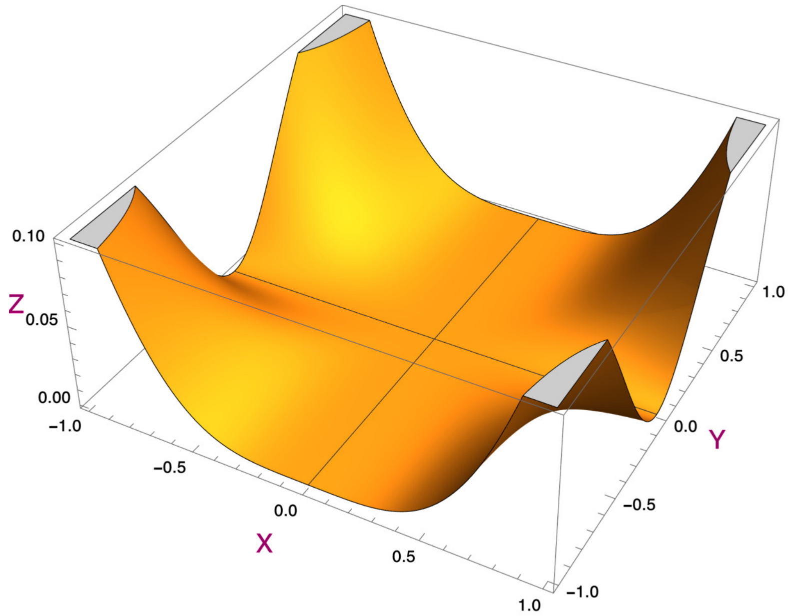


Figure 211. We see here a three dimensional figure of the graph of the function. It is a standard figure of a continuous function.

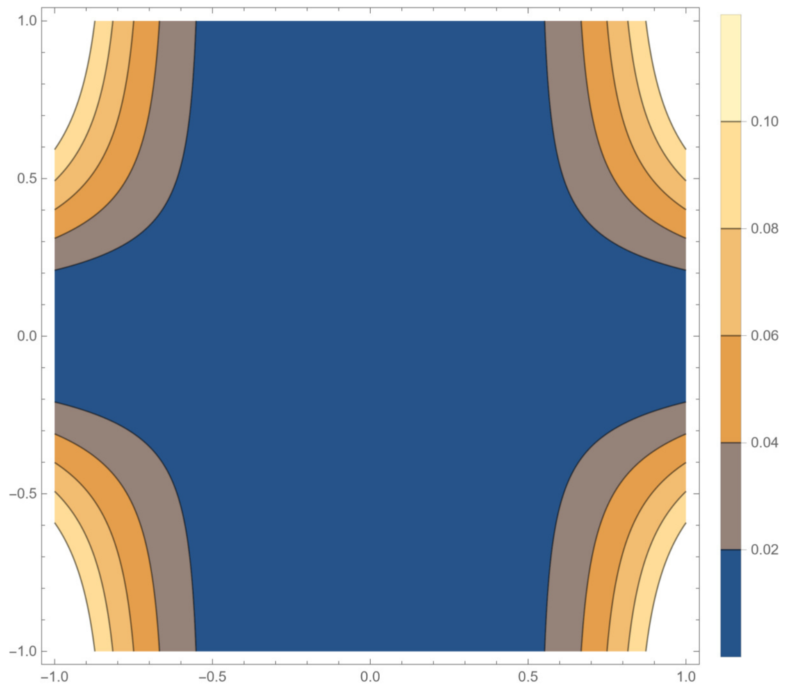


Figure 212. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

Exercise 78.

Is the following function continuous in $(0,0)$?

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2 - y^2} & \text{if } x^2 \neq y^2; \\ 0 & \text{if } x = y \text{ or } x = -y. \end{cases}$$

Solution.

We restrict this function to the continuous curves with equations $x = y + \lambda y^2$.
So

$$f|_{x=y+\lambda y^2}(x,y) = \begin{cases} \dfrac{(\lambda y + 1)^2}{\lambda(\lambda y + 2)} & \text{if } y \neq 0 \text{ and } y \neq -2/\lambda; \\ 0 & \text{if } y = 0 \text{ or } y = -2/\lambda. \end{cases}$$

We calculate the limit for $y \to 0$.

$$\lim_{y \to 0} \frac{(\lambda\, y + 1)^2}{\lambda\,(\lambda\, y + 2)} = \frac{1}{2\,\lambda}.$$

We see that these restricted functions have different limits. But if $f(x, y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x, y)$ is not continuous.

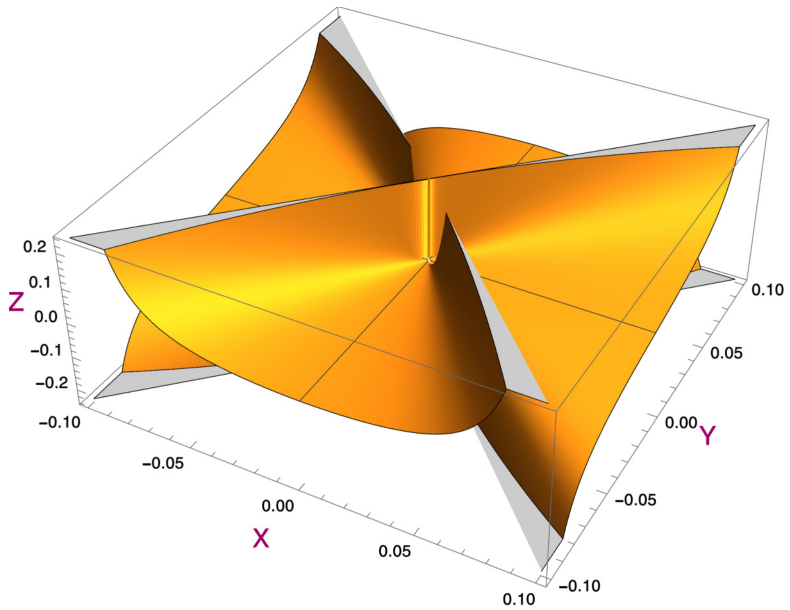


Figure 213. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

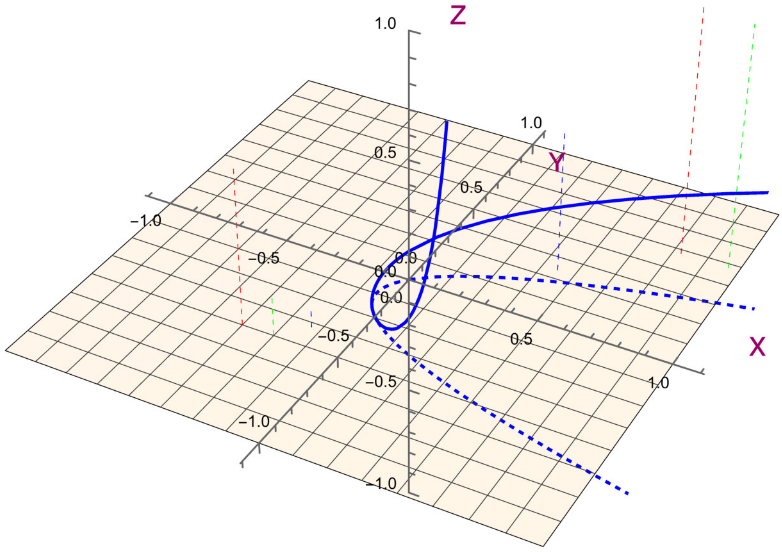


Figure 214. We have restricted the function here to $x = y + 1/2\,y^2$ and $x = y + y^2$ and $x = y + 3/2\,y$. We see in this figure clearly that the restrictions of the function to these curves are functions that have different limits in 0.

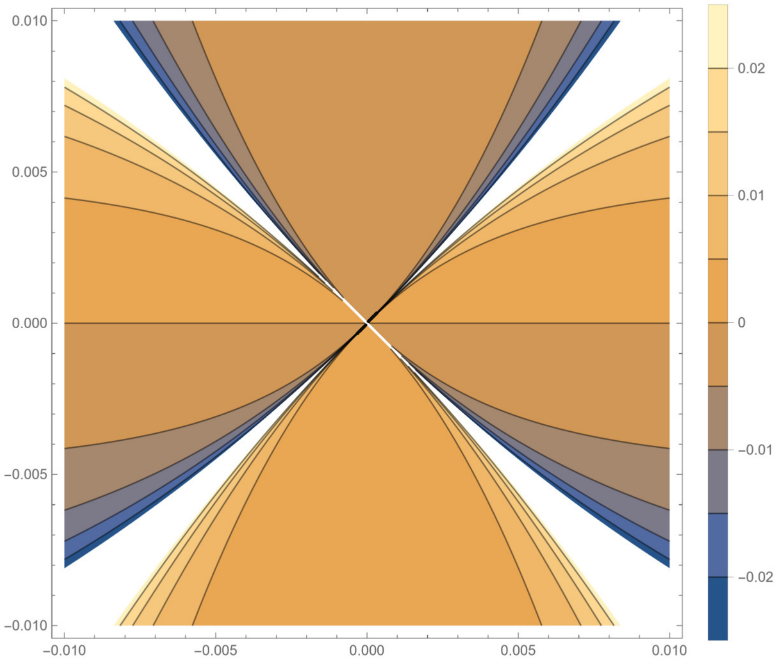


Figure 215. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Exercise 79.

Where is the following function continuous?

$$f(x,y) = \begin{cases} \dfrac{x-y}{x+y} & \text{if } x \neq -y; \\ 0 & \text{if } x = -y. \end{cases}$$

Solution.

Let us first investigate the function in points satisfying $x + y = 0$ but not equal to $(0,0)$. Let us say that this point has coordinates $(\mu, -\mu)$. If we restrict the function to the line $x = \mu$, then we have

$$f(\mu, y) = \frac{\mu - y}{\mu + y}.$$

This function is unbounded in any neighbourhood of $(\mu, -\mu)$ because the denominator is in that case zero and the numerator is not. So the function cannot be continuous there.

Let us now investigate the point $(0,0)$.

We restrict this function to the continuous curves with equations $y = -\lambda x$, with $\lambda \neq 1$. We observe then that

$$f|_{y=-\lambda x}(x,y) = \begin{cases} \dfrac{\lambda x + x}{x - \lambda x} = \dfrac{\lambda + 1}{1 - \lambda} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous in $(0,0)$.

We remark further that the function is continuous for all points (x,y) that do not satisfy $x + y = 0$.

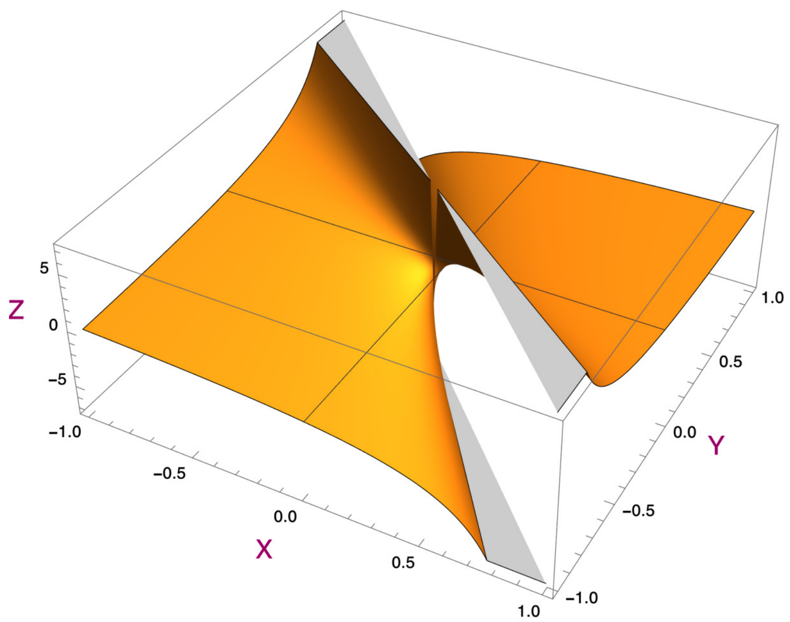


Figure 216. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

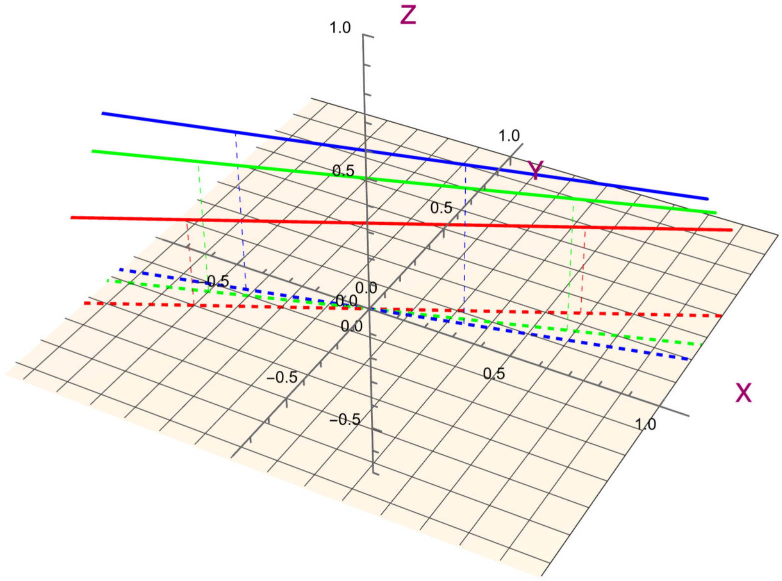


Figure 217. We have restricted the function here to $y = 1/4\,x$ and $y = 1/2\,x$ and $y = 1/3\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

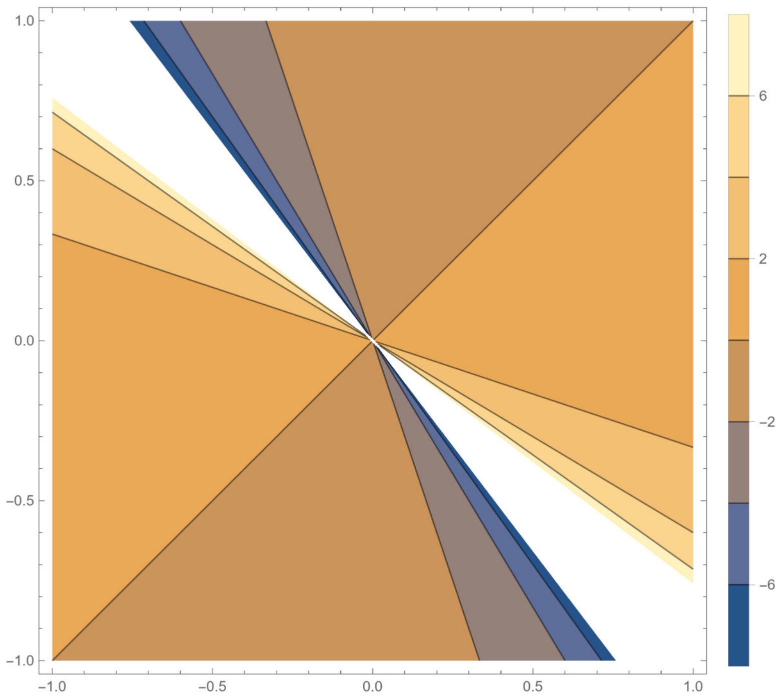


Figure 218. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0, 0)$. This looks discontinuous indeed.

3 appendix

We have written this paragraph to remind the reader about essential elementary inequalities of the exponential and logarithmic functions. We wanted to make also the point that these inequalities can be proved without notions of differentiability. Some of the inequalities are undeniably very easy to prove with using differentiability techniques. But this would make our text not topological in nature and that is not necessary.

We repeat here a possible definition of the exponential function $f(x) = e^x$.

$$e^x = \lim_{n\to+\infty}\left(1+\frac{x}{n}\right)^n .$$

We can see this definition for $x = 2$ in the following figure.

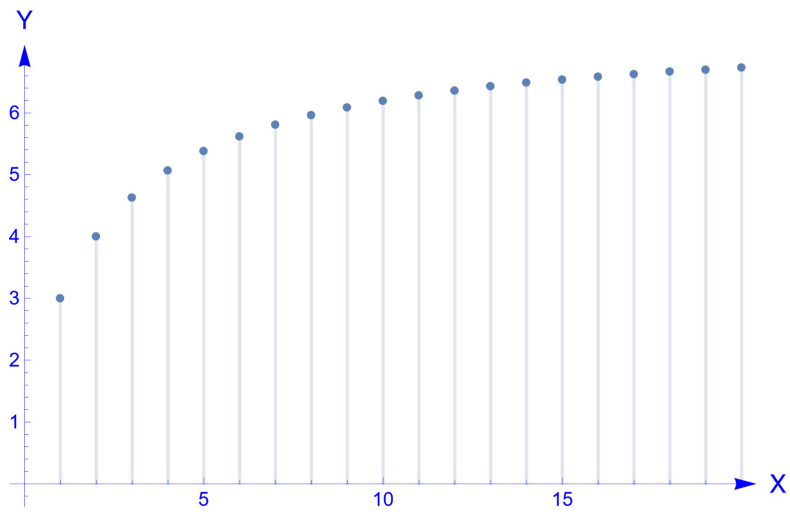


Figure 219. We see here a plot of the values $u_n = \left(1+\frac{x}{n}\right)^n$ when $x = 2$. The figure shows a rapid convergence and an increasing behaviour, but this gives a false or misleading impression. The convergence is in fact excruciatingly slow. Even for the value $n = 1.000.000$ we still do not have the correct value of the fifth decimal.

Theorem. ***The Jacob Bernoulli inequality.***
Let $n \in \mathbf{N}$ and $x \in \mathbf{R}$. Then

$$(1+x)^n \geq 1+nx \quad \text{if } x > -1.$$

Proof. We give a proof by induction. If $n = 1$, we have an equality. Suppose that we assume that the formula is true for n. We have to prove that it is valid for $n + 1$. We know that $(1 + x)^n \geq 1 + n\,x$, and we multiply both sides with $(1 + x)$ which is positive by assumption. $(1+x)^n\,(1+x) \leq (1+n\,x)\,(1+x) = 1+x+n\,x+n\,x^2 \geq 1+(n+1)\,x$. So the formula is valid for $n + 1$ also.

Let us illustrate this inequality with some interesting figures.
We see here a figure for $n = 3$.

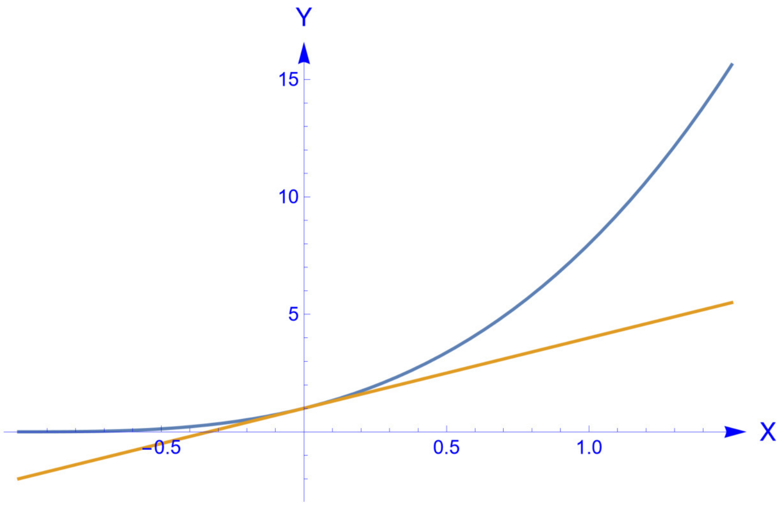


Figure 220. We see here a plot of the inequality for $n = 3$. We see the left hand side of the inequality in blue, the right hand side is in ochre. The function in the right hand side is in fact the tangent line in $x = 0$ of the function defined by the left hand side.

We see here a figure for $n = 7$.

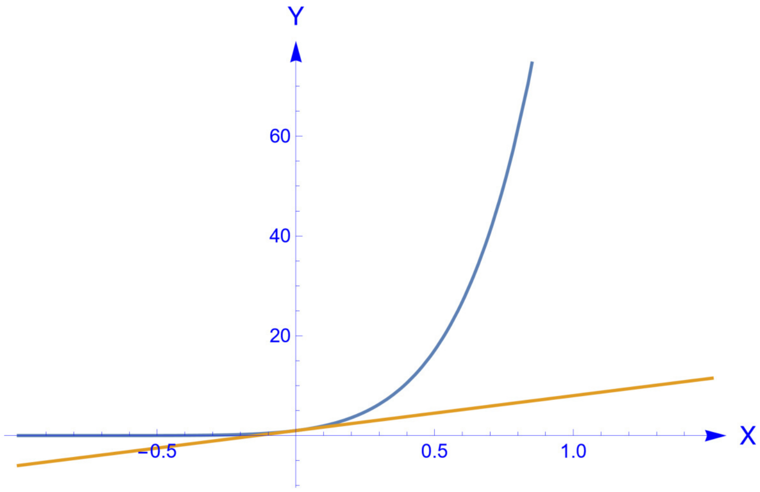


Figure 221. We see here a plot of the inequality for $n = 7$. We see the left hand side of the inequality in blue, the right hand side is in ochre. The function in the right hand side is in fact the tangent line in $x = 0$ of the function defined by the left hand side.

Remark. This inequality is valid for more values for x then stated in the theorem. We do not investigate this. The reason is that this inequality is used for a more general class of exponents then natural numbers and it is in that case only valid for $x > -1$. The proof is then a little bit more complicated. But we will not need this.

$\square$

Theorem. ***The first fundamental inequality of the exponential function.***
Let $x \in \mathbf{R}$. *Then*

$$1 + x \leq e^x.$$

Proof. We have by definition $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$. But by the Jacob Bernoulli inequality we have $1 + x \leq \left(1+\frac{x}{n}\right)^n$ if $x/n > -1$ which is always the case for large n. We can take the limit of the two sides of the inequality and this taking of the limit preserves the inequality. $\square$

Remark. The reader will see that this is equivalent with saying that the tangent line to the graph of $f(x) = e^x$ lies below the graph of the exponential function. See the figure below.

We do not need derivatives for this inequality. The only thing we actually need is that the limit $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ exists. But for this we do not need derivatives either. The proof of this is can also be entirely based on Bernoulli's inequality.

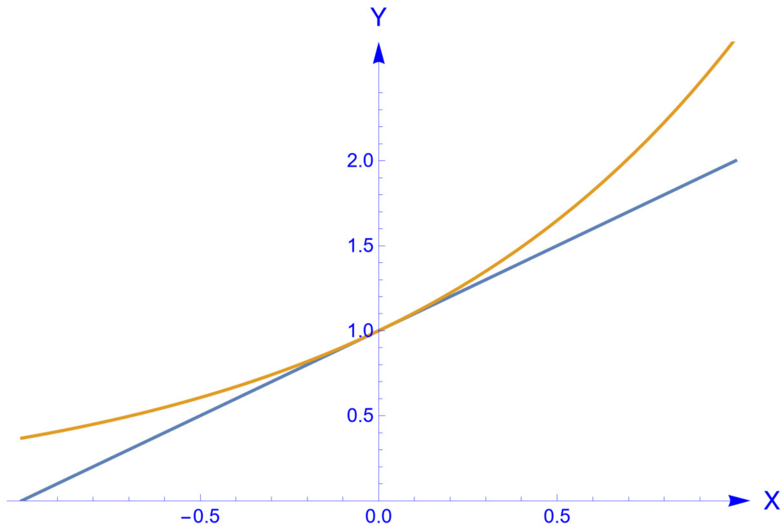


Figure 222. We see here a plot of the first fundamental inequality of the exponential function. We see the left hand side of the inequality in blue, the right hand side is in ochre. The function in the left hand side is in fact the tangent line in $x = 0$ of the function defined by the right hand side. The point is that we do not need derivatives for the proof of this inequality.

Theorem. ***The second fundamental inequality of the exponential function.***
Let $x \in \mathbf{R}$. Then

$$e^x \leq \frac{1}{1-x} \quad \text{if } x < 1.$$

Proof. We know by the first fundamental inequality that $1 + x \leq e^x$. Put $x = -z$. Then $1 - z \leq e^{-z}$. If $1 - z > 0$ or $z < 1$, then we invert both sides of the inequality to get $e^z \leq \frac{1}{1-z}$, which is the inequality we need. □

Let us illustrate this second fundamental inequality of the exponential function.

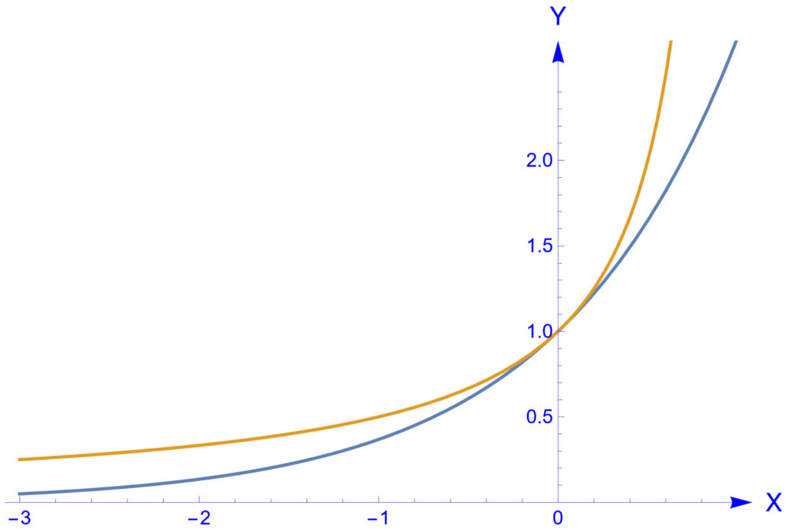


Figure 223. We see here a plot of the second fundamental inequality of the exponential function. We see the left hand side of the inequality, this is the graph of the exponential function in blue, the right hand side is in ochre. The point is once again that we do not need derivatives for the proof of this inequality.

Let us illustrate the two fundamental inequalities of the exponential function combined.

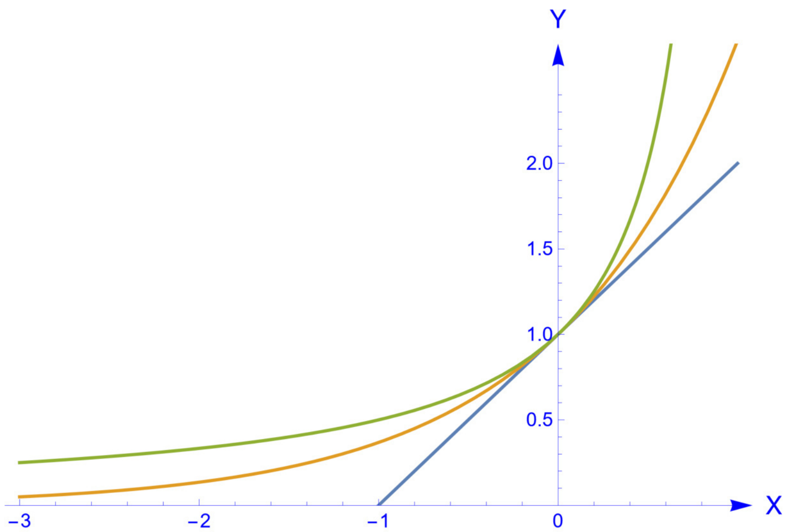


Figure 224. This is an illustration of $1 + x \leq e^x \leq \frac{1}{1-x}$ for $x < 1$. We see the graph of $1 + x$ in blue, the graph of e^x in ochre and the graph of $1/(1 - x)$ in green.

Theorem. ***The first fundamental inequality of the logarithmic function.***
Let $x \in \mathbf{R}$. Then

$$\ln(x+1) \geq \frac{x}{x+1} \quad \text{if } x > -1.$$

Proof. We know that $1 + x \leq e^x$. We restrict $1 + x > 0$ or $x > -1$. We take then the logarithm of the two sides. This is an operation that preserves the inequality. We have now that $\ln(1+x) \leq x$. We substitute $x = -z/(z+1)$. This implies that $z > -1$ because $x > -1$. Thus $\ln(1-z/(z+1)) \leq (-z/(z+1))$ or $\ln(1/(z+1)) \leq -z/(z+1)$. This gives us $-\ln(z+1) \leq -z/(z+1)$ or $\ln(z+1) \geq z/(z+1)$ where $z > -1$. □

Let us illustrate this first fundamental inequality of the logarithmic function.

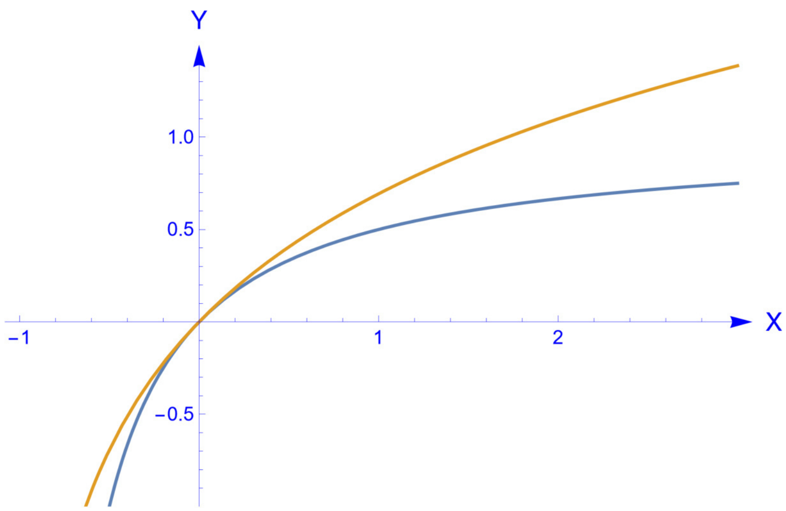


Figure 225. The graph of $x/(x+1)$ is drawn in blue and that of $\ln(1+x)$ is drawn in ochre.

Theorem. ***The second fundamental inequality of the logarithmic function.***
Let $x \in \mathbf{R}$. Then

$$\ln(1+x) \leq x \quad \text{if } x > -1.$$

Proof. We start from the first fundamental inequality of the exponential function. We have then $1 + x \leq e^x$. We restrict x to $x > -1$ and take the logarithm which preserves inequality. We have then $\ln(1+x) \leq x$ with $x > -1$. □

Let us illustrate this second fundamental inequality of the logarithmic function.

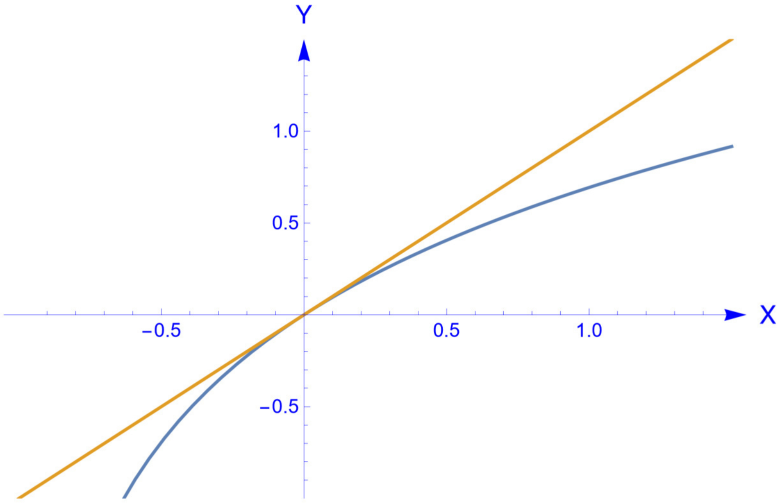


Figure 226. We see the graph of $\ln(1 + x)$ in blue and the graph of x in ochre.

Let us present a figure of the combined logarithmic inequalities.

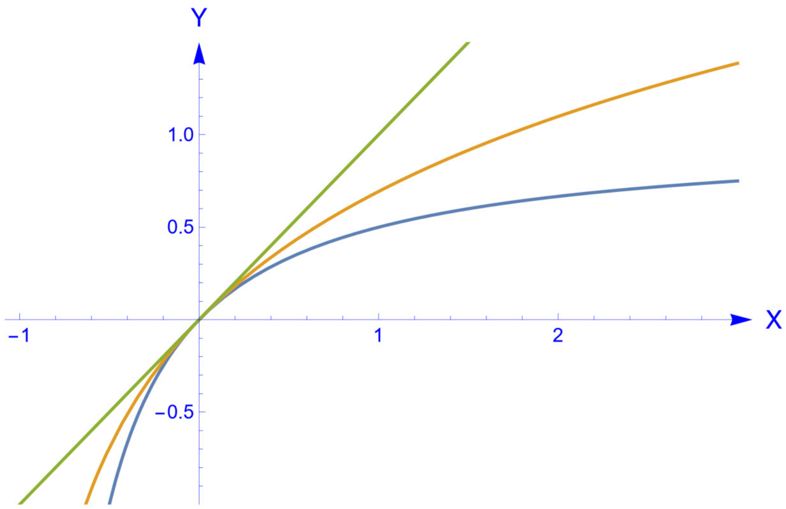


Figure 227. This is an illustration of the inequality $\frac{x}{x+1} \leq \ln(x + 1) \leq x$ with $x > -1$. The graph of $\frac{x}{x+1}$ is in blue, the graph of $\log(x + 1)$ is in ochre, the graph of x is in green.

4 contact the author

Comments are welcome. My email address is:
Dirk.Bollaerts@protonmail.com