Exercises on the differentiability of functions of two variables

With illustrations

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Introduction

This text is written for first year undergraduates in a course of calculus. It is not exclusively written for mathematicians. So the main target reading audience is mathematics, physics and science students, engineering and economics students. The text is also meant for mathematics students, although it is to be expected that they will rather work with more difficult functions and other topologies.

This text is not a linear text. It is to be expected that many students search for a function of a particular type that is related to the function they are studying. A consequence is that explanations in the solutions will overlap each other and are repeated.

All exercises have the same solution structure as explained below. So if one is only interested in differentiability alone, one can read only the fifth section of the answer which is the section on differentiability for each exercise. If one is only interested in the partial derivatives or the directional derivatives, then one can read section two and three of the solution of every exercise.

If somebody is looking for a function that satisfies some conditions but not other conditions, one can look only for the overview of the exercise which is for every exercise section eight of the solution.

A list of all exercises is given at the start of this text document. This can be of help for a student who is looking for something that is related with his problem.

This text is written with care. It has not passed though through the usual channels of peer review. It was meant originally for personal use. A text with so many calculations is prone to errors. So take care and ask your instructor for help in the case of doubt. This text is offered as it is and no promises are made. Murphy's law is universally applicable and for mathematicians it is not only a law, it is a fundamental axiom.

We avoided the big O notation. It is in our opinion not known in general calculus classes.

We avoided also exercises of the type where the solution uses primarily the tool of power series expansions. These are very nice exercises but again, in our opinion, they are not suited for first year undergraduates. We have used Taylor series but there are not many of them and they are used very sparingly.

We have used the ϵ-δ definition in all continuity proofs. Some instructors do not like this style and prefer limit notation. One can still use the ϵ-δ proofs. The inequalities that we have used in these proofs can be translated immediately in limit proofs by using among others the squeeze theorem.

Many students struggle at the start with the interrelations between the different calculations they have to make. In order to help with that, we have given an organigram with the logical interrelations.

We have tried to give some illustrations accompanying the calculations. The illustrations are only meant to give more meaning to the calculations and by no means vice versa. It is remarked that there are deficiencies in the illustrations caused by limited machine precision. We have nevertheless included those illustrations also. So in these cases the figures are more indicative then a complete and accurate picture of the mathematical reality. A wild example is the function e^{-1/x^2} in a neighbourhood of $x = 0$. The text is written so that a reader who hates illustrations will be able to skip the illustrations all together without loosing any meaning or reasoning.

There are two sections in every solution that can be skipped at a first reading. These are the alternative proofs for continuity in section four of every exercise and for differentiability in section six of every exercise. We thought that it could be interesting for some readers. In section seven of every exercise we will discuss an alternative proof for the differentiability which is the continuity of the partial derivatives. This alternative proof is so widespread that we did not indicate it with the label “optional". Note that all alternative proofs are implications and not equivalences. So a function can be continuous but no alternative proof following the prescribed lines explained in the text can be given. The same holds for the other alternative proofs. So in some sense, an alternative proof is a pure luxury.

Structure of the solutions

1. Continuity

We start with the continuity. We will either prove continuity or discontinuity.

2. Partial derivatives

There are no logical connections or implications between this topic and the continuity. So we will discuss partial derivatives in all cases independent from the existence of continuity. We will calculate

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

and

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

3. Directional derivatives

As is the case for partial derivatives, we will always try to compute the directional derivatives.

$$D_{(\mathbf{u},\mathbf{v})}(0,0) = \lim_{h\to 0} \frac{f(a+h\,u,b+h\,v) - f(a,b)}{h}.$$

Please note that partial derivatives are also directional derivatives. But there are many exercises in text books that ask for partial derivatives only, so that it is easier for some students to treat them separately. We will use the convention that (u,v) must be a normalised vector. We warn the reader that there are legitimate texts that do not follow this convention. We work with normalised vectors because that eliminates the fact that there can be many solutions for a direction dependent on the factor λ used for the direction $\lambda\,(u,v)$.

4. Alternative proof of continuity

This is not suited for a first reading. If that is the case, skip immediately to the section on “differentiability".

There is a possible alternative proof for continuity in terms of the first partial derivatives. If the first partial derivatives exist and are bounded in a neighbourhood of (a, b), then the function is continuous in (a, b). Remark that this is a logical implication and by no means an equivalence.

Remark that other alternative proofs do exist. We will use only this criterion. To avoid any misunderstanding: if we say that an alternative proof is not possible we mean that this alternative criterion is not possible.

5. Differentiability

We prove or disprove the differentiability first by using the definition. We will skip of course this subsection of the solution if the function does not satisfy at least one of the necessary properties: if the function is not continuous or if one of the directional derivatives does not exist, then we skip this subsection. The function can indeed only be differentiable if there is continuity and all directional derivatives exist. These two conditions are absolutely necessary but are evidently not sufficient for guaranteeing differentiability.

6. Alternative proof of differentiability

This is not suited for a first reading or if the reader is not acquainted with Lipschitz continuity. If that is the case, skip immediately to the section on "Continuity of the partial derivatives".

If the function is differentiable, then we try to give an alternative proof of this differentiability. If the function is not differentiable, then we will write the word "irrelevant". We have calculated already a lot before writing the section about an alternative proof, including all the directional derivatives and a formula for it. Then it turns out that there is not much work to be done to give an alternative proof. The function f is differentiable if

1. All directional derivatives exist.
2. All directional derivatives can be written in the form $D_{(\mathbf{u},\mathbf{v})}(a, b) = \nabla f(a, b) \cdot (u, v)$

   where the dot is the notation for the inner product, (u, v) is a normal direction vector and (a, b) is the point in which we discuss the differentiability.
3. The function is Lipschitz continuous in a neighbourhood of (a, b).

It is only the Lipschitz continuity that is left to prove because we have calculated the rest in the preceding sections. Consult the appendix of this document for references. We remember that a function is Lipschitz continuous in a set S if there exists a number K such that for all elements (x_1, y_1) and (x_2, y_2) in the set S, the inequality $|f(x_1, y_1) - f(x_2, y_2)| \leq K\,||(x_1, y_1) - (x_2, y_2)||$ is valid. The number K is called the Lipschitz constant. Remark that K is independent of (x_1, y_1) and (x_2, y_2).

7. Continuity of the partial derivatives.

Almost all texts give the following criterion for differentiability. If the partial derivatives are continuous, then the function is differentiable. Remark that this is a logical implication and not an equivalence. It is essentially a luxury criterion. If the partial derivatives are continuous, then we have much more then differentiability only. Remark that it follows that the partial derivatives can be discontinuous while the function is differentiable.

8. Overview

We summarise all results in a table.

9. Possible further investigations

We have sometimes doubts about the continuity of the second order partial derivatives if the two first order partial derivatives exist. Then we give some hints about possible further investigations but we will refrain of these in this text.

Organigram of the logical structure of the interrelationships

The logical interrelations are a big problem for every starting student. The confusion arises because of the fact that there are no logical relations at all when one starts computing some of the main properties of the function. There are indeed no relations between continuity, partial derivatives and directional derivatives. But if one has differentiability, then suddenly we have a lot of structure. Differentiability guarantees continuity and the existence of partial derivatives and all directional derivatives. And to make the situation really confusing, the much used alternative criterion for differentiability, and we mean by this the continuity of the partial derivatives works only one way. There is only an implication and not by any means an equivalence. So continuity of the partial derivatives implies differentiability but it is in fact much stronger then differentiability alone. It takes care of what one could call the continuity of the tangential spaces themselves.

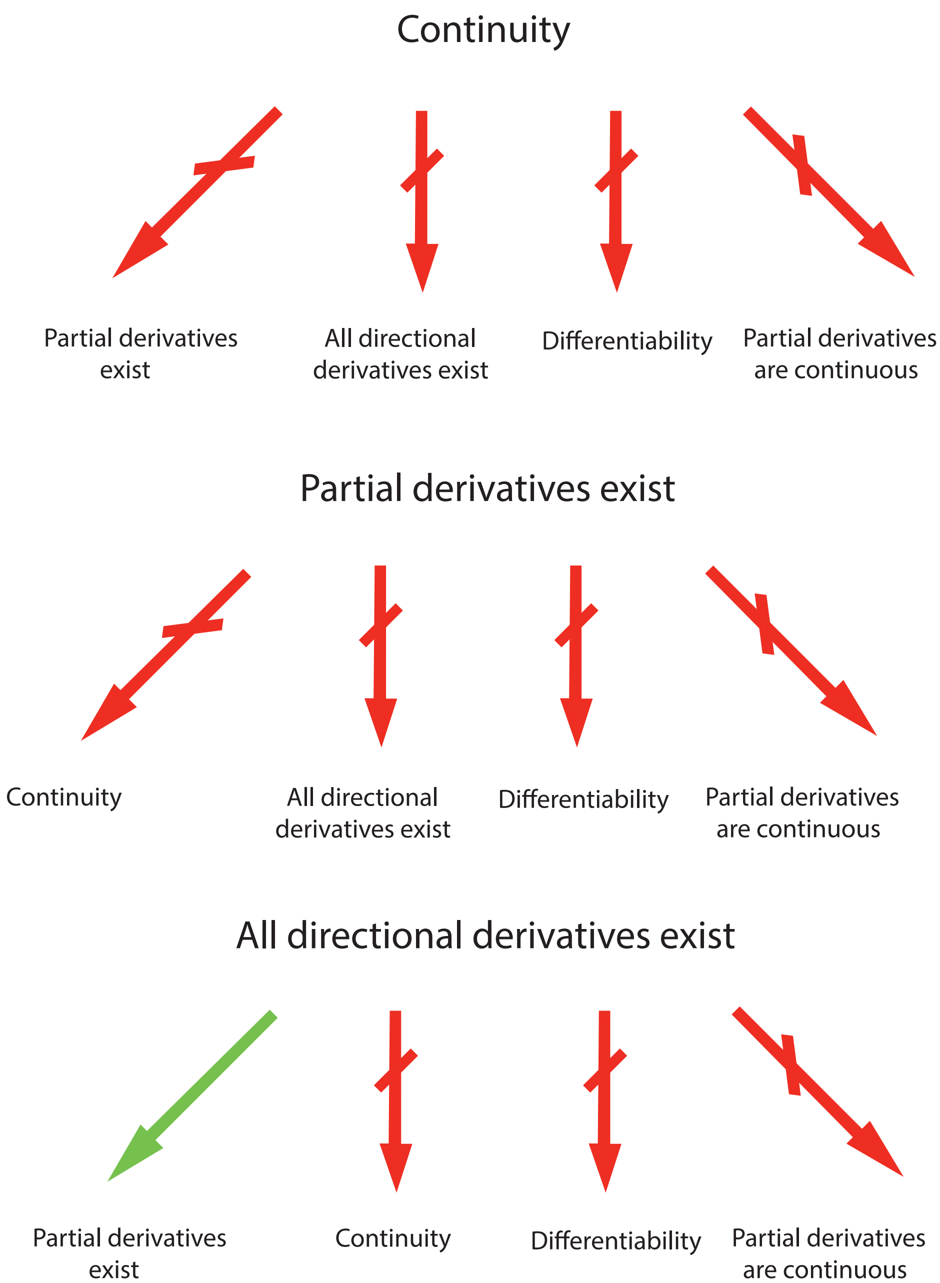

Figure 1. We see here an organigram of the logical relations. A red crossed arrow must be read as “does not necessarily imply".

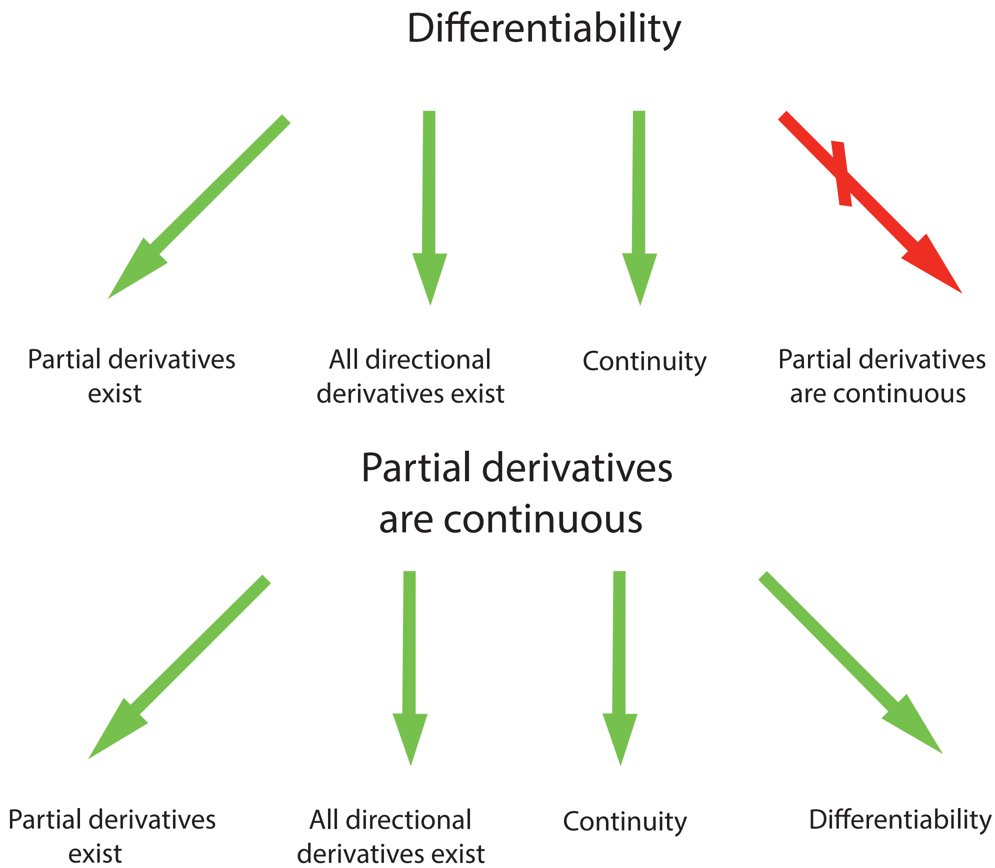

Figure 2. We see here the second part of an organigram of the logical relations. A red crossed arrow must be read as “does not necessarily imply".

List of all exercises

We list all exercises in the following pages. This can help the readers who are interested in searching for special functions that capture their current interest. In order to keep the list short and concise, we give only the main parts of the definitions of the functions.

Table 1: We give a list of the main part of the definitions of the functions in the exercises in this table. The exercises that are marked with "see below" are based on definitions that are too long to put them in the table and they are placed immediately below the table.

1.	$\left(x^2+y^2\right)\sin\left(\dfrac{1}{x^2+y^2}\right)$	**2.**	$\dfrac{x^3\,y}{x^4+y^2}$
3.	$\dfrac{3\,x^2\,y-y^3}{x^2+y^2}$	**4.**	$\dfrac{y^3-x^8\,y}{x^6+y^2}$
5.	$x^3\sin\left(\dfrac{1}{x^2}\right)+y^3\sin\left(\dfrac{1}{y^2}\right)$	**6.**	$\dfrac{x\,\lvert y\rvert}{x^2+y^2}$
7.	$\dfrac{y\,\lvert x\rvert^{3/2}}{x^3+y^2}$	**8.**	$\dfrac{\sin\left(\lvert x\,y\rvert+x^2\right)}{x}$
9.	$\dfrac{x^2\,y^2}{x^2+y^2}$	**10.**	$\dfrac{x\,y^4}{x^2+y^2}$
11.	$\dfrac{x\,y^2}{x^2+y^4}$	**12.**	$\dfrac{y\,\left(x^2-y\right)^2}{x^6}$
13.	$\dfrac{x^3\,y-x\,y^3}{x^2+y^2}$	**14.**	$\dfrac{x^2\,y}{x^6+y^2}$
15.	$\dfrac{x^2\,\lvert y\rvert^{5/4}}{x^4+y^2}$	**16.**	$\sqrt[3]{x^2\,y}$
17.	$x^2\,y\,\sin\left(\dfrac{1}{x}\right)$	**18.**	$\dfrac{x\,y}{\sqrt{x^2+y^2}}$
19.	$\dfrac{x^2\,y}{x^2+y^2}$	**20.**	$\dfrac{x^3+y^3}{x^2+y^2}$
21.	$\dfrac{x^3\,y}{x^6+y^2}$	**22.**	$\dfrac{x^5\,y}{x^8+y^4}$
23.	$\dfrac{x^5+y^4}{\left(x^2+y^2\right)^2}$	**24.**	$\dfrac{x\,y^2}{\sqrt{x^2+y^2}\,\left(x^2+y^4\right)}$

25.	$\dfrac{x\,y\,(x^2-y^2)}{x^2+y^2}$	**26.**	$\dfrac{y\,\sin(x\,y)}{x^2+y^2}$
27.	$\dfrac{y^3}{x^2+y^2}$	**28.**	$\left(x^2+y^2\right)\log\left(x^2+y^2\right)$
29.	$\dfrac{x^2+y^2}{x^2+y^4}$	**31.**	$\dfrac{x^2\,y^2}{x^2\,y^2+(x-y)^2}$
32.	$x^2\left(\sin\left(\dfrac{1}{x+y}\right)+2\right)$	**33.**	$x\,\sin\left(\dfrac{1}{x}\right)+y\,\sin\left(\dfrac{1}{y}\right)$
34.	$x^2\,\sin^2\left(\dfrac{y}{x}\right)$	**35.**	$\dfrac{\sin^2(x+y)}{\lvert x\rvert+\lvert y\rvert}$
36.	$\lvert x\,y\rvert$	**37.**	$y\,\sin\left(\dfrac{x}{\sqrt{\lvert y\rvert}}\right)$
38.	$\sin(y)\,\operatorname{sgn}(\sin(x))$	**39.**	$\dfrac{\sin\left(4\,x\sqrt{\lvert y\rvert}\right)}{\sqrt{\lvert x\,y\rvert}}$
39.	$\max\{x,y\}.$	**40.**	$y\sqrt{\lvert x\rvert}$
41.	$x\,\sin\left(\dfrac{1}{x^2+y^2}\right)$	**42.**	$\max\{\lvert x\rvert,\lvert y\rvert\}$
43.	$\left(x^2+y^2\right)\sin\left(\dfrac{1}{x+y}\right)$	**44.**	$\left(x^2+y^2\right)\,\sin\left(\dfrac{1}{x^4+y^4}\right)$
45.	$\left(x^2+y^2\right)\,\sin\left(\dfrac{1}{x^2+y^2}\right)$	**46.**	$x\,y^2\,\sin\left(\dfrac{1}{y}\right)$
47.	$\dfrac{\lvert y\rvert\,2^{-\frac{\lvert y\rvert}{x^2}}}{x^2}$	**48.**	$\dfrac{x\,y}{\lvert x\rvert}+x\,\sin\left(\dfrac{1}{y}\right)$
49.	$x^3\,\sin\left(\dfrac{1}{x^2}\right)+y^3\,\sin\left(\dfrac{1}{y^2}\right)$	**50.**	$\dfrac{y\sqrt{x^2+y^2}}{\lvert y\rvert}$
51.	$\dfrac{x^2\,y\sqrt{\lvert y\rvert}}{x^4+y^2}$	**52.**	$\sqrt[3]{x}\,\sqrt[3]{y}$

53.	$\sin\left(\dfrac{x^2+y^2}{x^4+y^4}\right)$	**54.**	$\sqrt{\sin^2(x)+\sin^2(y)}$
55.	$\dfrac{\lvert x\rvert+\lvert y\rvert-\lvert x+y\rvert}{(x^2+y^2)^{1/5}}$	**56.**	$\dfrac{\lvert x\rvert+\lvert y\rvert-\lvert x+y\rvert}{(x^2+y^2)^{1/2}}$
57.	$\sin\left(\dfrac{1}{x^2+y^2}\right)$	**58.**	See below
59.	$\min\{x,y\}=\dfrac{x+y}{2}-\dfrac{\lvert x-y\rvert}{2}$	**60.**	$\lvert x\rvert+\lvert y\rvert$
61.	$\dfrac{x\,y}{x^2+y^2}$	**62.**	$\left(\sqrt[3]{x}+\sqrt[3]{y}\right)^3$
63.	$\sqrt{\sin^2(x)+\sin^2(y)}$	**64.**	$\dfrac{(x+y)^2}{x^2+y^2}$
65.	$\dfrac{\log\left(x^2\,y^2\right)}{x^2+y^2}$	**66.**	See below
67.	$x+y$	**68.**	$x^2\left(1-\cos\left(\dfrac{y}{x}\right)\right)$
69.	$\dfrac{\sin(x-y)}{\sqrt{\lvert x\rvert}+\sqrt{\lvert y\rvert}}$	**70.**	$\dfrac{x\,y-\sin(x)\,\sin(y)}{x^2+y^2}$
71.	$\dfrac{\log\left(\lvert x\rvert+e^{\lvert y\rvert}\right)}{\sqrt{x^2+y^2}}$	**72.**	$\dfrac{(x-y)\,y\,e^{-\frac{1}{x^2}}}{(x-y)^2+2\,e^{-\frac{2}{(x)^2}}}$
73.	$\sqrt{4-x^2-y^2}$	**74.**	See below
75.	See below	**76.**	$\lfloor x+y\rfloor$
77.	$\dfrac{\cos(x\,y)-1}{x^2\,y^2}$	**78.**	See below

Here are the exercises that could not be fitted in the table and were marked “See below”.

58.

$$f(x,y)=\begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } y=0 \text{ and } x\neq 0,\\ y\sin\left(\frac{1}{y}\right) & \text{if } x=0 \text{ and } y\neq 0,\\ 0 & \text{elsewhere.}\end{cases}$$

66.

$$f(x,y)=\begin{cases} x^2\sin\left(\frac{1}{x}\right)+y^2\sin\left(\frac{1}{y}\right) & \text{if } xy\neq 0,\\ x^2\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0 \text{ and } y=0,\\ y^2\sin\left(\frac{1}{y}\right) & \text{if } y\neq 0 \text{ and } x=0,\\ 0 & \text{if } x=0 \text{ and } y=0.\end{cases}$$

74.

$$f(x,y)=\begin{cases} 1 & \text{if } x^2=y \text{ and } x\neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

75.

$$f(x,y)=\begin{cases} (x+1)^2+(y+1)^2-2 & \text{if } x<y \text{ and } y<2x \text{ and } x>0,\\ 0 & \text{elsewhere.}\end{cases}$$

78.

$$f(x,y)=\begin{cases} x^2+y^2 & \text{if } x<y \text{ and } y<2x \text{ and } x>0,\\ 0 & \text{elsewhere.}\end{cases}$$

Exercise 1.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability,
5. continuity of the partial derivatives

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} (x^2+y^2)\sin(x^2+y^2) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

1.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \left(x^2+y^2\right) \sin\left(\frac{1}{x^2+y^2}\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned} \left| \left(x^2+y^2\right) \sin\left(\frac{1}{x^2+y^2}\right) \right| &\leq x^2+y^2 \\ &\leq \sqrt{x^2+y^2}^2. \end{aligned}$$

It is sufficient to take $\delta = \epsilon^{1/2}$. We can find a δ, so we conclude that the function is continuous.

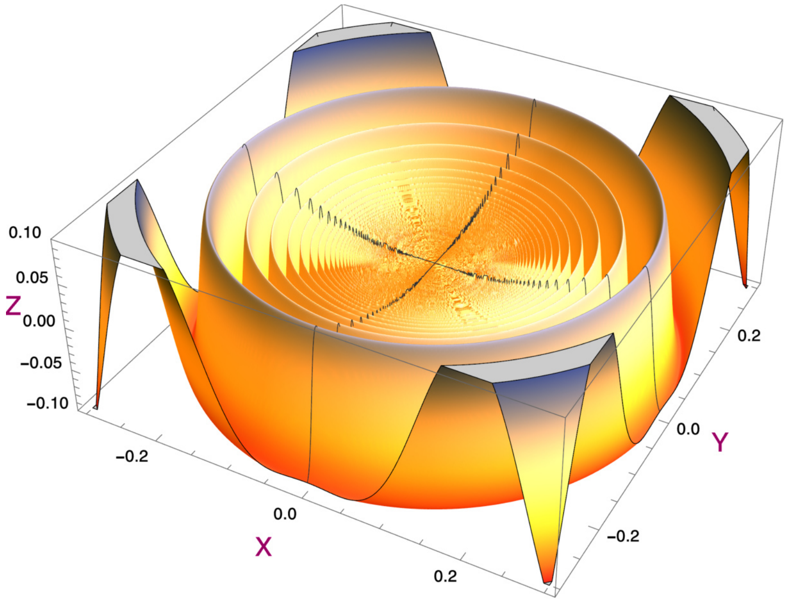

Figure 3. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

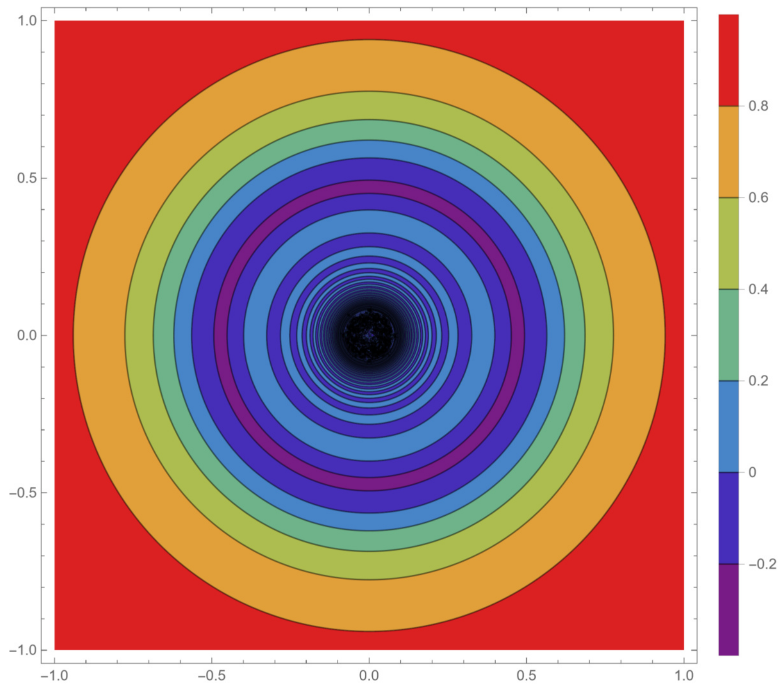

Figure 4. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

1.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f\big|_{y=0}(x,y) = \begin{cases} f(x,0) = x^2 \sin\left(\dfrac{1}{x^2}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$
$$= \lim_{h\to 0} h \sin\left(\frac{1}{h^2}\right)$$
$$= 0.$$

This limit is zero because we can squeeze like this: $0 \le |h \sin\left(\frac{1}{h^2}\right)| \le |h|$. So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).

We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = y^2 \sin\left(\frac{1}{y^2}\right) & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h}$$
$$= \lim_{h\to 0} h \sin\left(\frac{1}{h^2}\right)$$
$$= 0.$$

We have reasoned as before. So the partial derivative to y does exist. We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

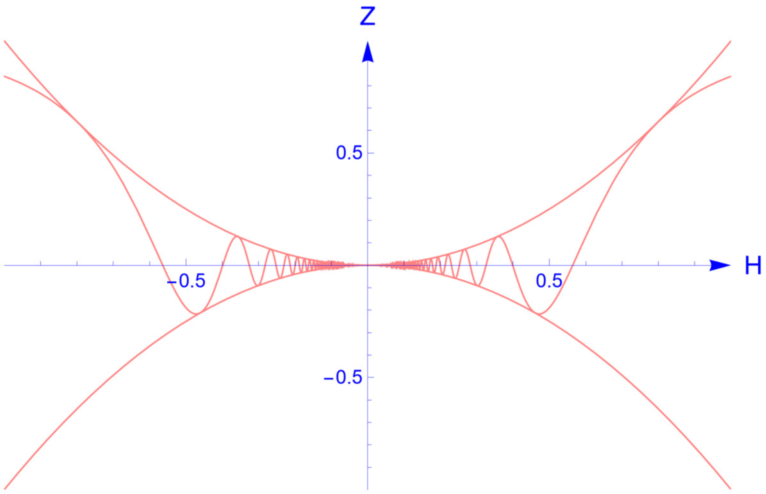

Figure 5. We see here a figure of the graph of the function restricted to the horizontal X-axis through $(0,0)$. We have plotted here the function $f(h,0)$.

The figure of the function restricted to the vertical Y-axis is completely analogous.

1.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0)=\lim_{h\to 0}\frac{f(0+h\,u,0+h\,v)-f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate

$$
\begin{aligned}
D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\
&= \lim_{h\to 0} h\left(u^2+v^2\right)\sin\left(\frac{1}{h^2\,(u^2+v^2)}\right) \\
&= \lim_{h\to 0} h \sin\left(\frac{1}{h^2}\right) \\
&= 0.
\end{aligned}
$$

One can prove this last fact by using the squeezing theorem. We have $0 \le |h\ \sin\left(\frac{1}{h^2}\right)| \le |h|$.

We have used in the calculation that (u,v) is a normal vector so that $u^2+v^2=1$.

So our directional derivatives do always exist.

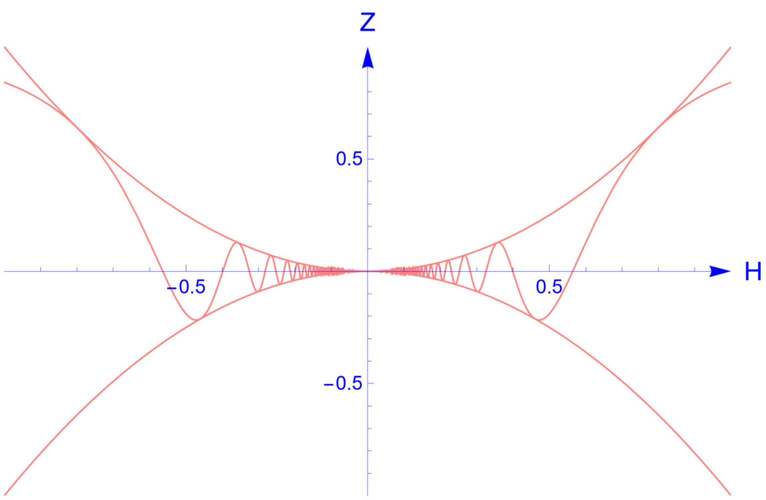

Figure 6. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. This result was to be expected because we have a rotational symmetry around the Z-axis. We have drawn here the function $f(h\,u, h\,v)$.

1.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y)$$
$$= \begin{cases} 2\,x\left(\sin\left(\dfrac{1}{x^2+y^2}\right) - \dfrac{\cos\left(\dfrac{1}{x^2+y^2}\right)}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y)$$
$$= \begin{cases} 2\,y\left(\sin\left(\dfrac{1}{x^2+y^2}\right) - \dfrac{\cos\left(\dfrac{1}{x^2+y^2}\right)}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is unbounded.

We see that the first term

$$2\,x\,\sin\left(\frac{1}{x^2+y^2}\right)$$

is certainly bounded. So we take a look at the second term.

$$-\frac{2\,x\,\cos\left(\frac{1}{x^2+y^2}\right)}{x^2+y^2}.$$

We restrict this term to $y = x$ and have then

$$-\frac{\cos\left(\frac{1}{2\,x^2}\right)}{x}.$$

We define then the sequence $x_k = \frac{1}{2\,\sqrt{\pi}\,\sqrt{k}}$, $k \in \mathbf{N}_0$, that converges to 0. But the values of those points in the second term are $-2\,\sqrt{\pi}\,\sqrt{k}$ and these values are unbounded. This proves that the partial derivative to x is certainly unbounded. Every neighbourhood in $(0,0)$ contains infinitely many points of the line with equation $y = x$ arbitrarily close to $(0,0)$. In order to illustrate this, look at the figure that follows.

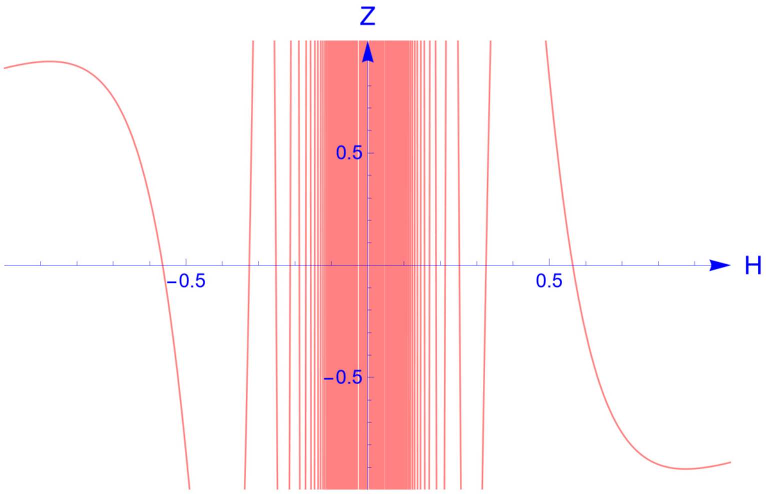

Figure 7. We see here the function $-\dfrac{2x\cos\left(\frac{1}{x^2+y^2}\right)}{x^2+y^2}$ restricted to $y = x$. We cannot observe the unbounded behaviour, but the figure indicates that this behaviour is possible. The function is in fact unbounded. We can create a sequence of points converging to $(0,0)$ where the cosine equals 1 and observe that the function values of these points go to infinity. But we see also that the function has in any neighbourhood an infinite amount of zeros. So an create a sequence of points converging to $(0,0)$ where the cosine equals 0. So the limit does obviously not exist.

We conclude that this function is locally unbounded. We do not repeat this argument for the partial derivative to y by symmetry considerations.

Because the two partial derivatives are unbounded in a neighbourhood of $(0,0)$, we do not have this particular alternative proof for the continuity.

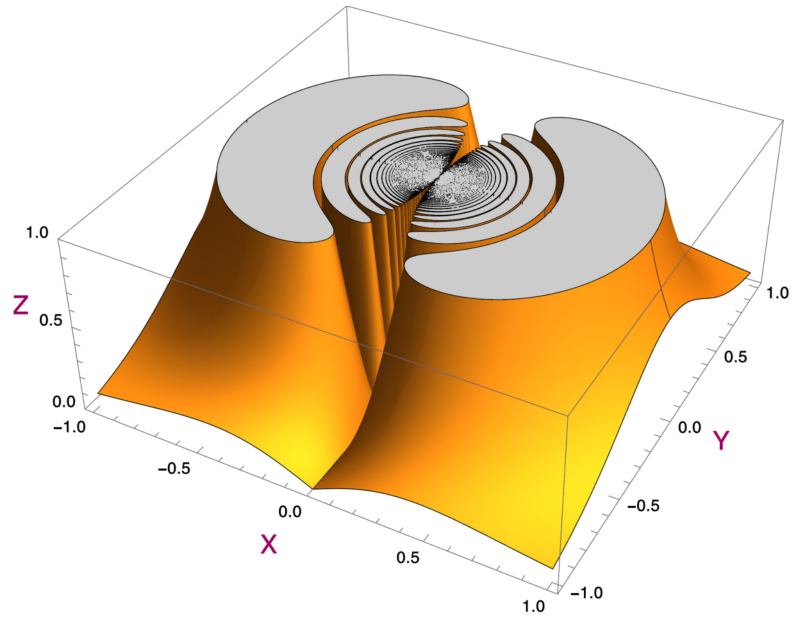

Figure 8. We see here the absolute value of the first partial derivative $\left|\dfrac{\partial f}{\partial x}\right|$. We can observe the unboundedness from this picture.

Because of the symmetry of the graph of this function, it is not necessary to give the plot of the absolute value of the partial derivative to y.

1.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks purely based upon the calculations we made. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

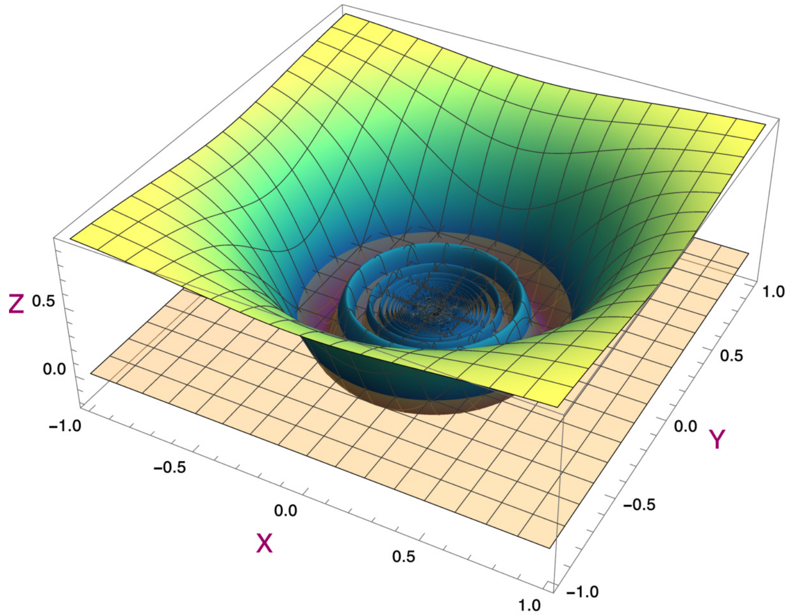

Figure 9. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function quite nicely but some caution is justified. We have still the oscillations.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

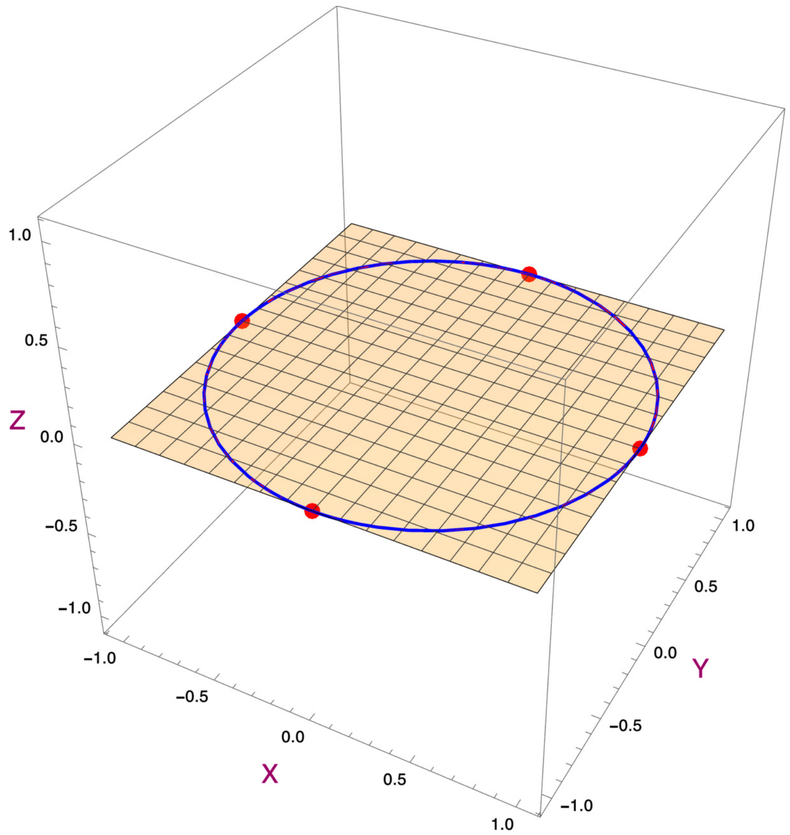

Figure 10. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We cannot see the circle of the unit vectors (u, v) in the X-Y plane because it is covered by the the blue circle consisting the vectors $(u, v, D_{(u,v)}(0,0))$. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We have now that all our calculations indicated that a tangent plane is plausible. We have still to be careful because of the rather strong oscillations.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h, k)$ in $(0, 0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}$$

where $(h,k) \neq (0,0)$. Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient!

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So in our case we have to prove that the function

$$q(h,k) = \begin{cases} \sqrt{h^2+k^2}\,\sin\left(\dfrac{1}{h^2+k^2}\right) & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left|\sqrt{h^2+k^2}\,\sin\left(\frac{1}{h^2+k^2}\right) - 0\right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\left|\sqrt{h^2+k^2}\,\sin\left(\frac{1}{h^2+k^2}\right)\right| \le \sqrt{h^2+k^2}.$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous. So the function $f(x,y)$ is differentiable.

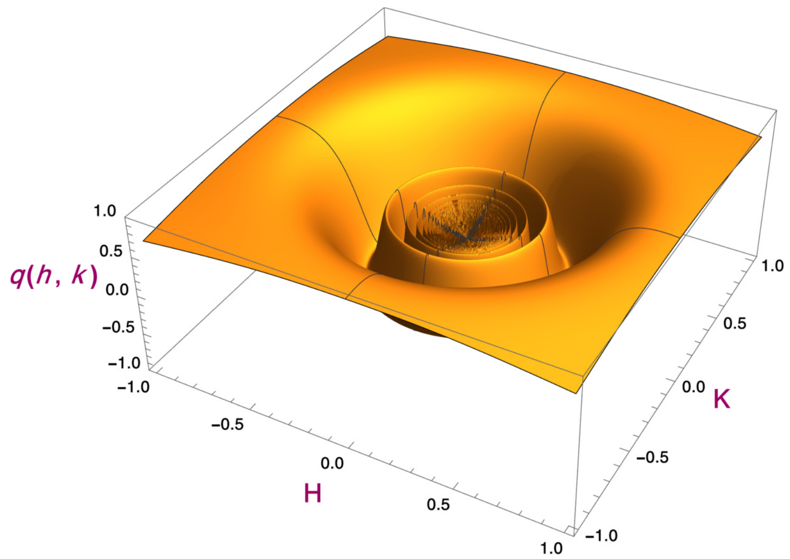

Figure 11. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

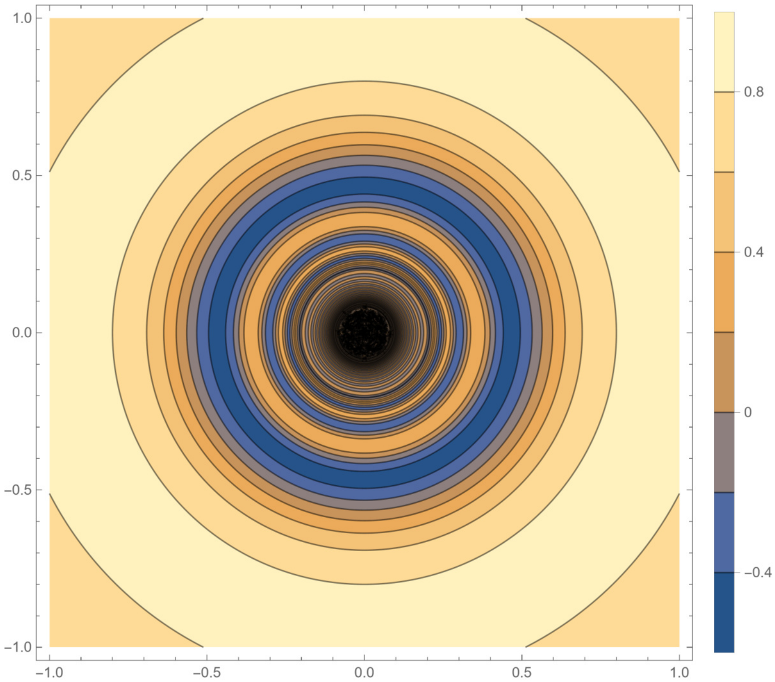

Figure 12. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

1.6 Alternative proof of the differentiability (optional)

This is not suited for a first reading or in the case that the student is not acquainted with Lipschitz continuity. Skip this section in that case and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\,u + \frac{\partial f}{\partial y}(0,0)\,v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the func-

tion $f(x,y)$ is locally Lipschitz continuous in $(0,0)$. We cite here the criterion that we will use.

A function is differentiable in (a,b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.
2. The directional derivatives have the following form in the point (a,b): $D_{(u,v)}(a,b) = \nabla f(a,b)\cdot(u,v) = \frac{\partial f}{\partial x}(a,b)\,u + \frac{\partial f}{\partial y}(a,b)\,v$.
3. The function is locally Lipschitz continuous in (a,b). This means that there exists at least one neighbourhood of (a,b) and a number $K > 0$ such that for all (x_1,y_1) and (x_2,y_2) in the neighbourhood, we have $|f(x_1,y_1) - f(x_2,y_2)| < K\,||(x_1,y_1) - (x_2,y_2)||$. Remark that the same K must be valid for all (x_1,y_1) and (x_2,y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very demanding condition but this is not unexpected because the differentiable function f must be "locally flat" and thus "locally linear" which is partly expressed by this Lipschitz continuity condition.

Let us try to prove that the Lipschitz condition does not hold. We have proven in section 1.4 that the first partial derivative is not bounded in any neighbourhood of $(0,0)$. This implies that the function is not Lipschitz continuous.

We conclude that an alternative proof following this criterion cannot exist.

1.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability. We know that if both the partial derivatives are continuous, then the function is differentiable. We have already calculated the partial derivatives, so it is not a lot of work to see if they are continuous or not.

If both the partial derivative derivatives are continuous, then we have an alternative proof that the derivative exists. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. We prove in this case more then the existence of the derivative. See the appendix for more information.

We have seen in section 1.4 that the function $\frac{\partial f}{\partial x}(x, y)$ is not bounded in any neighbourhood of $(0, 0)$. We have already proven that in section 1.4. But a continuous function is necessarily bounded in at least one neighbourhood. So the function $\frac{\partial f}{\partial x}(x, y)$ is not continuous.

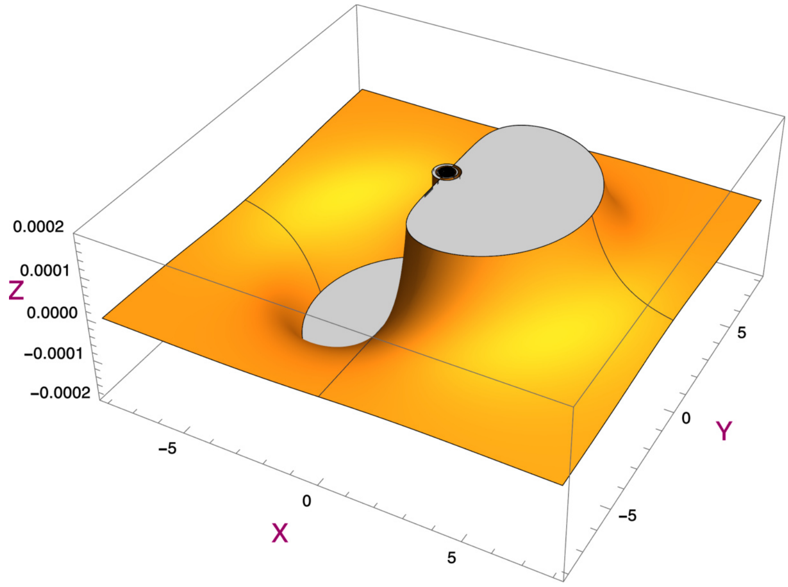

Figure 13. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial x}(x, y)$. This partial derivative is not bounded and not continuous if one approaches it on the X-axis. One can find another figure explaining this phenomenon below. Remark that the plot is in this case almost impossible and certainly not a careful representation.

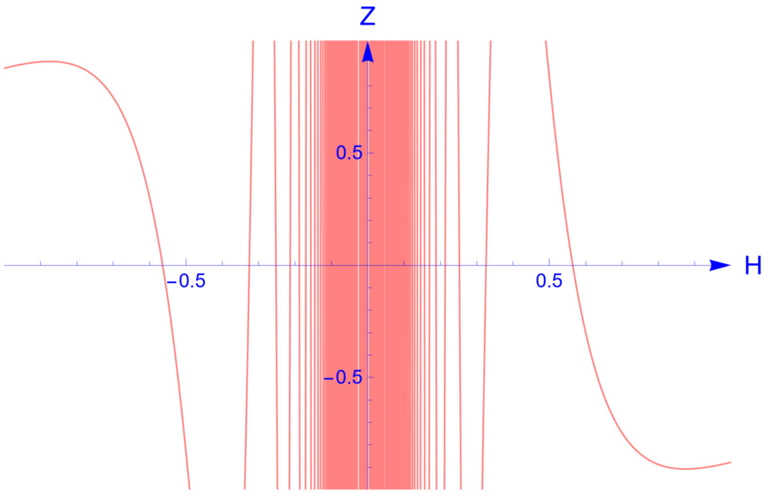

Figure 14. We see here a plot of the first partial derivative restricted to the X-axis. This function is quite pathological and clearly has no limit in $x = 0$. The meaning of this all says that the slope of the tangent line is changing in an incredible fast way and has no tendency to go smoothly to the tangent line in $x = 0$. We have drawn the function $f(h, 0)$.

1.8 Overview

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

Exercise 2.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability,
5. continuity of the partial derivatives

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

2.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{x^3 y}{x^4 + y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\left| \frac{x^3 y}{x^4 + y^2} \right| \leq |x| \frac{x^2 |y|}{x^4 + y^2}$$
$$\leq |x| \frac{1}{2}$$
$$\leq \frac{1}{2} \sqrt{x^2 + y^2}.$$

We have used the following reasoning in the inequalities. Because $0 \leq (a - b)^2$ we have $2\,a\,b \leq a^2 + b^2$. So $a\,b/(a^2 + b^2) \leq 1/2$ and by substituting $a = x^2$ en $b = |y|$, we have $x^2\,|y|/(x^4 + y^2) \leq 1/2$.

It is sufficient to take $\delta = 2\,\epsilon$. We can find a δ, so we conclude that the function is continuous.

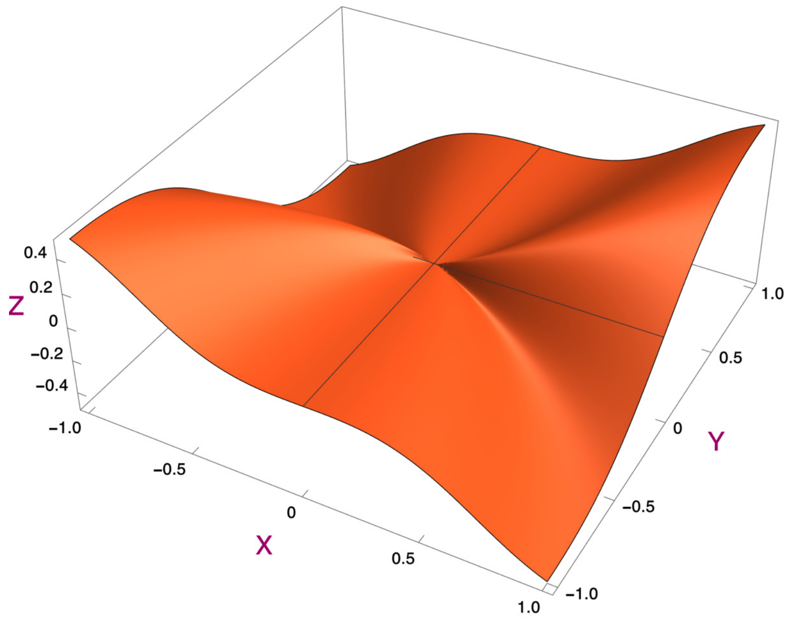

Figure 15. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

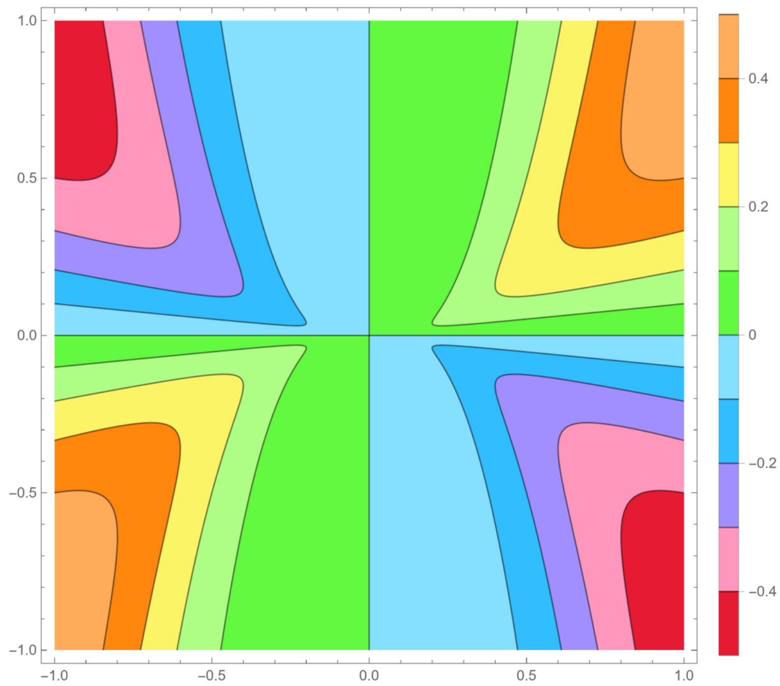

Figure 16. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

2.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to x exists.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

2.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(\mathbf{u},\mathbf{v})}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the limits of

$$\frac{f(0+hu,0+hv) - f(0,0)}{h} = \frac{hu^3v}{h^2u^4+v^2}.$$

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{h\,u^3\,v}{h^2\,u^4 + v^2} \\ &= 0. \end{aligned}$$

This limit is zero if $v \neq 0$, but we covered that exceptional case before.

So the directional derivatives do always exist.

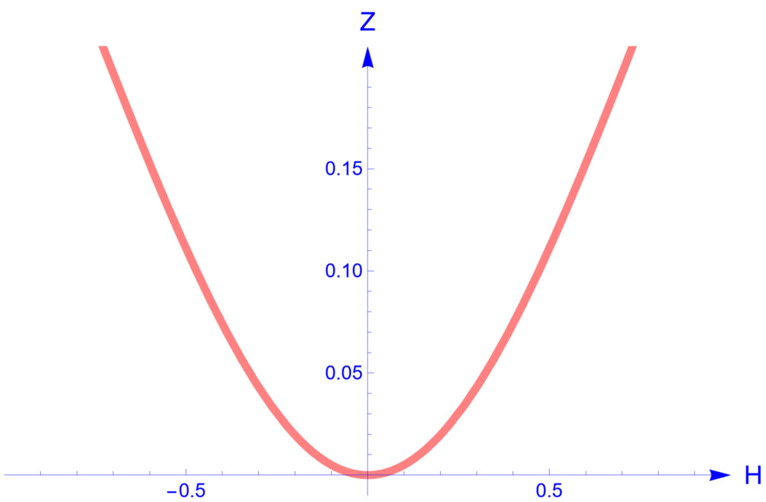

Figure 17. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn here the function $f(h\,u, h\,v)$.

2.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we

have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{y\left(3x^2y^2 - x^6\right)}{\left(x^4+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{x^3\left(x^4 - y^2\right)}{\left(x^4+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial y}\right|$ is unbounded. We have that if we restrict the function to $y = \lambda x^2$ the following

$$\frac{\partial f}{\partial y}(x,\lambda x^2) = \frac{x^3\,(x^4 - \lambda^2 x^4)}{(\lambda^2 x^4 + x^4)^2} = \frac{1-\lambda^2}{(\lambda^2+1)^2\,x}.$$

So we do have unbounded behaviour in any neighbourhood of $(0,0)$. Because the partial derivative to y is unbounded in a neighbourhood of $(0,0)$, we do not have this alternative proof for the continuity.

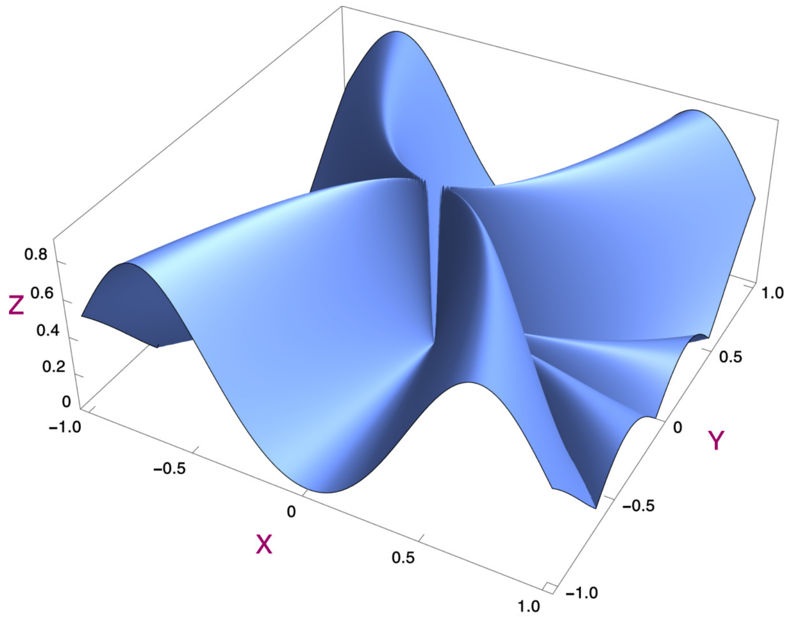

Figure 18. We see here the absolute value of the first partial derivative $\left|\frac{\partial f}{\partial x}\right|$. We can observe the probable boundedness from this picture. We have to take care however. Pictures can be devious. We did not investigate this further by calculation because of the unboundedness of the second partial derivative which we did check by calculation.

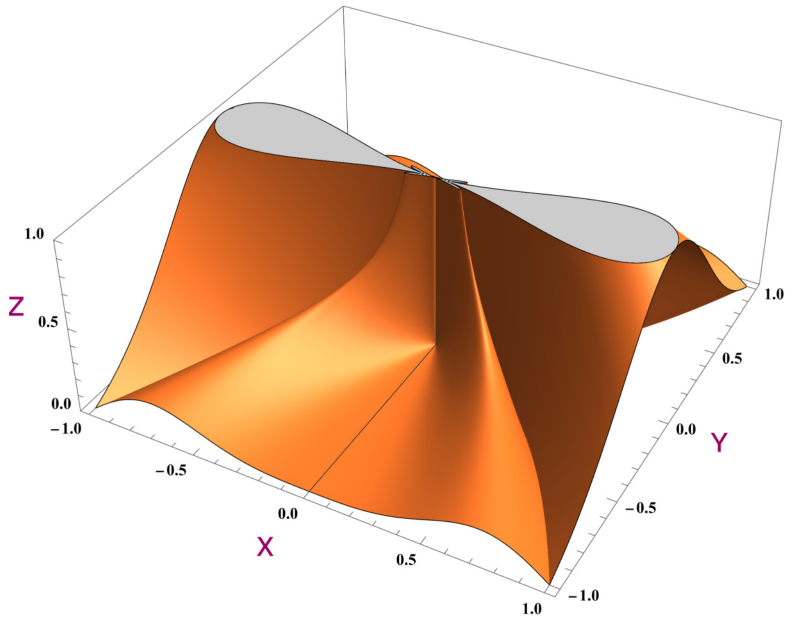

Figure 19. We see here the absolute value of the second partial derivative $\left|\frac{\partial f}{\partial y}\right|$. We can observe a possible unboundedness from this picture.

2.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

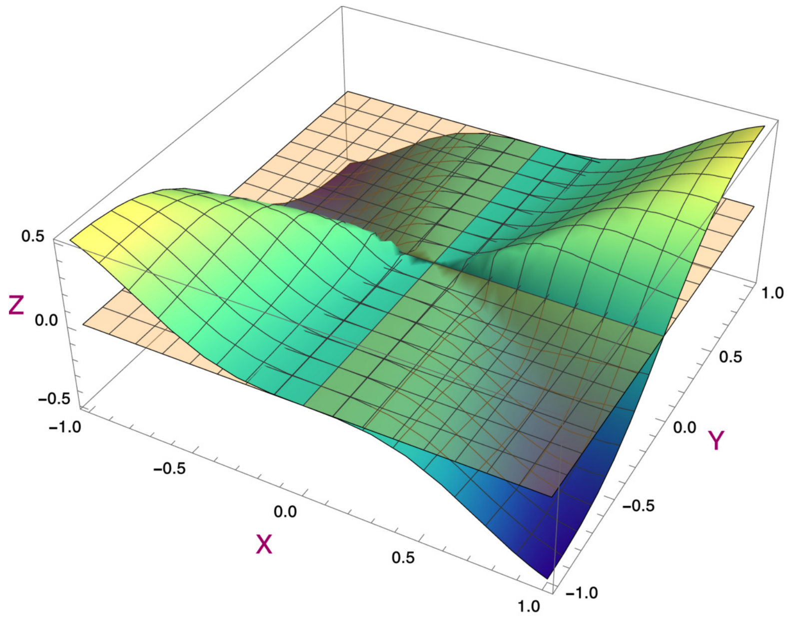

Figure 20. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane does not fit the function very nicely or that at least some question marks remain. It will be seen by further calculations that there is no tangent plane.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

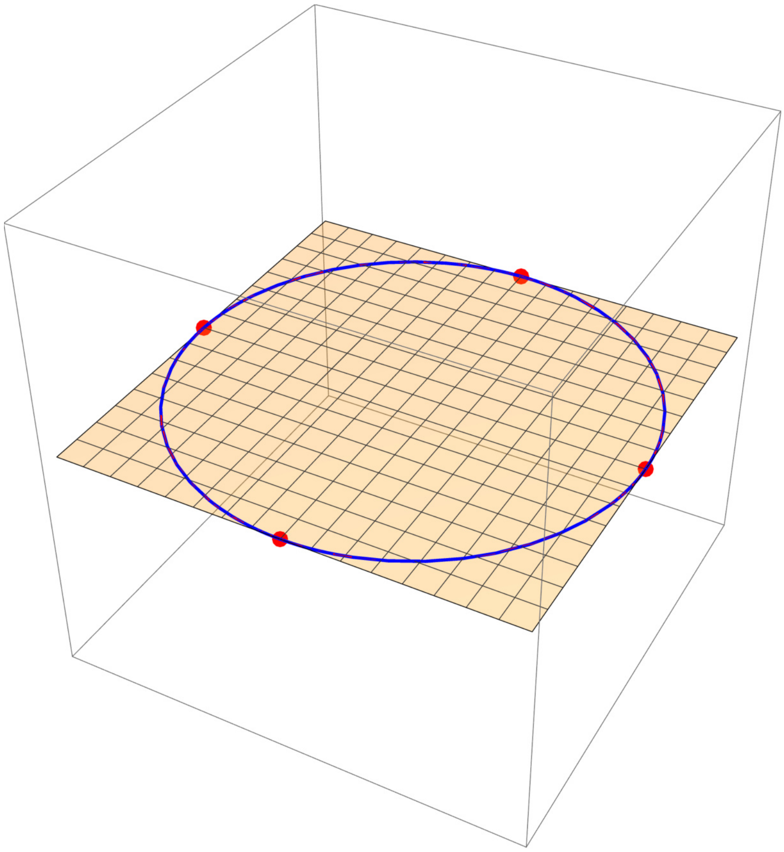

Figure 21. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient!

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h^3\,k}{\sqrt{h^2+k^2}\left(h^4+k^2\right)} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0). \end{cases}$$

is continuous in $(0,0)$.

We restrict the function $q(h,k)$ to the continuous curves with equations $k = \lambda\,h^2$. We observe then that

$$q\big|_{k=\lambda h^2}(h,k) = \begin{cases} q(h,\lambda\,h^2) = \dfrac{h\lambda}{\left(\lambda^2+1\right)\sqrt{h^4\lambda^2+h^2}} & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

We calculate the limit.

$$\lim_{h\to 0} \frac{h\,\lambda}{\left(\lambda^2+1\right)\sqrt{h^4\,\lambda^2+h^2}} = \lim_{h\to 0} \frac{\operatorname{sgn}(h)\,\lambda}{(1+\lambda^2)\sqrt{1+h^2\,\lambda^2}}.$$

We see that these restricted functions have no limits if $\lambda \neq 0$. But if $q(h,k)$ is continuous, all these limit values should be $q(0,0) = 0$. So this function $q(h,k)$ is not continuous in $(0,0)$. The function $f(x,y)$ is not differentiable in $(0,0)$.

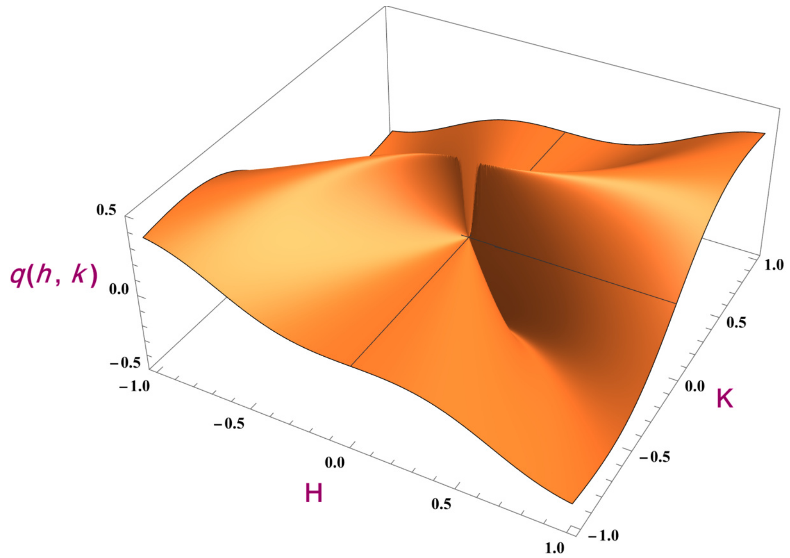

Figure 22. We see here a three dimensional figure of the graph of the function $q(h,k)$. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

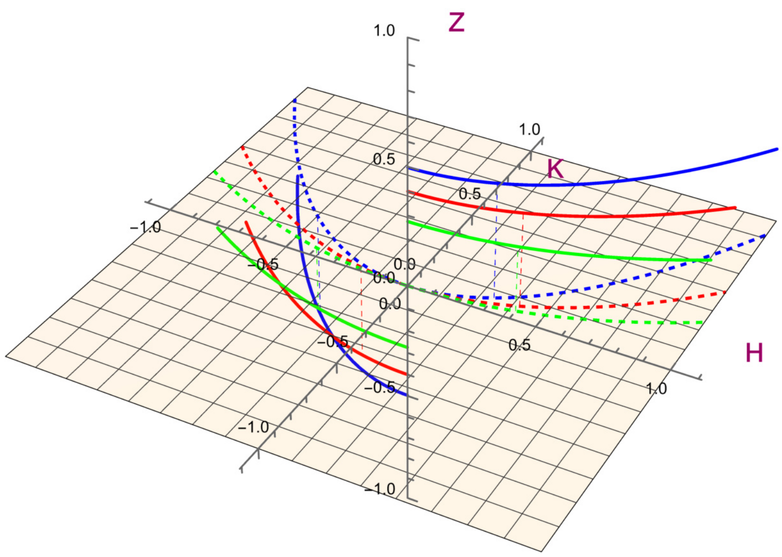

Figure 23. We have restricted the function $q(h,k)$ here to $k = 1/2\,h$ and $k = 3/10\,h$ and $k = 9/10\,h$. We see in this figure clearly that the restrictions of the function to these lines are functions that have no limits in 0.

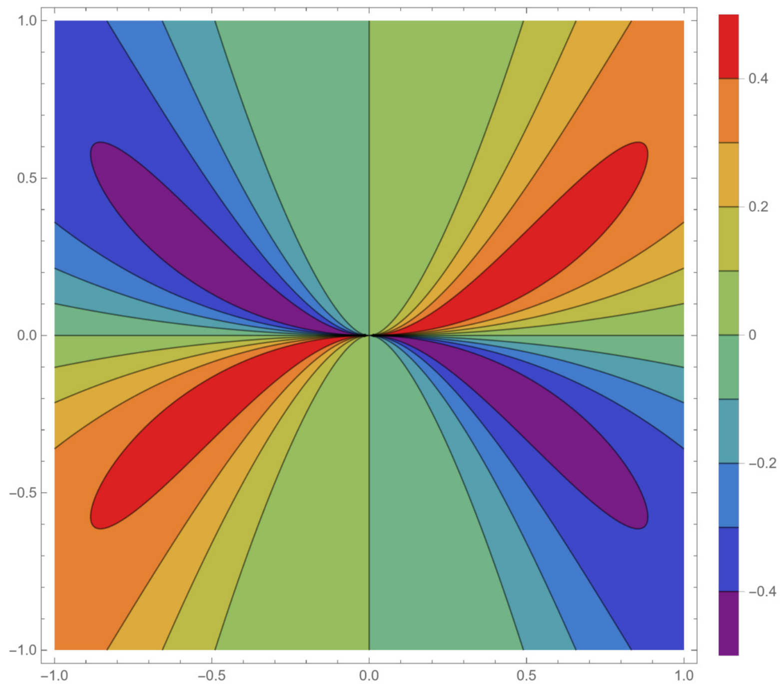

Figure 24. We see here a figure of the contour plot of the function $q(h,k)$. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

2.6 Alternative proof of differentiability (optional)

This section is **irrelevant** for this exercise, because the function is not differentiable

2.7 Continuity of the partial derivatives

Irrelevant.

2.8 Overview

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

2.9 One step further

We have used in the calculations for differentiability that we had some magical curves $k = \lambda h^2$ which behaved very strangely when mapped by $q(h,k)$ to the Z-direction. We want to see what is going on with these curves. Let us define the 3-dimensional curve in parametric form that projects in the (h,k)-plane to our curve $k = \lambda h^2$: $(x(t), y(t), z(t)) = (t, \lambda t^2, f(t, \lambda t^2)) = \left(t, \lambda t^2, \frac{\lambda t}{\lambda^2+1}\right)$.

Remark that this curve is differentiable because that is not obvious by the non differentiability of the function f. This curve lies completely in the surface defined by the function. It is clear that the tangent vector lies in the tangent plane if the function is differentiable. Now we have a candidate tangent plane, we draw that and see what is going on with the tangent vector $(x'(t), y'(t), z'(t)) = \left(1, 2\lambda t, \frac{\lambda}{\lambda^2+1}\right)$. We also know that the candidate tangent plane is the only possible tangent plane because it takes care of a good fit in the X-direction and the Y-direction, which is the absolute minimum that a tangent plane must do. So if $t = 0$, then we have the tangent vector $\left(1, 0, \frac{\lambda}{\lambda^2+1}\right)$. If $\lambda = 1$, this leaves us with the tangent vector $\left(1, 0, \frac{1}{2}\right)$. We will see that this vector does not lie in the candidate tangent plane. So we see that the candidate tangent plane is not a real tangent plane. The function is not differentiable. Please consult the figure of this situation.

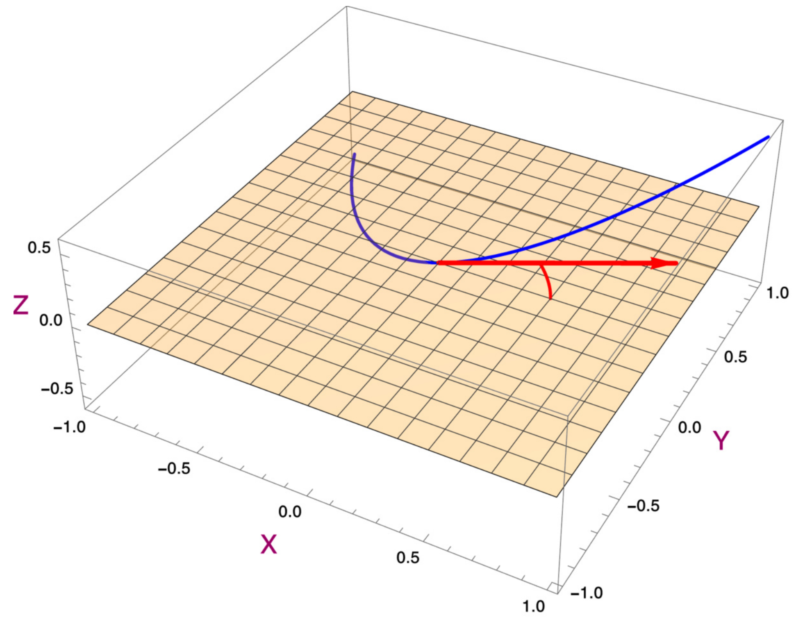

Figure 25. The tangent vector is not in the only possible candidate tangent plane.

Exercise 3.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability,
5. continuity of the partial derivatives

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{3x^2y - y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

3.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{3x^2y - y^3}{x^2+y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\left|\frac{3x^2y - y^3}{x^2 + y^2}\right| \le \frac{3x^2|y| + |y|^3}{x^2 + y^2}$$

$$\le \frac{3\sqrt{x^2 + y^2}^2\sqrt{x^2 + y^2} + \sqrt{x^2 + y^2}^3}{\sqrt{x^2 + y^2}^2}$$

$$\le \frac{4\sqrt{x^2 + y^2}^3}{\sqrt{x^2 + y^2}^2}$$

$$\le 4\sqrt{x^2 + y^2}.$$

It is sufficient to take $\delta = \epsilon/4$. We can find a δ, so we conclude that the function is continuous.

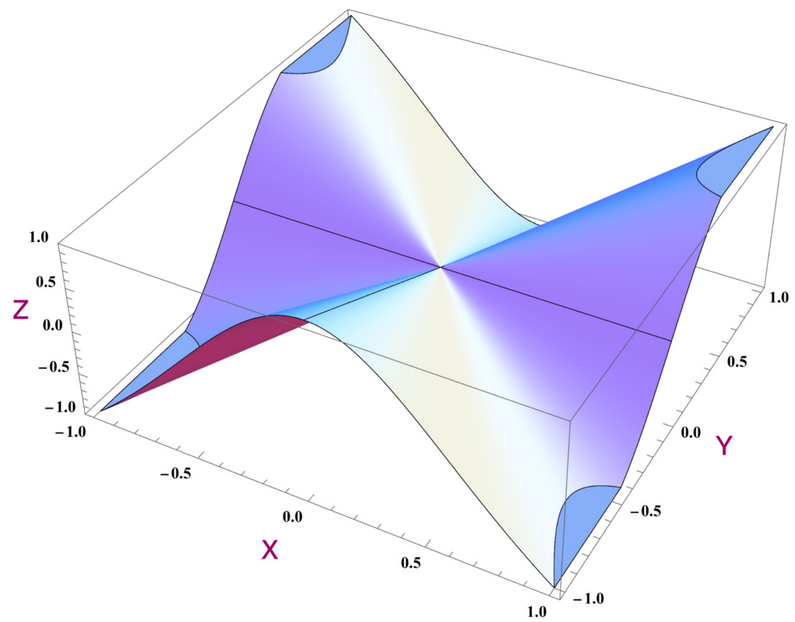

Figure 26. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

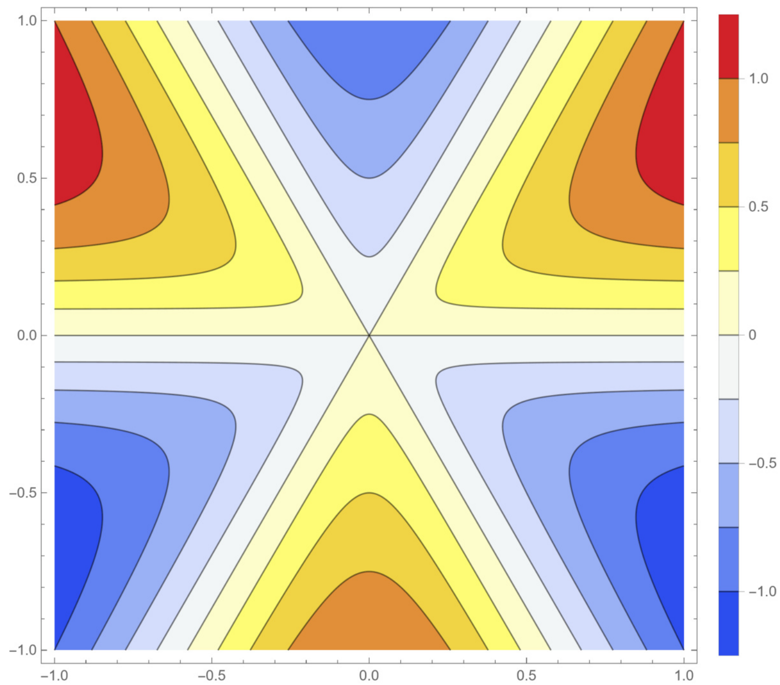

Figure 27. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

3.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).
We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h \to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = -y & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} \\ &= \lim_{h \to 0} -1 \\ &= -1. \end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = -1.$$

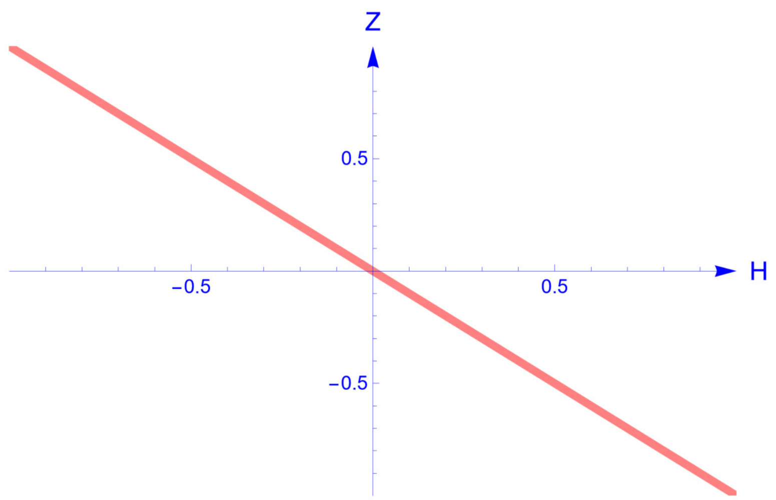

Figure 28. We see here a figure of the graph of the function restricted to the vertical Y-axis through $(0,0)$. We have drawn here the function $f(0,h)$.

3.3 Directional derivatives

Let (u, v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2 + v^2} = 1$. Then the directional derivative in $(0, 0)$ in the direction (u, v), notation $D_{(\mathbf{u},\mathbf{v})}(0, 0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0 + h\,u, 0 + h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1, 0)$ and $(0, 1)$.

In the case of our function we have to calculate

$$\begin{aligned}\lim_{h\to 0} \frac{f(0 + h\,u, 0 + h\,v) - f(0,0)}{h} &= \lim_{h\to 0} \frac{3\,h^3\,u^2\,v - h^3\,v^3}{h\,(h^2\,u^2 + h^2\,v^2)} \\ &= -\frac{v\,\left(v^2 - 3\,u^2\right)}{u^2 + v^2}.\end{aligned}$$

$$\begin{aligned}D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0 + h\,u, 0 + h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} -v\,\left(v^2 - 3\,u^2\right) \\ &= -v\,\left(v^2 - 3\,u^2\right).\end{aligned}$$

We have used that (u, v) is a normal vector and so $u^2 + v^2 = 1$.

So the directional derivatives do always exist.

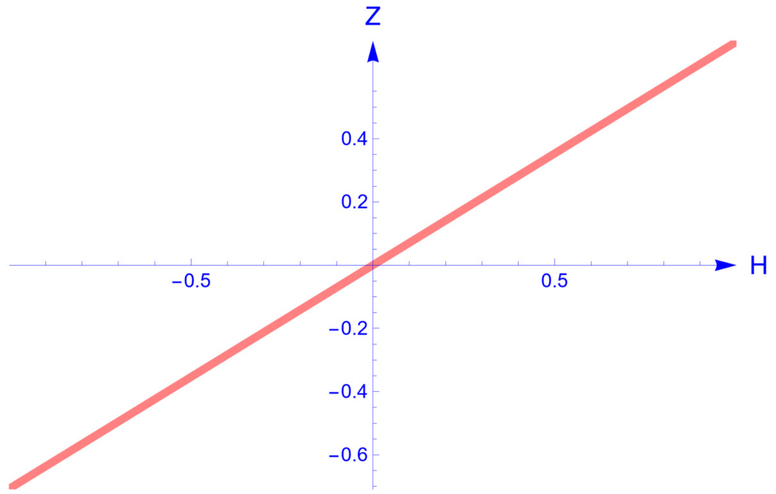

Figure 29. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = (1/\sqrt{2}, 1/\sqrt{2})$. We have drawn here the function $f(hu, hv)$.

3.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{8xy^3}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} -\dfrac{-3x^4+6x^2y^2+y^4}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ -1 & \text{if } (x,y) = (0,0). \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

$$\begin{aligned}\left|\frac{\partial f}{\partial x}\right| &\leq \frac{8\,|x|\,|y|^3}{(x^2+y^2)^2}\\ &\leq \frac{8\sqrt{x^2+y^2}\sqrt{x^2+y^2}^{\,3}}{\sqrt{x^2+y^2}^{\,4}}\\ &\leq 8.\end{aligned}$$

Let us try to prove that $\left|\frac{\partial f}{\partial y}\right|$ is bounded.

$$\begin{aligned}\left|\frac{\partial f}{\partial y}\right| &\leq \left|\frac{-3\,x^4+6\,x^2\,y^2+y^4}{(x^2+y^2)^2}\right|\\ &\leq \frac{3\sqrt{x^2+y^2}^{\,4}+6\sqrt{x^2+y^2}^{\,2}\sqrt{x^2+y^2}^{\,2}+\sqrt{x^2+y^2}^{\,4}}{\left(\sqrt{x^2+y^2}\right)^4}\\ &\leq \frac{10\sqrt{x^2+y^2}^{\,4}}{\sqrt{x^2+y^2}^{\,4}}\\ &\leq 10.\end{aligned}$$

We have chosen here the restriction $\sqrt{x^2+y^2} < 1$.

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

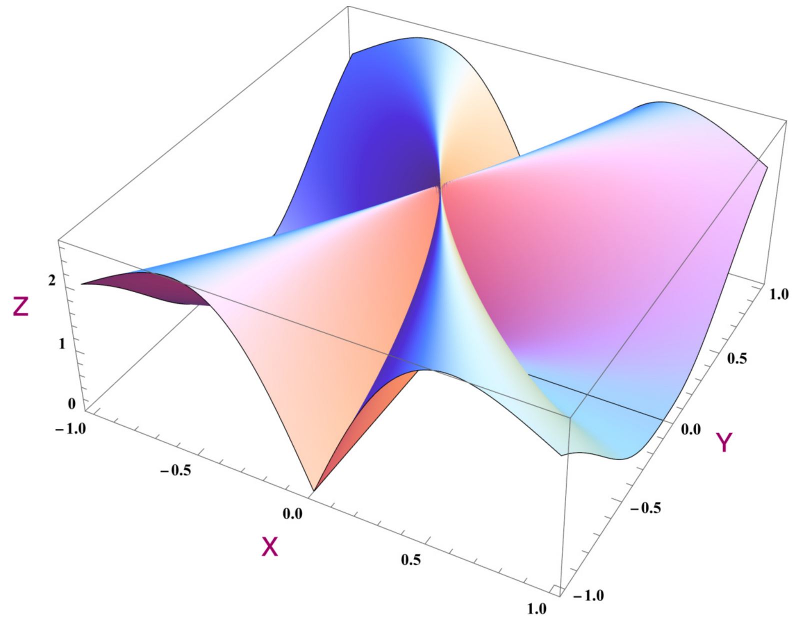

Figure 30. We see here the absolute value of the first partial derivative $\left|\frac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

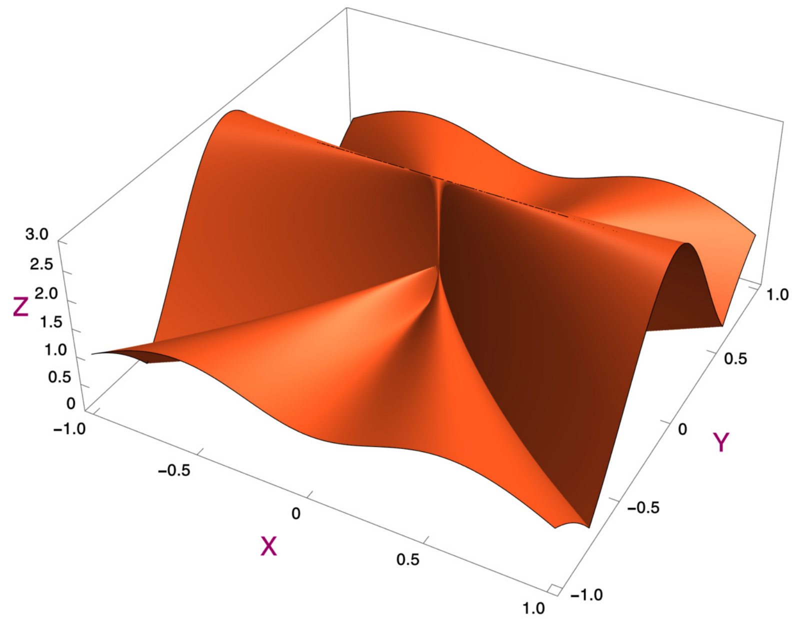

Figure 31. We see here the absolute value of the second partial derivative $\left|\frac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture.

3.5 Differentiability

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

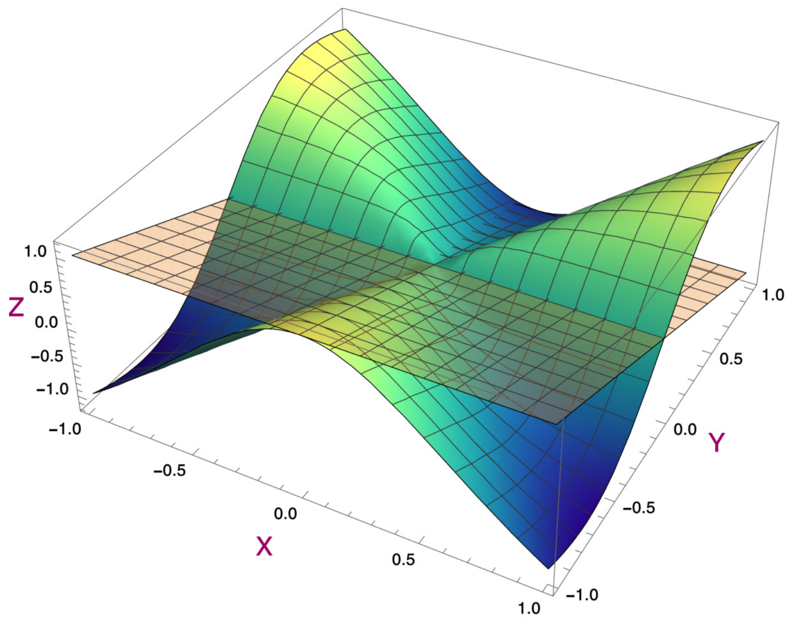

Figure 32. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane does not fit the function very nicely. It is indeed no tangent plane following our future calculations.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

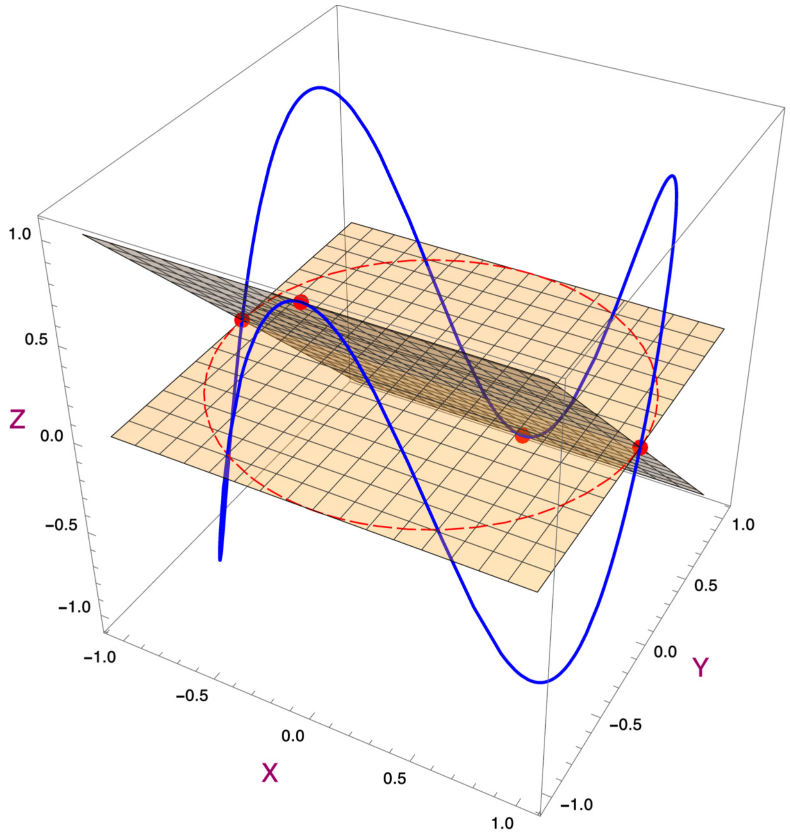

Figure 33. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue curve the vectors $(u, v, D_{(u,v)}(0,0))$. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ do not sweep out a nice ellipse lying in the candidate tangent plane! Only four points are in the candidate tangent plane. On this visual basis, we conclude that the function is not differentiable, but we stated that we want to give an alphabetical proof following the definition.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient!

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{4\,h^2\,k}{\left(h^2+k^2\right)^{3/2}} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We restrict the function $q(h,k)$ to the continuous curves with equations $k = \lambda\,h$. We observe then that

$$q\big|_{k=\lambda h}(h,k) = \begin{cases} \dfrac{4\,h^3\,\lambda}{\left(h^2\,\lambda^2+h^2\right)^{3/2}} = \dfrac{4\,h^3\,\lambda}{\left(\lambda^2+1\right)^{3/2}\ |h|^3} & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

We see that these restricted functions have no limits. But if $q(h,k)$ is continuous, all these limit values should be $q(0,0) = 0$. So this function $q(h,k)$ is not continuous in $(0,0)$. The function $f(x,y)$ is not differentiable in $(0,0)$.

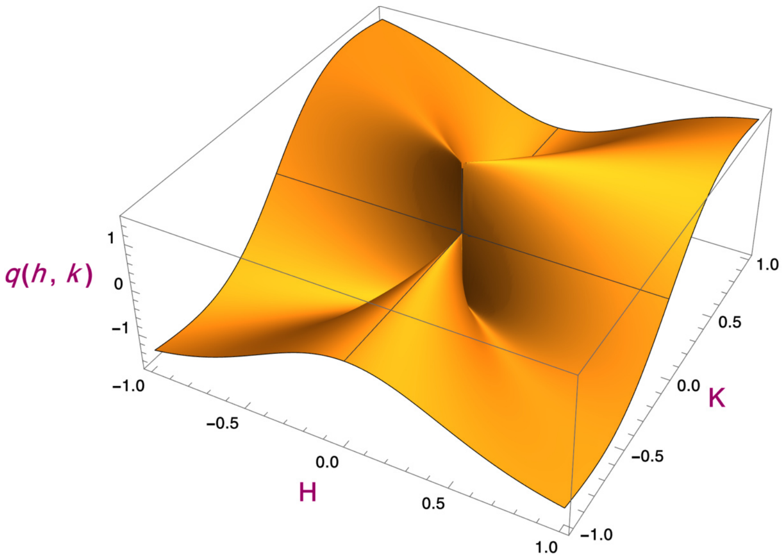

Figure 34. We see here a three dimensional figure of the graph of the function $q(h,k)$. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

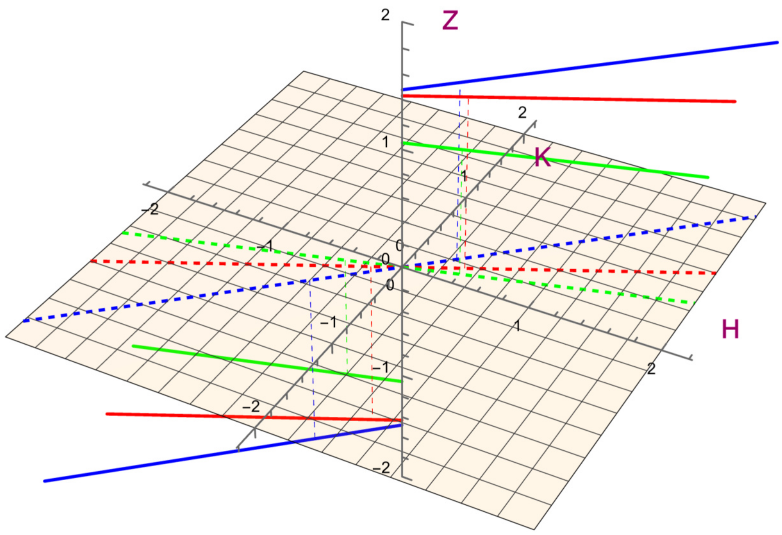

Figure 35. We have restricted the function $q(h,k)$ here to $k = 1/2\,h$ and $k = 3/10\,h$ and $k = 9/10\,h$. We see in this figure clearly that the restrictions of the function to these lines are functions that have no limits in 0.

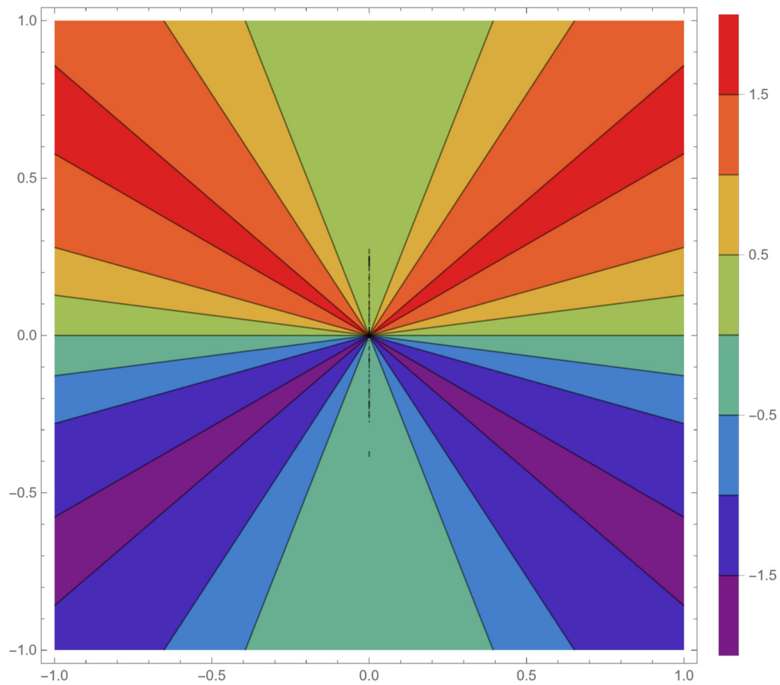

Figure 36. We see here a figure of the contour plot of the function $q(h,k)$. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Alternative proof for the non differentiability

Suppose that we already met in the course the differentiation rule of the composition of two differentiable functions. This is also called the chain rule. Then we have proven the following. If the function is differentiable in (a,b), then the directional derivative can be calculated is follows.

$$D_{(u,v)}f(a,b) = \frac{\partial f}{\partial x}(a,b)\,u + \frac{\partial f}{\partial y}(a,b)\,v\,.$$

Important remark. This formula is only valid if the function is differentiable. One of the most common mistakes is that one uses this formula in the case of non differentiability. It seems to be easy to calculate quickly the partial derivatives if they exist and then use this formula.

We have calculated the directional derivatives and we saw that

$$D_{u,v}f(0,0) = -v\,\left(v^2 - 3\,u^2\right)$$

and this is certainly not the linear function in u and v which we should have in the case of differentiability. So we conclude again with this alternative proof that the function is not differentiable.

3.6 Alternative proof of differentiability (optional)

This is irrelevant because the function is not differentiable.

3.7 Continuity of the partial derivatives

This is irrelevant because the function is not differentiable.

3.8 Overview

$$f(x,y) = \begin{cases} \dfrac{3x^2y - y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

3.9 One step further

We have used in the calculations for differentiability that we had some magical curves $k = \lambda h$ which behaved very strangely when mapped by $q(h,k)$ to the Z-direction. We want to see what is going on with these curves. Let us define the 3-dimensional curve in parametric form that projects in the (h,k)-plane to our curve $k = \lambda h$: $(x(t), y(t), z(t)) = (t, \lambda t, f(t, \lambda t)) = \left(t, \lambda t, -\frac{\lambda(\lambda^2-3)t}{\lambda^2+1}\right)$.

This curve lies completely in the surface defined by the function. It is clear that the tangent vector lies in the tangent plane if the function is differentiable. Now we had a candidate tangent plane, we draw that and see what is going on with the tangent vector $(x'(t), y'(t), z'(t)) = \left(1, \lambda, -\frac{\lambda(\lambda^2-3)}{\lambda^2+1}\right)$.

We also know that the candidate tangent plane is the only possible tangent plane because it takes care of a good fit in the X-direction and the Y-direction, which is the absolute minimum that a tangent plane must do. So if $t = 0$, then we have the tangent vector $\left(1, \lambda, -\frac{\lambda(\lambda^2-3)}{\lambda^2+1}\right)$.

If $\lambda = 1$, this leaves us with the tangent vector $(1, 1, 1)$. We will see that this vector does not lie in the tangent plane. So we see once again that the candidate tangent plane is not a real tangent plane. Please consult the figure of this situation.

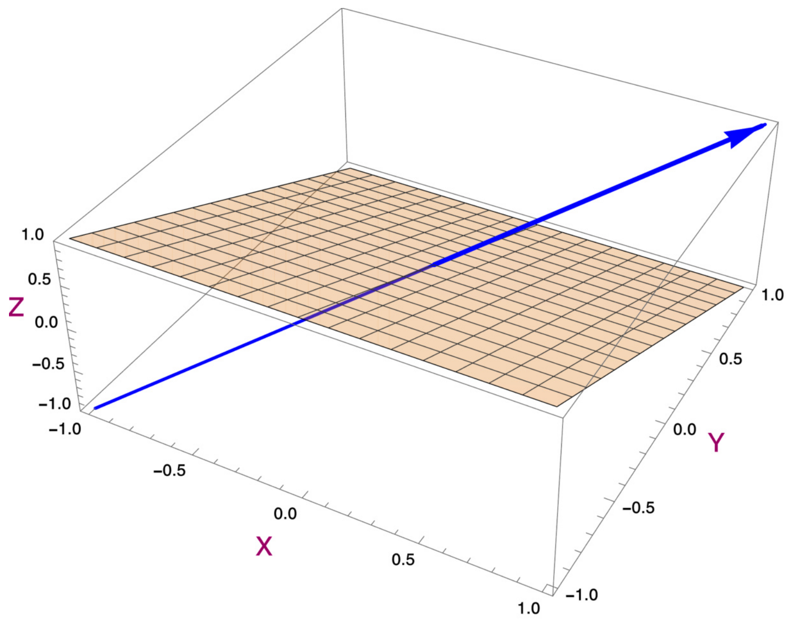

Figure 37. The tangent vector is not in the only possible candidate tangent plane. Because the curve is in this example a line, we have that this tangent vector is on the line itself. This tangent vector intersects the plane transversally.

It can maybe be worthwhile to take a look at the gradient vector field. We see that this gradient vector field exists in this case. But it is not continuous because in the case that the gradient vector field is continuous, then the function is differentiable. So we wonder if we can find an indication for this fact.

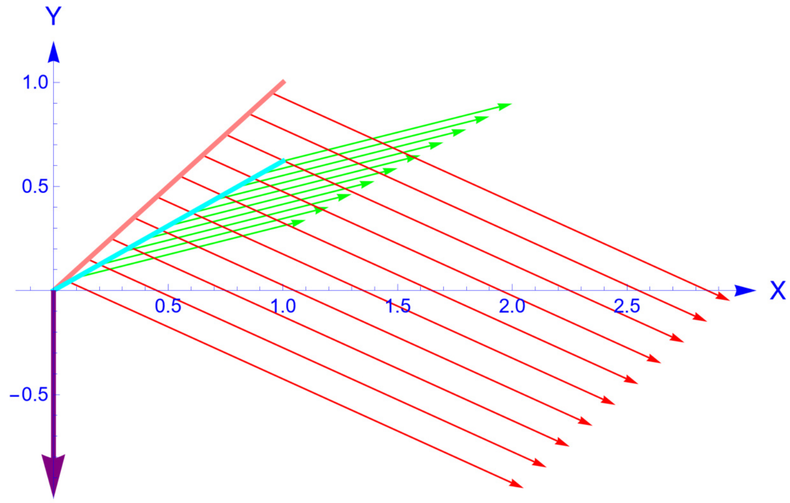

Figure 38. We made here the following sketch. We have drawn the graphics of $y = x$ in pink and $y = 0.62\,x$ in cyan. We have sketched the gradient vector field $\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$ which is not continuous as can be seen as follows. Observe the gradient vector field on the pink curve, these are the red vectors. Observe the gradient vector field on the cyan curve, these are the green vectors. The purple vector is the gradient vector in $(0,0)$. The red vectors converge to a vector with a non zero x-component. This component is equal to 2. The green vectors converge to a vector that has a x-component that is two times smaller then the x-component of the vector to which the red vector field converges if $x \to 0$. This is clearly impossible if the vector field is continuous. Moreover, the x-component of the limit vector should in both cases be zero if the gradient vector field is continuous. We conclude that this gradient vector field is not continuous. The function is differentiable in that case, which it is not. Please note however that a sketch of the gradient vector field is inherently a sketch of discrete data. So the utmost care must be taken in order to make it a little bit trustworthy.

Exercise 4.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{y^3 - x^8 y}{x^6 + y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

4.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{y^3 - x^8 y}{x^6 + y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$
\begin{aligned}
\left|\frac{y^3 - x^8 y}{x^6 + y^2}\right| &\le \frac{|y|^3 + x^8|y|}{x^6 + y^2} \\
&\le \frac{|y|^3 + (|x|^3)^{8/3}|y|}{(x^3)^2 + y^2} \\
&\le \frac{\sqrt{(|x|^3)^2 + y^2}^{\,3} + \sqrt{(|x|^3)^2 + y^2}^{\,8/3}\sqrt{(|x|^3)^2 + y^2}}{(x^3)^2 + y^2} \\
&\le \sqrt{(|x|^3)^2 + y^2} + \sqrt{(|x|^3)^2 + y^2}^{\,5/3} \\
&\le \sqrt{(|x|^3)^2 + y^2}\left(1 + \sqrt{(|x|^3)^2 + y^2}^{\,2/3}\right) \\
&\le 2\sqrt{(|x|^3)^2 + y^2}.
\end{aligned}
$$

We have that $|x|^3 \le \sqrt{(|x|^3)^2 + y^2}$ or $|x| \le \sqrt{(|x|^3)^2 + y^2}^{\,1/3}$ and that $|y| \le \sqrt{(|x|^3)^2 + y^2}$. We have applied these inequalities in the second inequality. Then we have chosen the restriction that $\sqrt{(|x|^3)^2 + y^2} < 1$ so that the last step is justified. In order to do that, we take $(|x|^3)^2 < 1/\sqrt{2}$ and $y^2 < 1/\sqrt{2}$. This is satisfied if we keep $|x| < \sqrt[6]{1/\sqrt{2}}$ and $|y| < \sqrt[2]{1/\sqrt{2}}$.
Then we choose $(|x|^3)^2 < \epsilon^2/8$ and $y^2 < \epsilon^2/8$. This means that we have now the open square defined by

$$
\left\{(x, y) \text{ such that } |x|, |y| < \min\left\{\sqrt[6]{1/\sqrt{2}},\ \sqrt[2]{1/\sqrt{2}},\ \sqrt[6]{\epsilon^2/8},\ \sqrt[2]{\epsilon^2/8}\right\}\right\}.
$$

We have now found an open square. In order to have an open disk, we can take as radius δ half the length of an edge of the square. We can find a δ, so we conclude that the function is continuous.

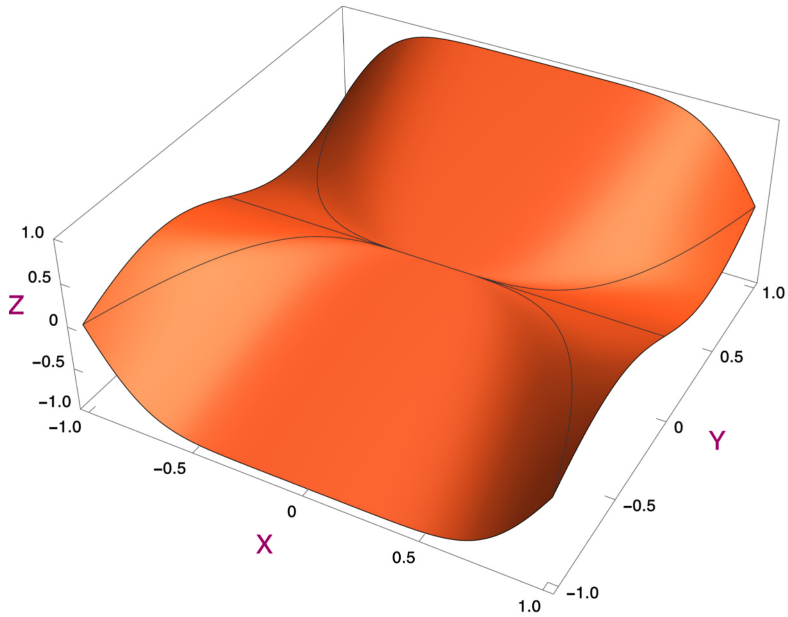

Figure 39. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

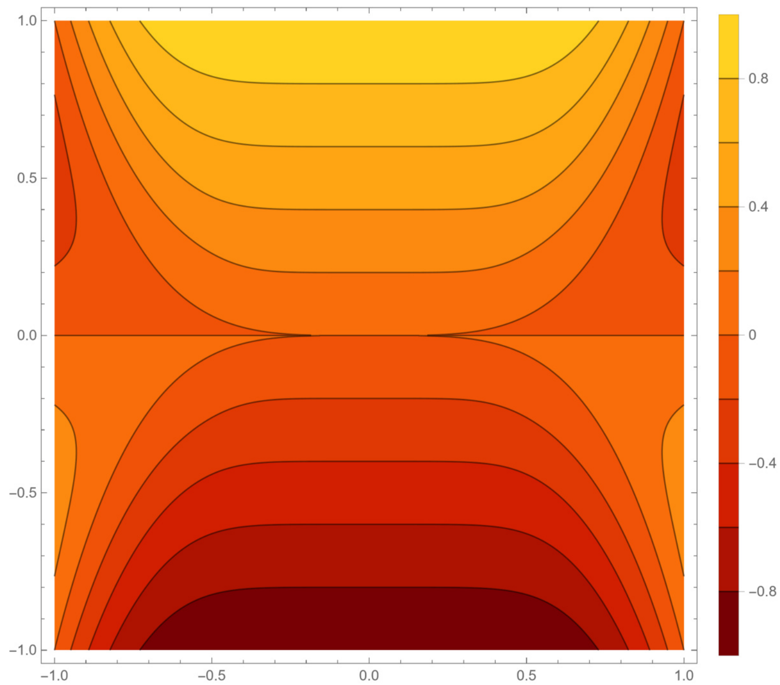

Figure 40. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

4.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).
We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = y & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= 1. \end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 1.$$

4.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{v^3 - h^6\,u^8\,v}{h^4\,u^6 + v^2} \\ &= v. \end{aligned}$$

The calculation is only valid for $v \neq 0$ but we covered that case before. So the directional derivatives do always exist.

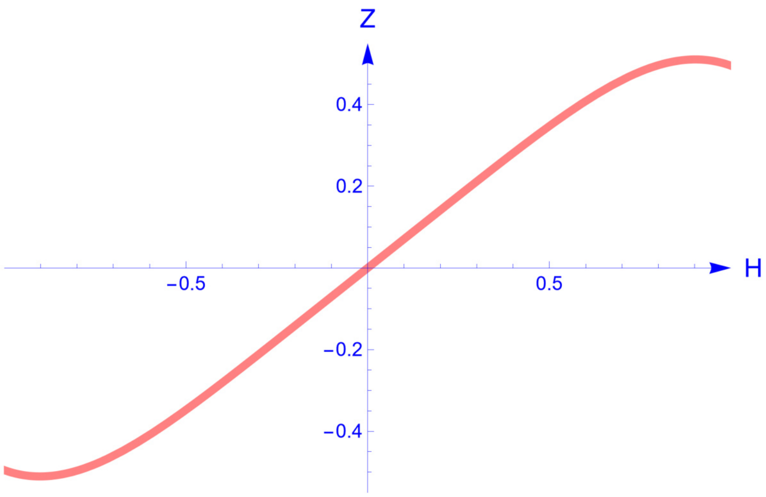

Figure 41. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

4.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} -2\,\dfrac{x^5\,y\,\left(x^8+4\,y^2\,x^2+3\,y^2\right)}{\left(x^6+y^2\right)^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0). \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{-x^{14}+x^8\,y^2+3\,x^6\,y^2+y^4}{\left(x^6+y^2\right)^2} & \text{if } (x,y)\neq(0,0),\\ 1 & \text{if } (x,y)=(0,0). \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded. From the third inequality on, we will write for space reasons $A=\sqrt{(|x|^3)^2+y^2}$.

$$\begin{aligned}
\left|\frac{\partial f}{\partial x}\right| &\leq \left|-\frac{2\,x^5\,y\,\left(x^8+4\,x^2\,y^2+3\,y^2\right)}{\left(x^6+y^2\right)^2}\right| \\
&\leq \left|\frac{2\,x^2\,\left(x^8+4\,x^2\,y^2+3\,y^2\right)}{2\,\left(x^6+y^2\right)}\right| \\
&\leq \left|\frac{A^{2/3}\,\left(A^{8/3}+4\,A^{2/3}\,A^2+3\,A^2\right)}{A^2}\right| \\
&\leq \left(5A^{2/3}+3\right)A^{2/3} \\
&\leq 8.
\end{aligned}$$

We have used the fact that $0\leq(a-b)^2$ or that $2\,a\,b\leq a^2+b^2$. So we have that $2\,a\,b\leq(a^2+b^2)$ or that $a\,b/(a^2+b^2)\leq 1/2$. We applied this in the first inequality by substituting $a=|x|^3$ and $b=|y|$. We have thus that $|x|^3\,|y|/(x^6+y^2)\leq 1/2$ and applied it in the first step.

We have also that $|x|^3 \leq \sqrt{(|x|^3)^2 + y^2}$ or $|x| \leq \sqrt{(|x|^3)^2 + y^2}^{1/3}$ and that $|y| \leq \sqrt{(|x|^3)^2 + y^2}$. We have applied these inequalities in the second step. From the third line on we use the abbreviation $A = \sqrt{(|x|^3)^2 + y^2}$. We choose then $A = \sqrt{(|x|^3)^2 + y^2} < 1$. This is certainly the case if $(|x|^3)^2 < 1/\sqrt{2}$ and $y^2 < 1/\sqrt{2}$. So we have that we can take $|x| < \sqrt[6]{1/\sqrt{2}}$ and $|y| < \sqrt[2]{1/\sqrt{2}}$. So in this rectangular open neighbourhood of $(0,0)$ we have the bound 8.

Let us try to prove that $\left|\frac{\partial f}{\partial y}\right|$ is bounded. Consult the previous inequalities for definitions and explanations.

$$\begin{aligned}
\left|\frac{\partial f}{\partial y}(x,y)\right| &\leq \left|\frac{-x^{14} + x^8 y^2 + 3x^6 y^2 + y^4}{(x^6 + y^2)^2}\right| \\
&\leq \frac{A^{14/3} + A^{8/3} A^2 + 3A^{6/3} A^2 + A^4}{A^4} \\
&\leq 2\left(A^{2/3} + 2\right) \\
&\leq 6.
\end{aligned}$$

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

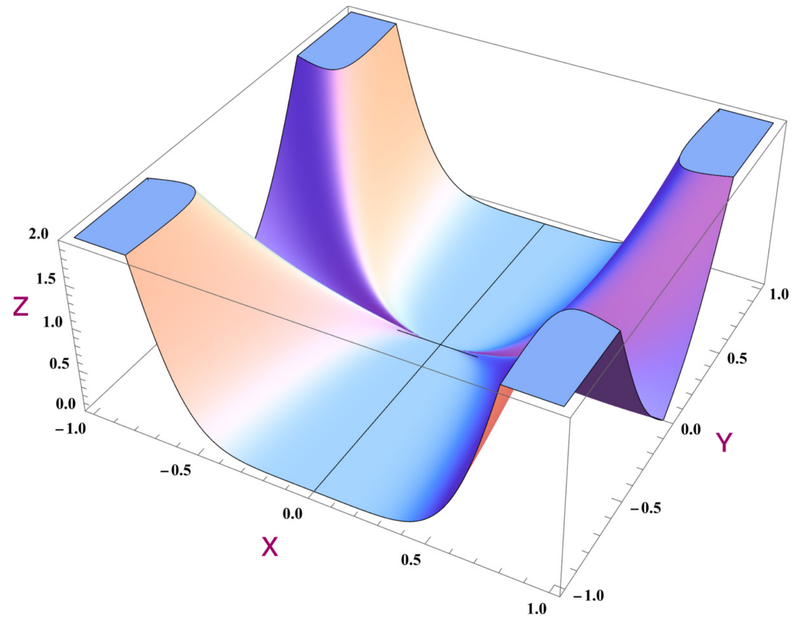

Figure 42. We see here the absolute value of the first partial derivative $\left|\frac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

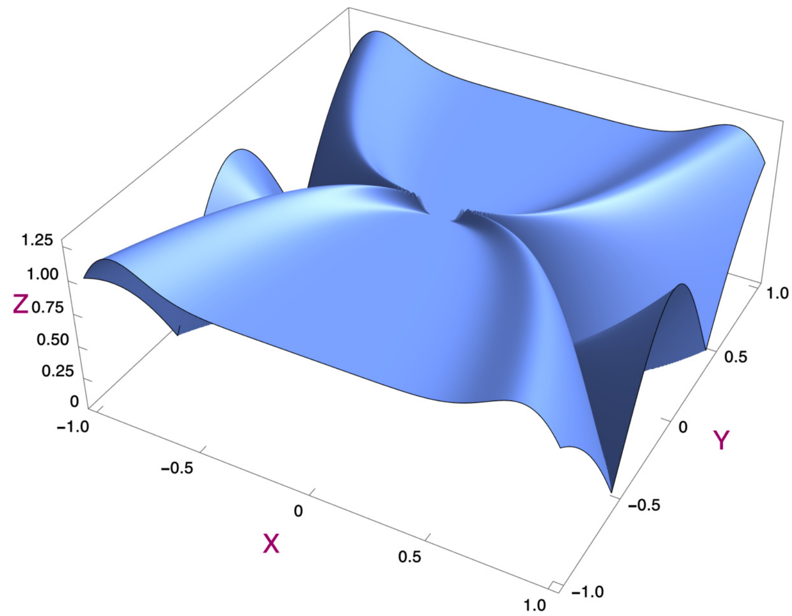

Figure 43. We see here the absolute value of the second partial derivative $\left|\frac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture.

4.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

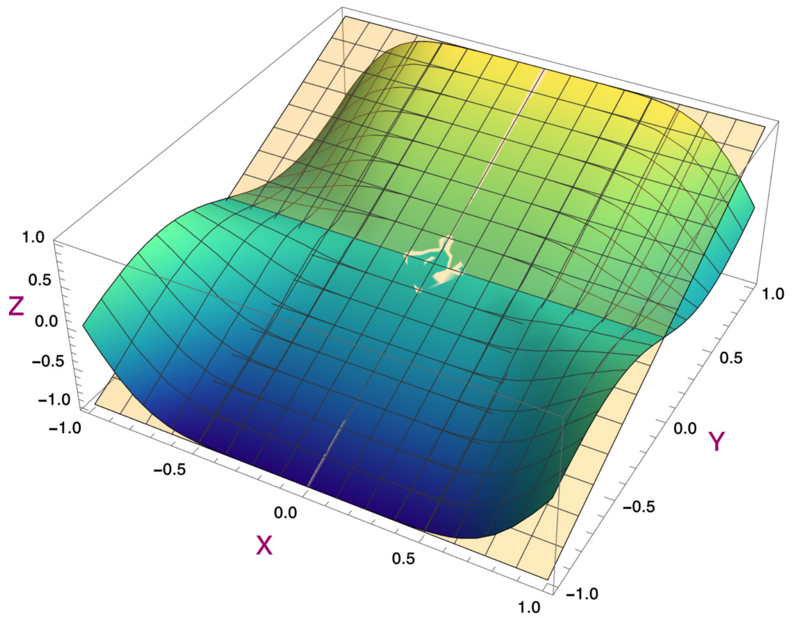

Figure 44. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function quite nicely. We are going to try to give a continuity proof.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we

know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

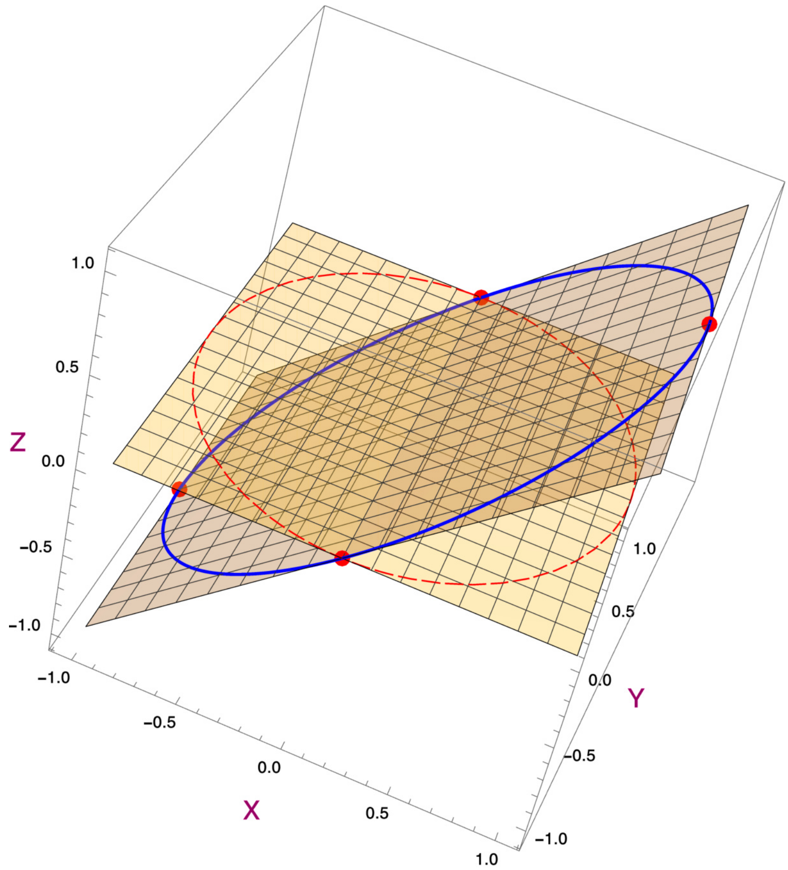

Figure 45. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h) - f(0)}{h}$, which is commonly called the differential quotient!

If we have

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} -\dfrac{h^6\left(h^2+1\right)k}{\sqrt{h^2+k^2}\left(h^6+k^2\right)} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left| -\frac{h^6\,(h^2+1)\,k}{\sqrt{h^2+k^2}\,(h^6+k^2)} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\left| -\frac{h^6\left(h^2+1\right)k}{\sqrt{h^2+k^2}\left(h^6+k^2\right)}\right| \leq \frac{|h|^3\left(h^2+1\right)}{2\sqrt{h^2+k^2}}$$
$$\leq \frac{2\,|h|^3}{2\sqrt{h^2+k^2}}$$
$$\leq \frac{\sqrt{h^2+k^2}^{\,3}}{\sqrt{h^2+k^2}}$$
$$\leq \sqrt{h^2+k^2}^{\,2}.$$

We have used the fact that $0 \leq (a-b)^2$ or that $2\,a\,b \leq a^2+b^2$. So we have that $2\,a\,b \leq (a^2+b^2)$ or that $a\,b/(a^2+b^2) \leq 1/2$. We applied this in the first inequality by substituting $a = h^3$ and $b = k$. We have also chosen $|h| < 1$ so that $h^2+1 < 2$.

It is sufficient to take $\delta = \min\{\sqrt{\epsilon}, 1\}$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous. And the consequence is that the function $f(x,y)$ is differentiable.

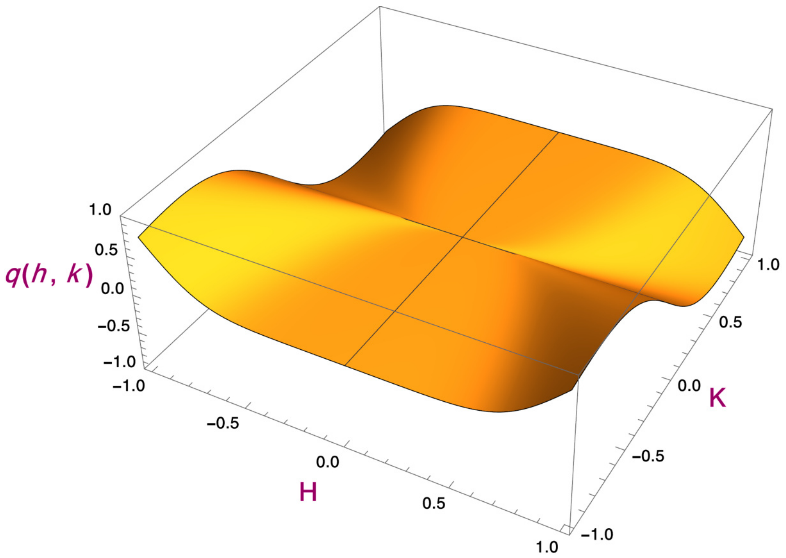

Figure 46. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

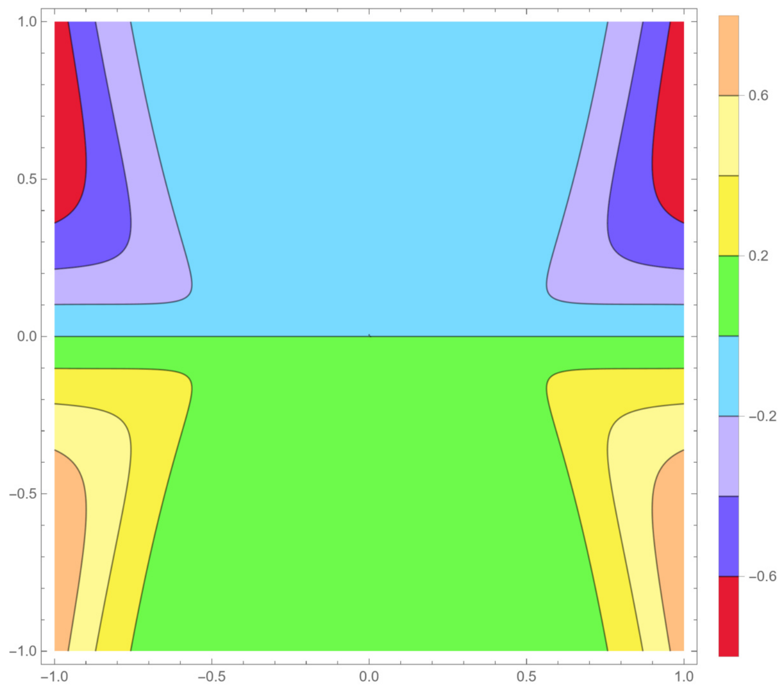

Figure 47. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

4.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\, u + \frac{\partial f}{\partial y}(0,0)\, v\,.$$

So we have now almost an intuitive tangent plane.

Discussion of an alternative proof for the differentiability

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the function $f(x,y)$ is locally Lipschitz continuous in $(0,0)$. We cite here the criterion that we will use.

A function is differentiable in (a,b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.
2. The directional derivatives have the following form: $D_{(u,v)}(a,b) = \nabla f(a,b) \cdot (u,v) = \frac{\partial f}{\partial x}(a,b)\, u + \frac{\partial f}{\partial y}(a,b)\, v$.
3. The function is locally Lipschitz continuous in (a,b). This means that there exists at least one neighbourhood of (a,b) and a number $K > 0$ such that for all (x_1, y_1) and (x_2, y_2) in the neighbourhood, we have $|f(x_1,y_1) - f(x_2,y_2)| < K\,||(x_1,y_1) - (x_2,y_2)||$. Remark that the same K must be valid for all (x_1, y_1) and (x_2, y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be locally flat and thus locally linear which is partly expressed by this Lipschitz continuity condition.

Let us try to prove the Lipschitz condition. We are going to use a well known method.

$$\begin{aligned}
&|f(x_1,y_1) - f(x_2,y_2)| \\
&\qquad = |f(x_1,y_1) - f(x_1,y_2) + f(x_1,y_2) - f(x_2,y_2)| \\
&\qquad \le |f(x_1,y_1) - f(x_1,y_2)| + |f(x_1,y_2) - f(x_2,y_2)|.
\end{aligned}$$

We focus now on the first term $|f(x_1,y_1)-f(x_1,y_2)|$. We fix now x_1 and look upon $f(x_1,y)$ as a function in one variable. So it springs to mind that we can estimate by using Lagrange's intermediate value theorem. So we have

$$|f(x_1,y_1) - f(x_1,y_2)| = \left|\frac{\partial f}{\partial y}(x_1,\xi)\right| \; |y_1 - y_2|$$

where ξ is a number lying in the open interval (y_1, y_2) if e.g. $y_1 \le y_2$. Remark that we already have proven that the partial derivatives are bounded in a neighbourhood, we can take that bound, say M_2 and that neighbourhood in this case also. So $\left|\frac{\partial f}{\partial y}(x_1,\xi)\right| \le M_2$. We work in a completely analogous way for the second term $|f(x_1,y_2) - f(x_2,y_2)|$. We have in that case $\left|\frac{\partial f}{\partial x}(\xi,y_2)\right| \le M_1$.

We take up our inequality again

$$\begin{aligned}
&|f(x_1,y_1) - f(x_2,y_2)| \\
&\qquad \le |f(x_1,y_1) - f(x_1,y_2) + f(x_1,y_2) - f(x_2,y_2)| \\
&\qquad \le |f(x_1,y_1) - f(x_1,y_2)| + |f(x_1,y_2) - f(x_2,y_2)| \\
&\qquad \le M_2\,|(y_1 - y_2)| + M_1\,|(x_1 - x_2)| \\
&\qquad \le M_2\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} + M_1\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \\
&\qquad \le (M_1 + M_2)\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}.
\end{aligned}$$

So this function is locally Lipschitz continuous. We have an alternative proof for the differentiability.

4.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof that the derivative exists. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. We prove in this case more then the existence of the derivative. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

We calculated the two partial derivatives

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} -2\,\dfrac{x^5\,y\,\left(x^8+4\,y^2\,x^2+3\,y^2\right)}{\left(x^6+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{-x^{14}+x^8\,y^2+3\,x^6\,y^2+y^4}{\left(x^6+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 1 & \text{if } (x,y) = (0,0). \end{cases}$$

We investigate the continuity of the partial derivatives.

Discussion of the continuity of the first partial derivative.
We are only going to show the visual results for the first partial derivative to x. All indications are in favour of the continuity. But we do not investigate it further. We have serious doubts about the partial derivative to y and investigate that first.

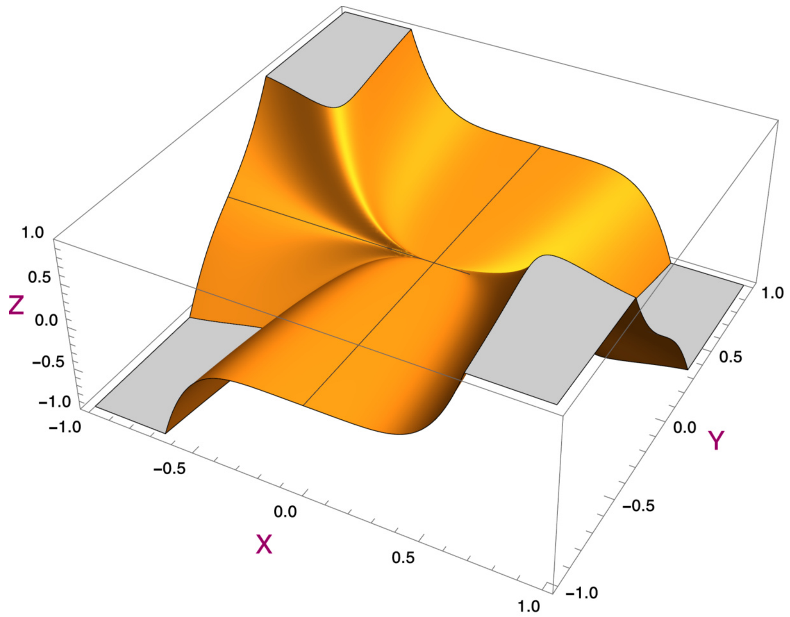

Figure 48. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial x}(x, y)$. This looks like a continuous function.

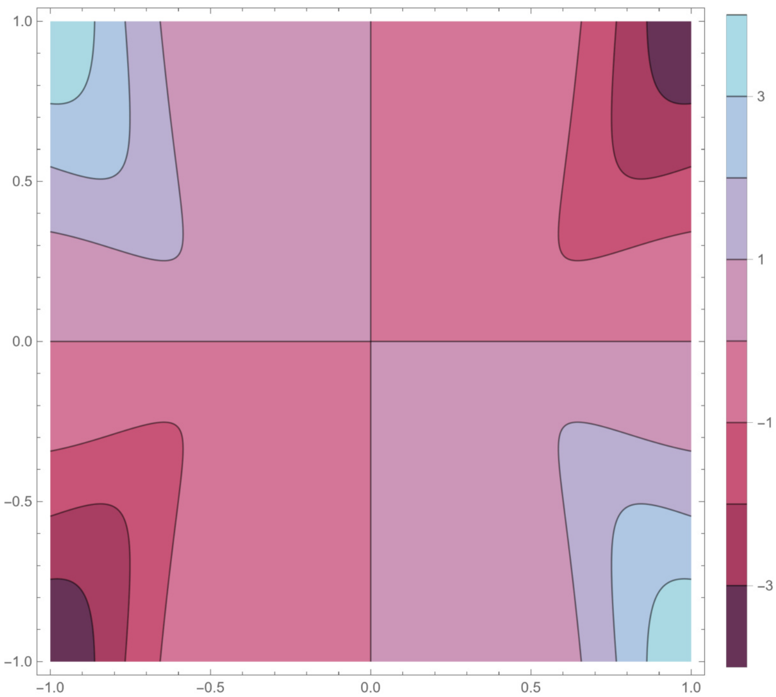

Figure 49. We see here a figure of the contour plot of $\frac{\partial f}{\partial x}(x,y)$. Only level curves of level around 0 come close to $(0,0)$.

Discussion of the continuity of the second partial derivative.

We restrict the function $\frac{\partial f}{\partial y}$ to the continuous curves with equations $y = \lambda x^3$. We observe then that

$$\left.\frac{\partial f}{\partial y}\right|_{y=\lambda x^3}(x,y) = \begin{cases} \dfrac{\lambda^4 + \lambda^2\left(x^2+3\right) - x^2}{\left(\lambda^2+1\right)^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So the limit for $x \to 0$ is

$$\lim_{x\to 0} \frac{\lambda^4 + \lambda^2\,(x^2+3) - x^2}{(\lambda^2+1)^2} = \frac{\lambda^4 + 3\,\lambda^2}{(\lambda^2+1)^2}.$$

We see that these restricted functions have many different limits. But if $\frac{\partial f}{\partial y}(x,y)$ is continuous, all these limit values should be the same. So this function $\frac{\partial f}{\partial y}(x,y)$ is not continuous.

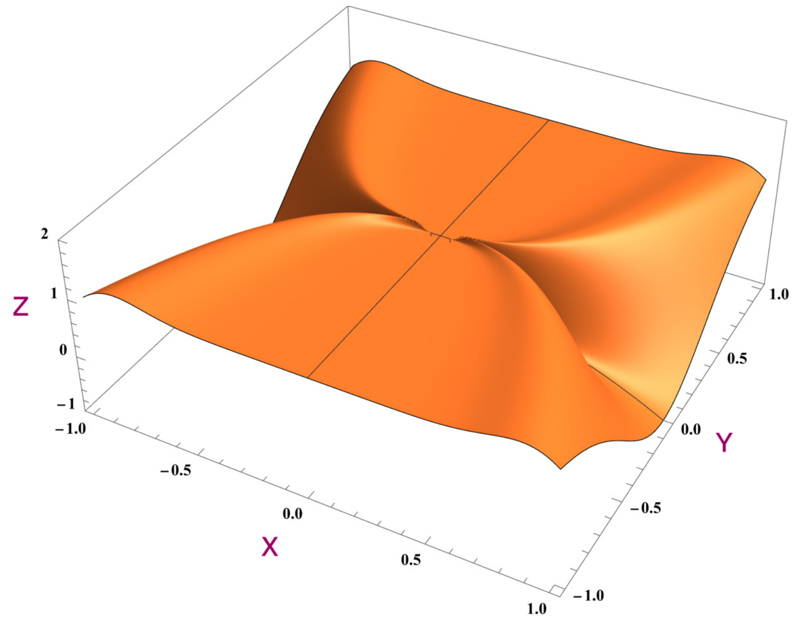

Figure 50. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial y}(x, y)$. The vertical line above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

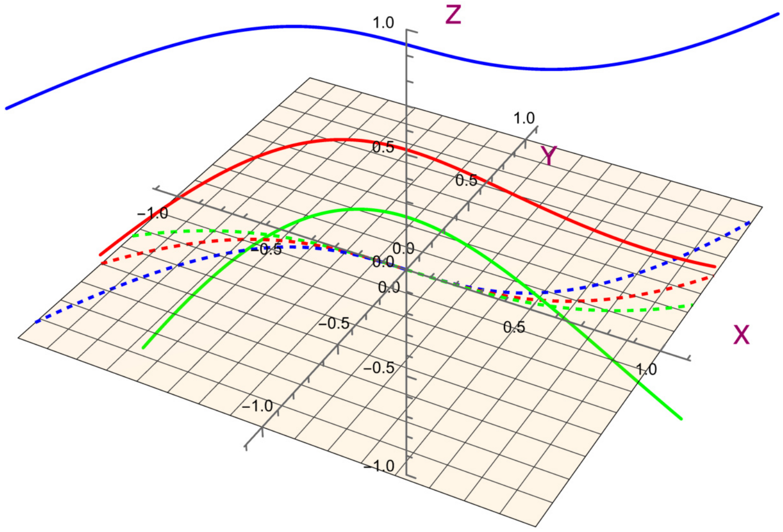

Figure 51. We have restricted the function $\frac{\partial f}{\partial y}(x, y)$ here to $y = 1/2\,x^3$ and $y = 3/10\,x^3$ and $y = 9/10\,x^3$. We see in this figure clearly that the restrictions of the function $\frac{\partial f}{\partial y}(x, y)$ to these lines are functions that have different limits in 0.

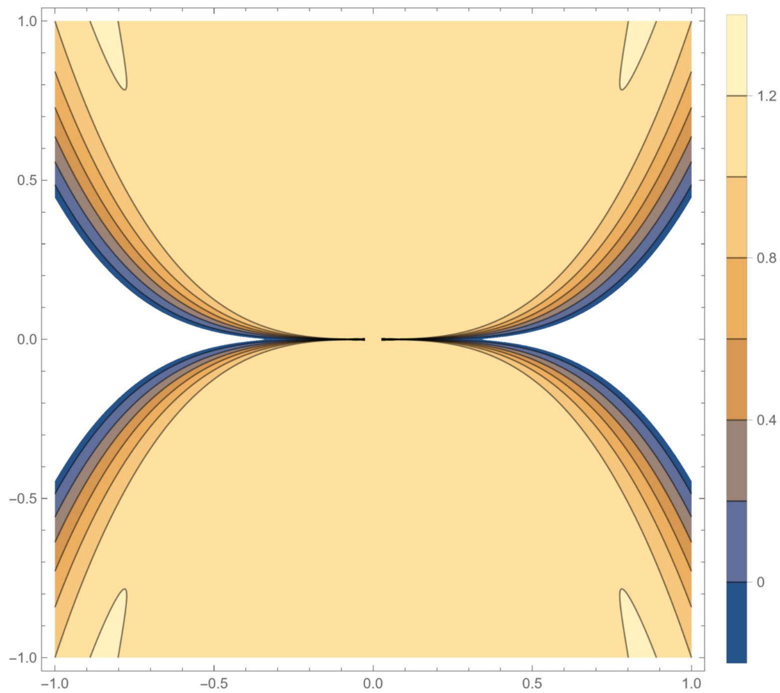

Figure 52. We see here a figure of the contour plot of the partial derivative $\frac{\partial f}{\partial y}(x, y)$. Many level curves of very different levels approach $(0, 0)$. This looks discontinuous indeed.

4.8 Overview

$$f(x,y) = \begin{cases} \dfrac{y^3 - x^8 y}{x^6 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

4.9 One step further

It can maybe be worthwhile to take a look at the gradient vector field. We see that this gradient vector field exists in this case. But the gradient vector field cannot be continuous because it implies that the function is differentiable. So we wonder if we can find an indication for this fact.

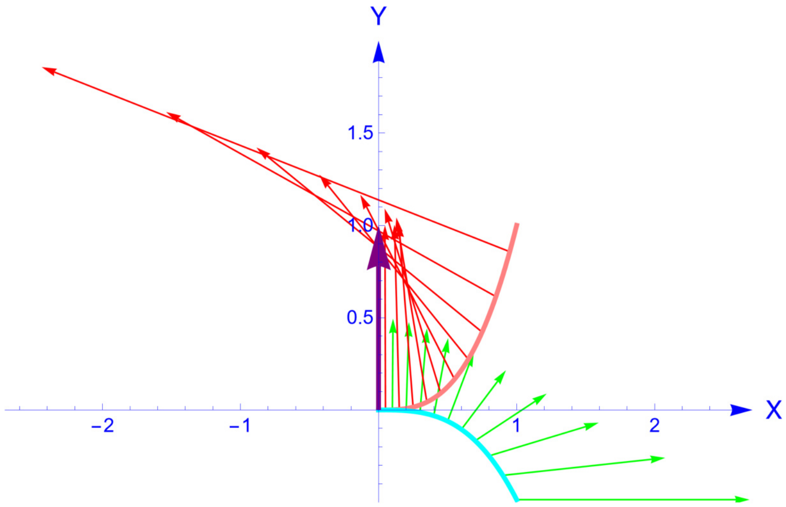

Figure 53. We made here the following sketch. We have drawn the graphics of $y = x^3$ in pink and $y = 0.37x^3$ in cyan. We have sketched the gradient vector field $\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$ which is not continuous as can be seen as follows. Observe the gradient vector field on the pink curve, these are the red vectors. Observe the gradient vector field on the cyan curve, these are the green vectors. The purple vector is the gradient vector in $(0,0)$. The red vectors converge to a vector with a non zero y-component. This component is equal to 1. The green vectors converge to a vector that has a y-component that is two times smaller then the y-component of the vector to which the red vectors converge if $x \to 0$. This is clearly impossible if the vector field is continuous. Moreover, the y-component of the limit vector should in both cases be 1 if the gradient vector field is continuous. We conclude that this gradient vector field is not continuous. The function is differentiable in that case, which it is not. Please note however that a sketch of the gradient vector field is inherently a sketch of discrete data. So the utmost care must be taken in order to make it a little bit trustworthy.

Exercise 5.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} x^3 \sin\left(\frac{1}{x^2}\right) + y^3 \sin\left(\frac{1}{y^2}\right) & \text{if } x\,y \neq 0, \\ 0 & \text{if } x\,y = 0. \end{cases}$$

5.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| x^3 \sin\left(\frac{1}{x^2}\right) + y^3 \sin\left(\frac{1}{y^2}\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| x^3 \sin\left(\frac{1}{x^2}\right) + y^3 \sin\left(\frac{1}{y^2}\right) \right| &\le |x|^3 \left|\sin\left(\frac{1}{x^2}\right)\right| + |y|^3 \left|\sin\left(\frac{1}{y^2}\right)\right| \\
&\le |x|^3 + |y|^3 \\
&\le \sqrt{x^2+y^2}^3 + \sqrt{x^2+y^2}^3 \\
&\le 2\sqrt{x^2+y^2}^3.
\end{aligned}$$

It is sufficient to take $\delta = (\epsilon/2)^{1/3}$. We can find a δ, so we conclude that the function is continuous.

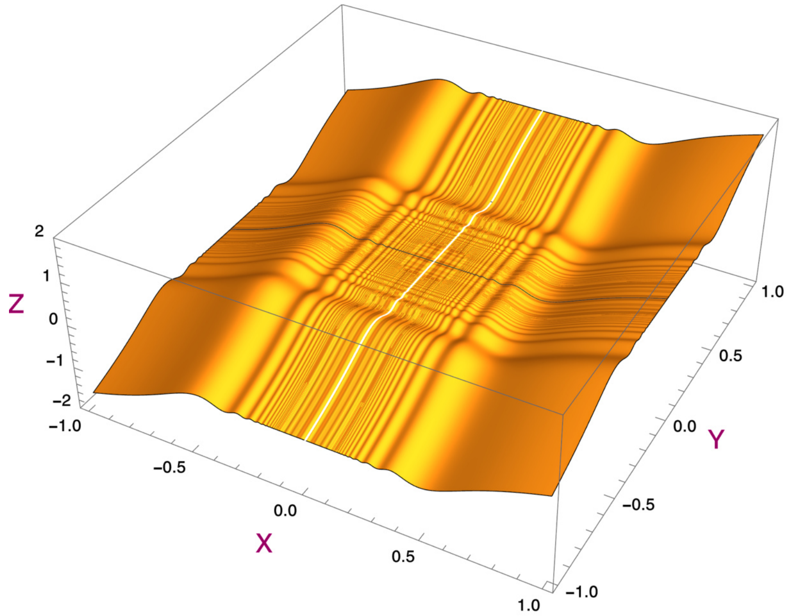

Figure 54. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

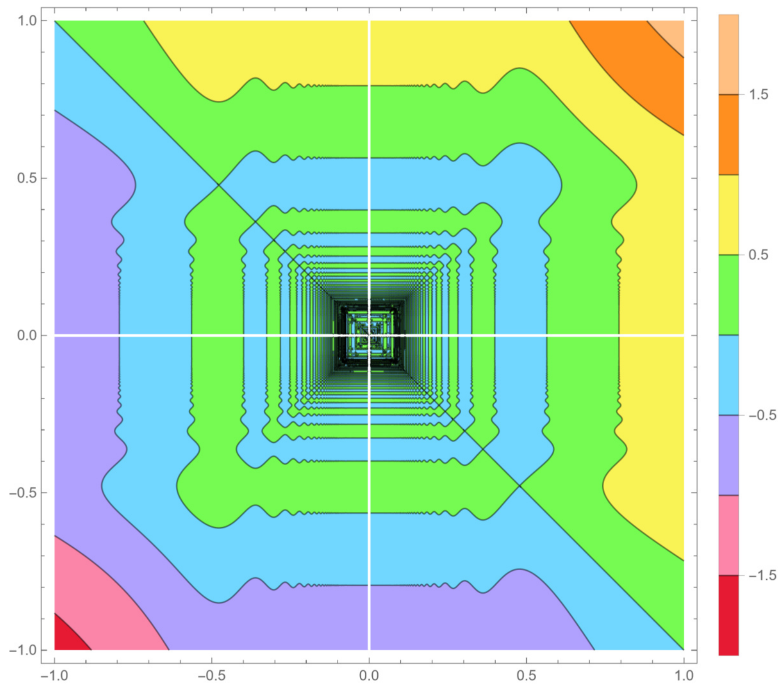

Figure 55. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

5.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f\big|_{y=0}(x,y)\begin{cases} f(x,0)=0 & \text{if } x\neq 0;\\ 0 & \text{if } x=0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} \\ &= 0. \end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

5.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\ &= \lim_{h\to 0} h^2 \left(u^3 \sin\left(\frac{1}{h^2u^2}\right) + v^3 \sin\left(\frac{1}{h^2v^2}\right)\right) \\ &= 0. \end{aligned}$$

We can assume that $u \neq 0$ and $v \neq 0$ because we computed the partial derivatives in the preceding section.

So the directional derivatives do always exist.

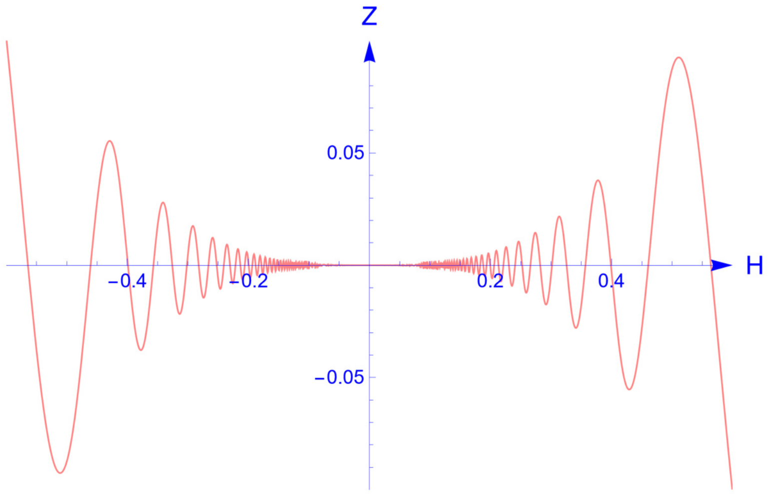

Figure 56. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$. It shows that the directional derivatives are 0.

5.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Looking for a definition of the partial derivatives. We have to be able to define the partial derivatives in at least one neighbourhood around $(0,0)$. We have no problems with points that are in the interior of the definition domains of the classical functions. We have there the classical calculation rules for defining those functions and the partial derivatives always exist there and are even continuous.

But in our case, we have to investigate the points that are in exceptional subsets in the definition of the function.

In this case these are the points $(a, 0)$ and $(0, b)$.

Let us look at a point $(a, 0)$ with $a \neq 0$. We are going to investigate the function in $(a, 0)$ in the Y-direction. This function is defined by

$$f(a, h) = \begin{cases} a^3 \sin\left(\frac{1}{a^2}\right) + h^3 \sin\left(\frac{1}{h^2}\right) & \text{if } h \neq 0, \\ 0 & \text{if } h = 0. \end{cases}$$

We see that in almost all cases of a, the function is not continuous and consequently not differentiable. So in the cases where $a^3 \sin\left(\frac{1}{a^2}\right) \neq 0$ the function is not differentiable. The points a where these happens, are countable. The conclusion is that the partial derivative $\frac{\partial f}{\partial y}(a, 0)$ does not exist for almost all a with $a \neq 0$.

We consult a figure for this observation.

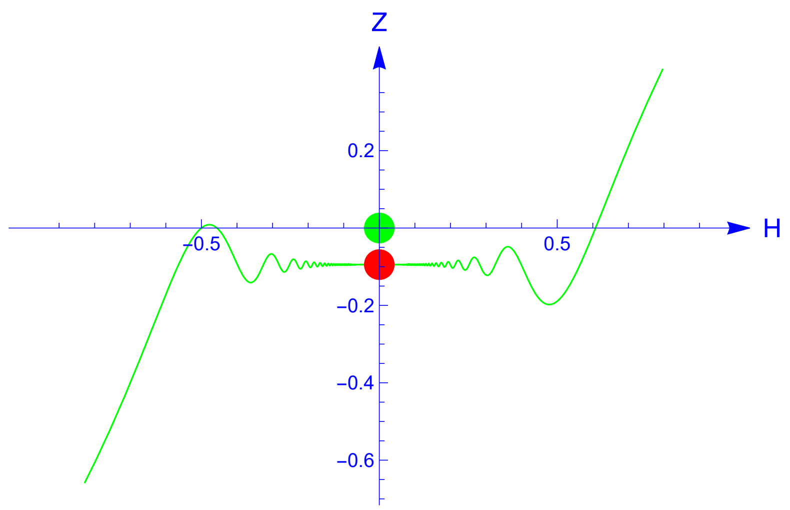

Figure 57. We see here a figure of the graph of the function restricted to the line through $(a, 0)$ with direction $(0, 1)$, this is the Y-direction. We see that this function is not continuous in $h = 0$. We have drawn this figure with the value $a = 1/2$. This is a figure of the function with function definition $f(a, h)$.

The partial derivatives do not all exist in any neighbourhood of $(0, 0)$. So the partial derivatives cannot be defined in any neighbourhood of $(0, 0)$. The conclusion is that an alternative proof following the lines described at the start of this section cannot be given. Other alternative proofs can of course exist.

We note that this observation is also relevant for section 7 if we get as far as section 7 because this can be irrelevant depending on previous sections. In section 7 we have to prove that the partial derivatives are continuous. But if there are no partial derivatives in any neighbourhood of (0,0), then we cannot prove also anything in that section.

We note also that a function can be perfectly differentiable, even as the partial derivatives are not defined in any *neighbourhood* of $(0,0)$. It is though necessary that the partial derivative in the *point* $(0,0)$ itself does exist in order that the function is differentiable in $(0,0)$. From this viewpoint, the existence of the partial derivatives in a neighbourhood of $(0,0)$ is pure luxury.

5.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

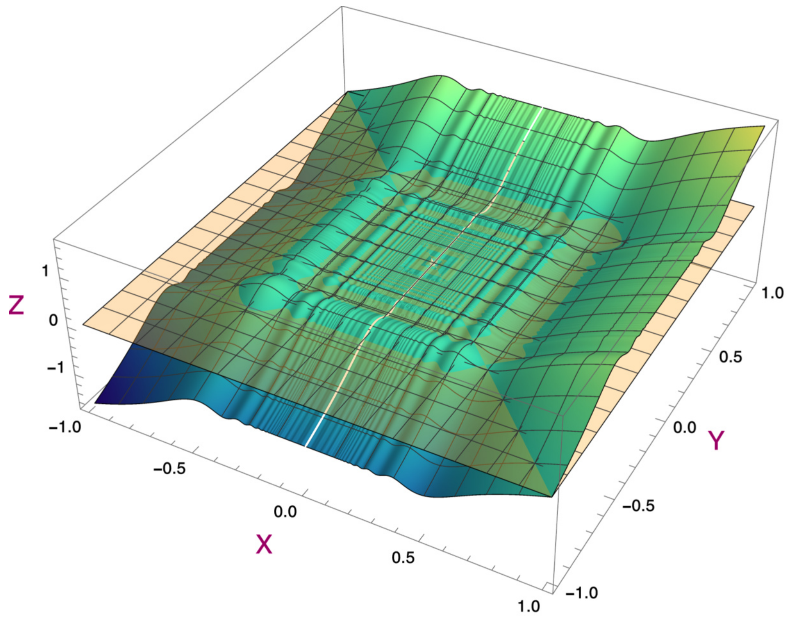

Figure 58. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x, y)$. We see here that the candidate tangent plane fits the function very nicely. But we have to careful due to the heavy oscillations.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

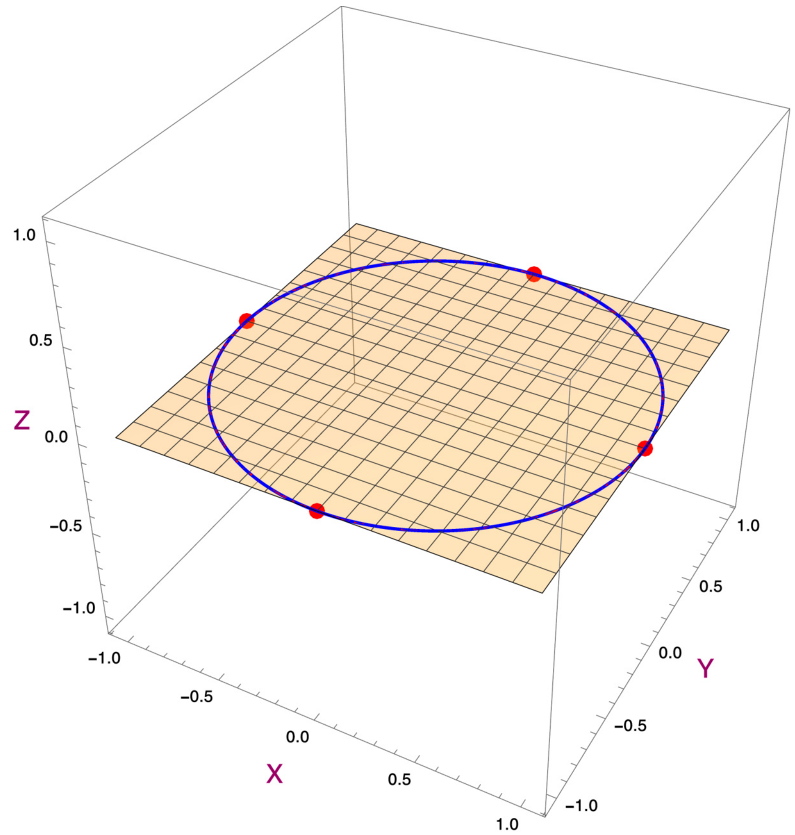

Figure 59. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k)$$

$$= \begin{cases} \dfrac{h^3 \sin\left(\dfrac{1}{h^2}\right) + k^3 \sin\left(\dfrac{1}{k^2}\right)}{\sqrt{h^2+k^2}} & \text{if } (h,k) \neq (0,0) \text{ and } h \neq 0 \text{ and } k \neq 0; \\ 0 & \text{if } (h,k) = (0,0) \text{ of } k = 0 \text{ or } h = 0 \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left| \frac{h^3 \sin\left(\frac{1}{h^2}\right) + k^3 \sin\left(\frac{1}{k^2}\right)}{\sqrt{h^2+k^2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{h^3 \sin\left(\frac{1}{h^2}\right) + k^3 \sin\left(\frac{1}{k^2}\right)}{\sqrt{h^2+k^2}} \right| &\le \frac{|h|^3 \left|\sin\left(\frac{1}{h^2}\right)\right| + |k|^3 \left|\sin\left(\frac{1}{k^2}\right)\right|}{\sqrt{h^2+k^2}} \\
&\le \frac{|h|^3 + |k|^3}{\sqrt{h^2+k^2}} \\
&\le \frac{\sqrt{h^2+k^2}^3 + \sqrt{h^2+k^2}^3}{\sqrt{h^2+k^2}} \\
&\le 2\,\frac{\sqrt{h^2+k^2}^3}{\sqrt{h^2+k^2}} \\
&\le 2\sqrt{h^2+k^2}^2.
\end{aligned}$$

It is sufficient to take $\delta = \sqrt{\epsilon/2}$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous. The function f is differentiable.

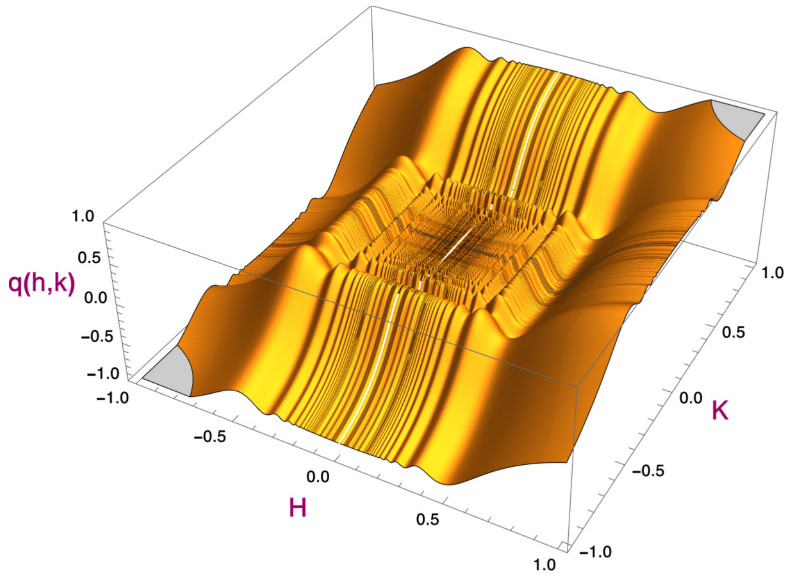

Figure 60. We see here a three dimensional figure of the graph of the function $q(h, k)$. This looks like a continuous function.

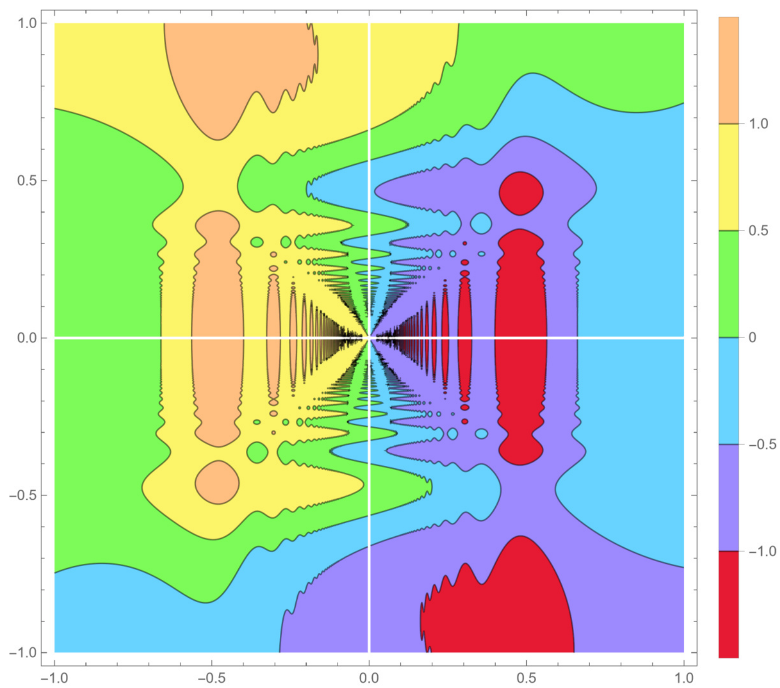

Figure 61. We see here a figure of the contour plot of the function $q(h, k)$. Only level curves of level around 0 come close to $(0, 0)$.

5.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

The function is not continuous in any neighbourhood of $(0,0)$. So it cannot be Lipschitz continuous. An alternative proof following these lines is thus not possible.

5.7 Continuity of the partial derivatives

The partial derivatives do not all exist in any neighbourhood of $(0,0)$. See section 4 for an explanation.

5.8 Overview

$$f(x,y) = \begin{cases} x^3 \sin\left(\frac{1}{x^2}\right) + y^3 \sin\left(\frac{1}{y^2}\right) & \text{if } x\,y \neq 0, \\ 0 & \text{if } x\,y = 0. \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

Exercise 6.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y)=\begin{cases}\dfrac{x\,|y|}{x^2+y^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

6.1 Continuity

We restrict the function to the continuous curves with equations $y = \lambda x$. We observe then that

$$f|_{y=\lambda x}(x,y)=\begin{cases}f(x,\lambda x)=\dfrac{x|\lambda x|}{\lambda^2x^2+x^2}=\dfrac{|\lambda x|}{x(\lambda^2+1)} & \text{if } x\neq 0;\\ 0 & \text{if } x=0.\end{cases}$$

We see that these restricted functions have no limits if $\lambda \neq 0$. But if $f(x,y)$ is continuous, all these limit values should exist. So this function $f(x,y)$ is not continuous.

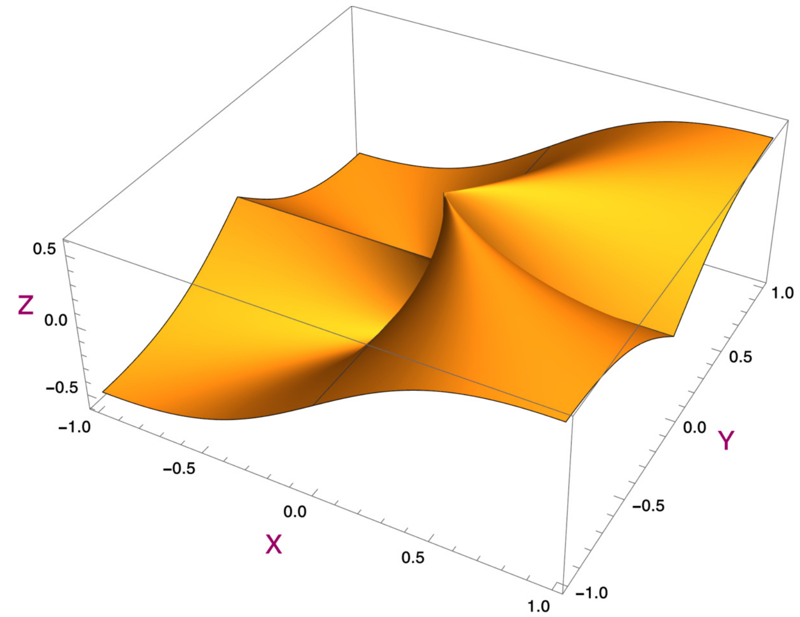

Figure 62. We see here a three dimensional figure of the graph of the function. The vertical line above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

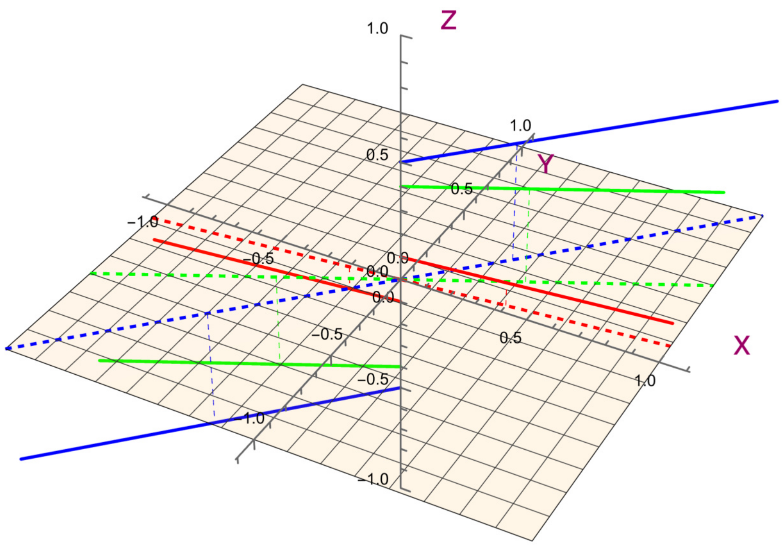

Figure 63. We have restricted the function here to $y = 1/10\,x$ and $y = 5/10\,x$ and $y = x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have no limits in 0.

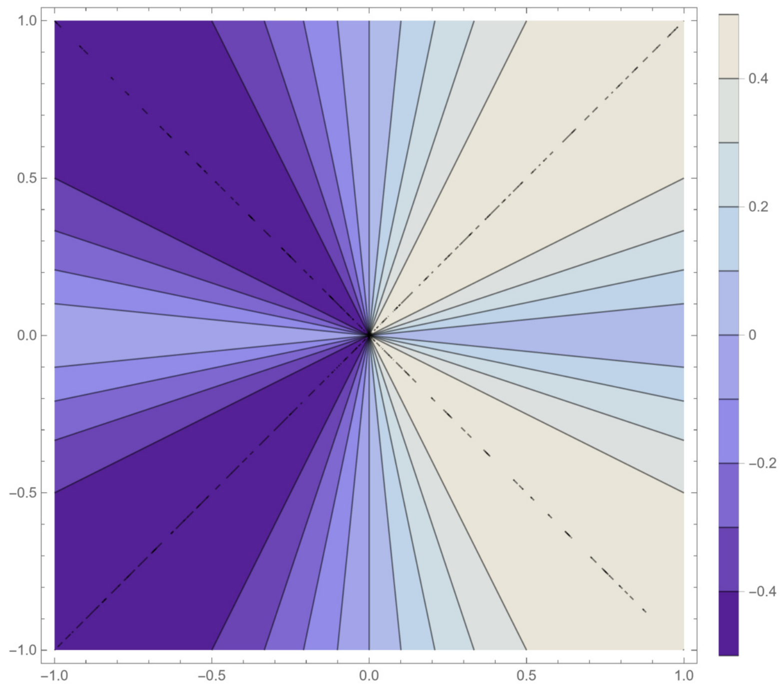

Figure 64. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

6.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f\,|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

6.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h \to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

But the functions $f(hu, hv) = \frac{u\,|h\,v|}{h\,(u^2+v^2)}$ are not even continuous if $uv \neq 0$. So they are not differentiable.

These limits do not exist if $u \neq 0$ and $v \neq 0$.

So the directional derivatives do not always exist.

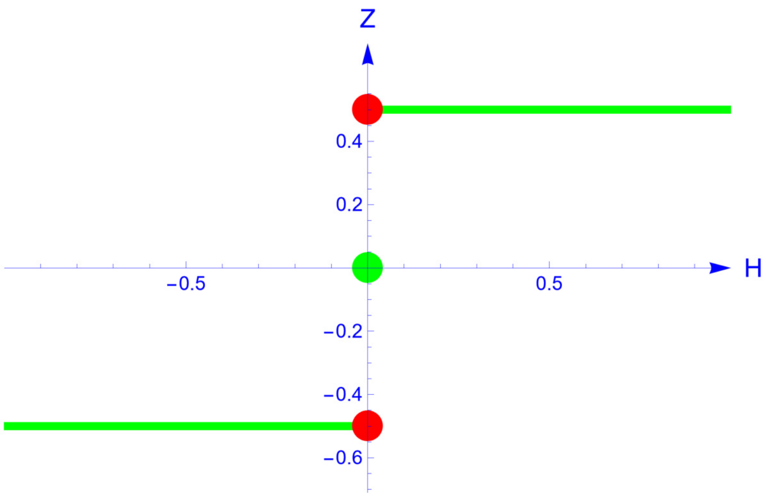

Figure 65. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$.

6.4 Alternative proof of continuity (optional)

This is irrelevant. The function is not continuous.

6.5 Differentiability

The function is not continuous. So the function is not differentiable.

6.6 Alternative proof of differentiability (optional)

Irrelevant.

6.7 Continuity of the partial derivatives

Irrelevant.

6.8 Overview

$$f(x,y) = \begin{cases} \dfrac{x\,|y|}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	no
partial derivatives exist	yes
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 7.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y)=\begin{cases}\dfrac{y\,|x|^{3/2}}{x^3+y^2} & \text{if } (x,y)\neq(0,0);\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

7.1 Continuity

We restrict the function to the continuous curves with equations $y = \lambda\,|x|^{3/2}$. We observe then that

$$f|_{y=\lambda x}(x,y)=\begin{cases}f(x,\lambda\,|x|^{3/2})=\dfrac{\lambda\,|x|^3}{\lambda^2\,|x|^3+x^3}=\dfrac{\lambda\ \mathrm{sgn}(x)}{1+\lambda^2\ \mathrm{sgn}\,x} & \text{if } x\neq 0;\\ 0 & \text{if } x=0.\end{cases}$$

We see that these restricted functions have no limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0)=0$. So this function $f(x,y)$ is not continuous.

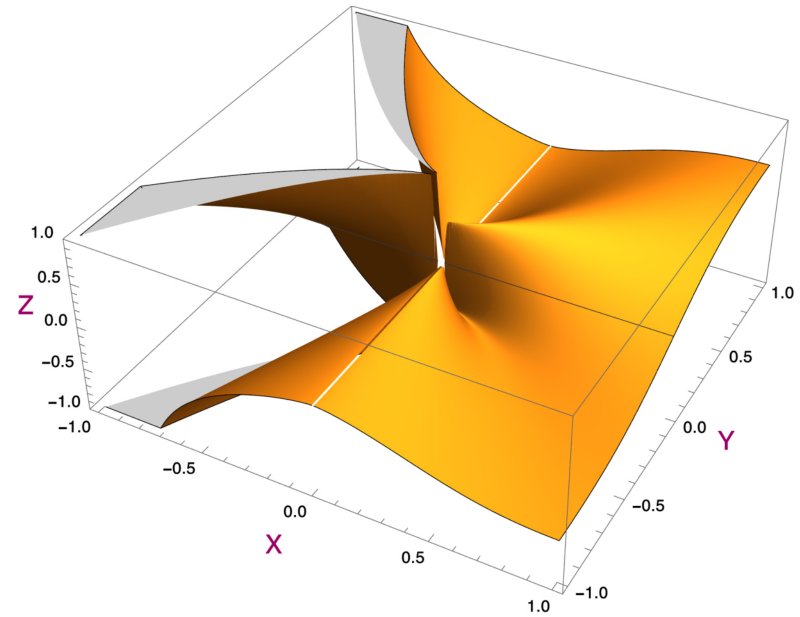

Figure 66. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

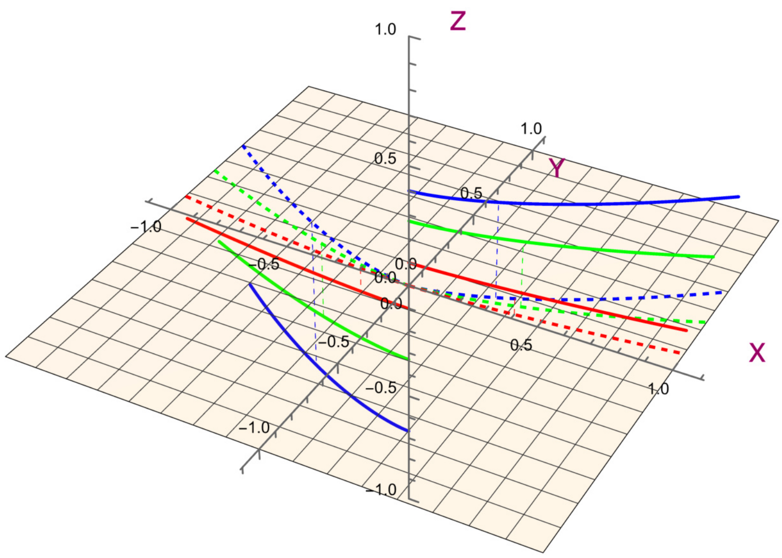

Figure 67. We have restricted the function here to $y = 1/10\,x^{3/2}$ and $y = 3/10\,x^{3/2}$ and $y = 1/2\,x^{3/2}$. We see in this figure clearly that the restrictions of the function to these lines are functions that have no limits in 0.

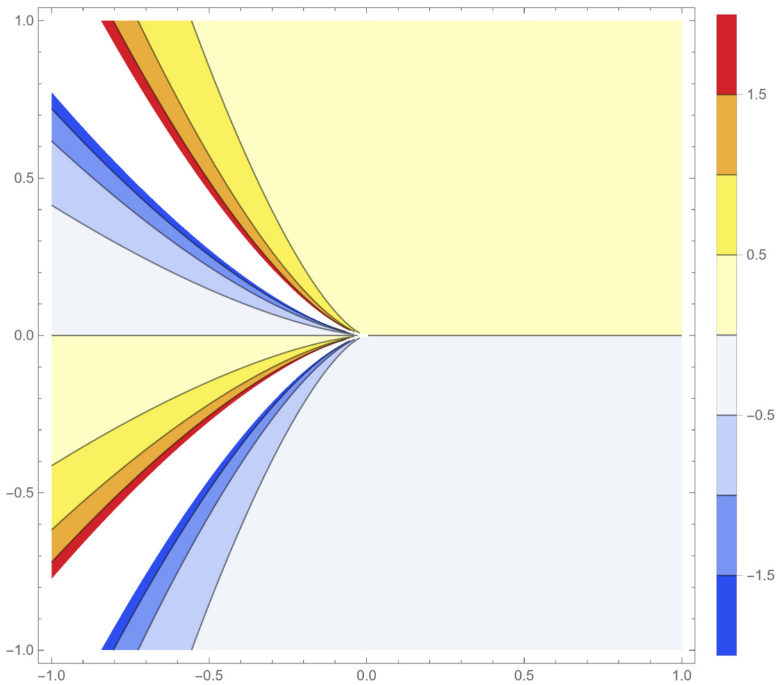

Figure 68. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

7.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

7.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{v\,|h\,u|^{3/2}}{h^2\,(h\,u^3 + v^2)}. \end{aligned}$$

We see that the limit is not finite if $u\,v \neq 0$. So the directional derivatives, excluding the partial derivatives, do not exist.

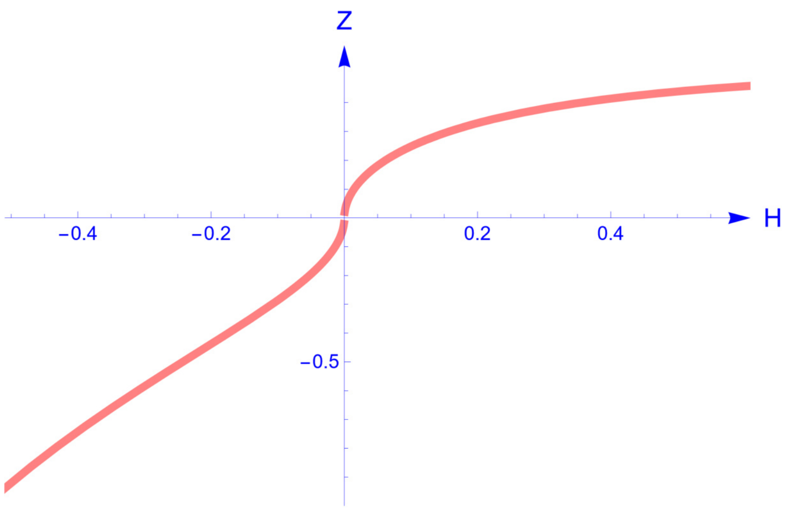

Figure 69. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$. We see the vertical tangent in 0. This causes the infinite behaviour of the slopes.

7.4 Alternative proof of continuity (optional)

Irrelevant. The function is not continuous.

7.5 Differentiability

There is no differentiability. The function is even not continuous.

7.6 Alternative proof for the differentiability

Irrelevant. The function is not differentiable.

7.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

7.8 Overview

$$f(x,y) = \begin{cases} \dfrac{y\,|x|^{3/2}}{x^3 + y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	no
partial derivatives exist	yes
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 8.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{\sin\left(|xy| + x^2\right)}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

8.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{\sin\left(|xy| + x^2\right)}{x} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$
\begin{aligned}
\left|\frac{\sin\left(|x\,y| + x^2\right)}{x}\right| &\leq \frac{|x\,y| + x^2}{|x|} \\
&\leq |y| + |x| \\
&\leq \sqrt{x^2 + y^2} + \sqrt{x^2 + y^2} \\
&\leq 2\sqrt{x^2 + y^2}.
\end{aligned}
$$

We have made use of the fact that $|\sin(x)| \leq |x|$.

It is sufficient to take $\delta = \epsilon/2$. We can find a δ, so we conclude that the function is continuous.

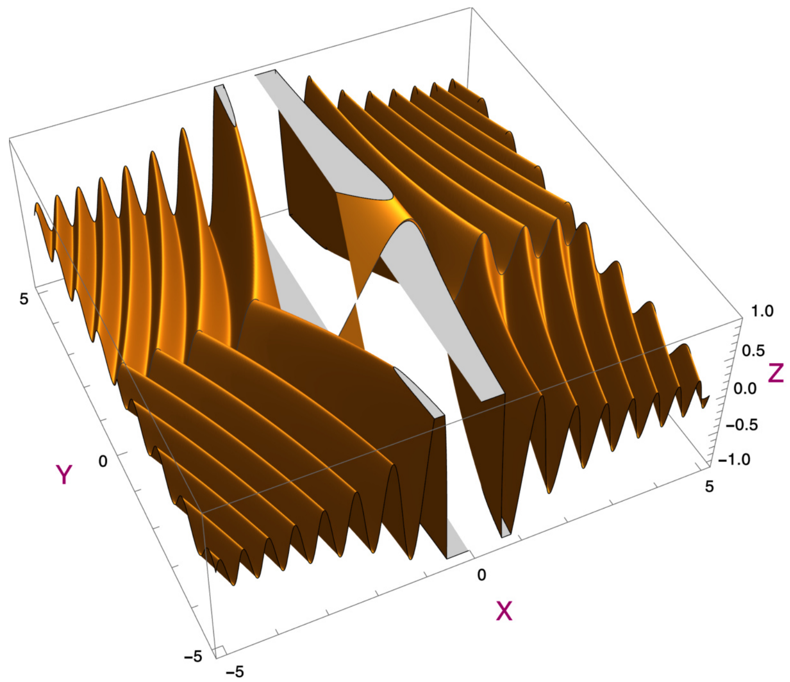

Figure 70. We see here a three dimensional figure of the graph of the function. This looks like a continuous function. This is a more global view. In the following picture, we have a more local view.

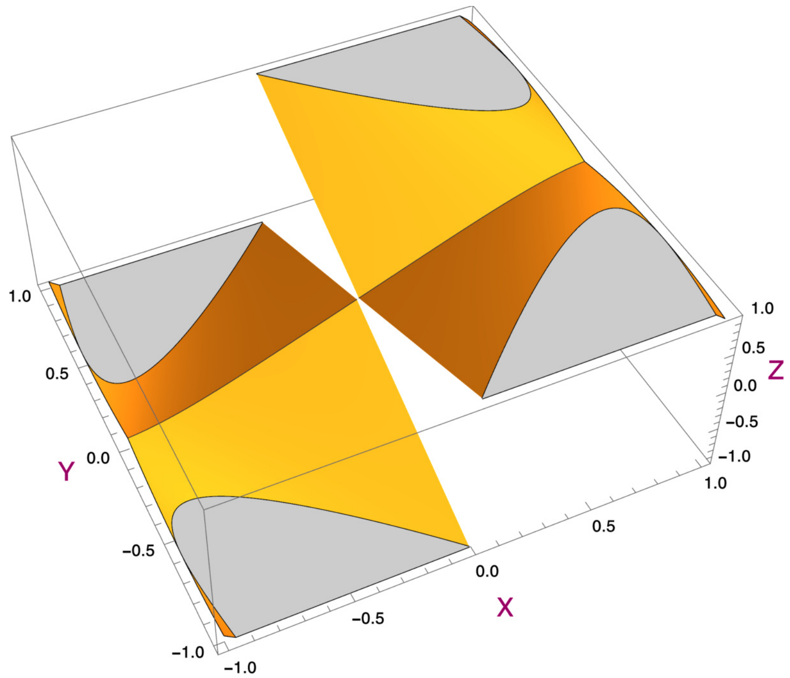

Figure 71. We see here a three dimensional figure of the graph of the function. This looks like a continuous function. Let us however not forget that we defined the function on the line with equation $x = 0$ as being 0. The drawing does not represent that part of the function and it could give the visual impression that the partial derivative to y is not defined in $x = 0$!

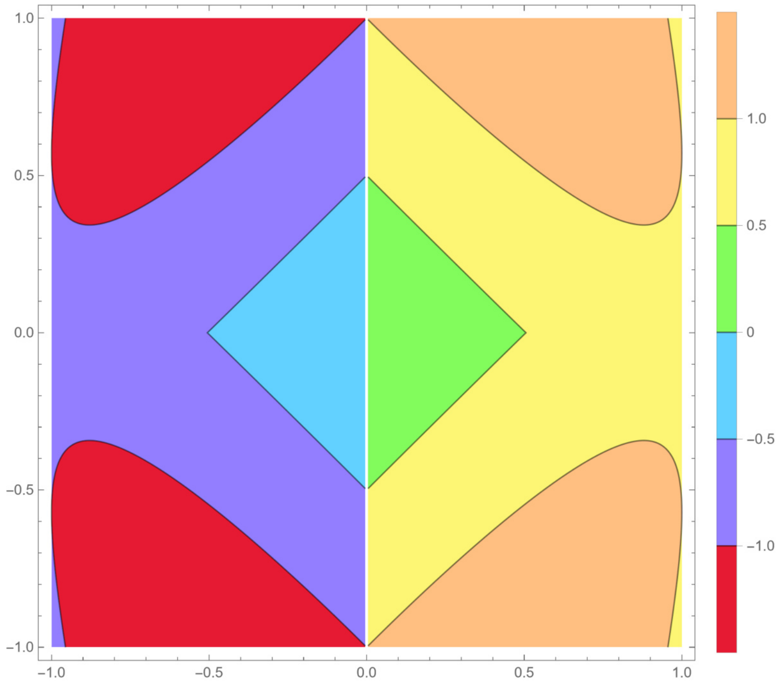

Figure 72. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

8.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = \dfrac{\sin\left(x^2\right)}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0}\frac{\sin(h^2)}{h^2}\\ &= 1.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y)=0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 1 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

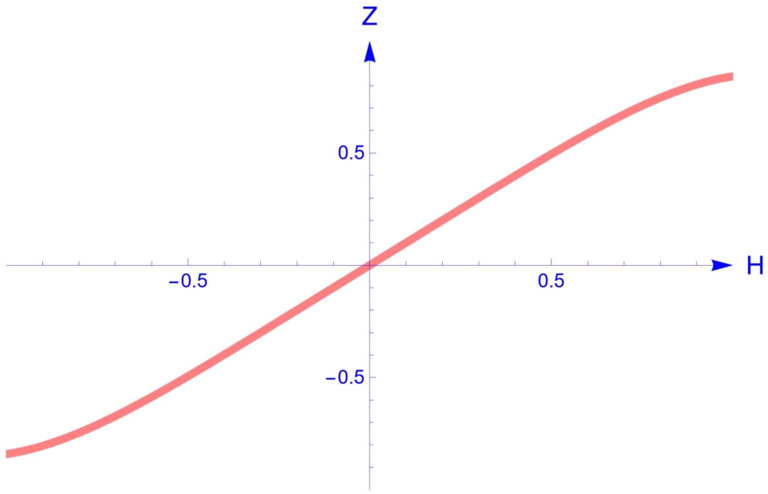

Figure 73. We see here a figure of the graph of the function $\frac{\sin(x^2)}{x}$ restricted to the horizontal X-axis through $(0,0)$. We have plotted here the function $f(h,0)$.

8.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$
\begin{aligned}
D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{\sin\left(|h^2\,u\,v| + h^2\,u^2\right)}{h^2\,u} \\
&= \lim_{h\to 0} \frac{\sin\left(|h^2\,u\,v| + h^2\,u^2\right)}{|h^2\,u\,v| + h^2\,u^2} \, \frac{|h^2\,u\,v| + h^2\,u^2}{h^2\,u} \\
&= u + |v| \operatorname{sgn}(u).
\end{aligned}
$$

The calculation is only valid if $u \neq 0$. But we covered that case before. So the directional derivatives do always exist.

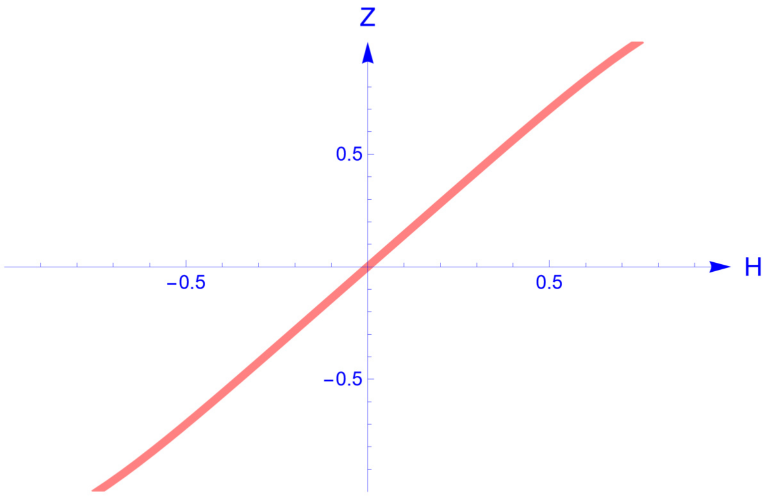

Figure 74. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

8.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Functions containing absolute values in their definition are essentially piecewise defined functions.

Our function is

$$f(x,y) = \begin{cases} \dfrac{\sin\left(x^2 + x\,y\right)}{x} & \text{if } x\,y \geq 0 \text{ and } x \neq 0, \\ \dfrac{\sin\left(x^2 - x\,y\right)}{x} & \text{if } x\,y < 0 \text{ and } x \neq 0. \\ 0 & \text{if } x = 0, \end{cases}$$

Let us investigate the existence of the partial derivative to x in a point $(0,b)$ with $b > 0$. Let us now consider the function $f(h,b)$ in a point (h,b) and consider b fixed and h the variable. Then we calculate the left limit and the right limit in $h = 0$. The function in h is now

$$f(h,b) = \begin{cases} -\dfrac{\sin\left(h\,b - h^2\right)}{h} & \text{if } h < 0, \\ \dfrac{\sin\left(h^2 + h\,b\right)}{h} & \text{if } h > 0, \\ 0 & \text{if } h = 0. \end{cases}$$

We calculate now the left limit and the right limit in $h = 0$.

$$\begin{aligned}
\lim_{\substack{h\to 0\\<}} -\frac{\sin\left(b\,h-h^2\right)}{h} &= \lim_{\substack{h\to 0\\<}} -\frac{\sin\left(h\,(b-h)\right)}{h}\\
&= \lim_{\substack{h\to 0\\<}} -\frac{\sin\left(h\,(b-h)\right)}{h(b-h)}\frac{h(b-h)}{h}\\
&= \lim_{\substack{h\to 0\\<}} -\frac{h(b-h)}{h}\\
&= \lim_{\substack{h\to 0\\<}} -(b-h)\\
&= -b.
\end{aligned}$$

$$\begin{aligned}
\lim_{\substack{h\to 0\\>}} \frac{\sin\left(bh+h^2\right)}{h} &= \lim_{\substack{h\to 0\\>}} \frac{\sin\left(h(b+h)\right)}{h}\\
&= \lim_{\substack{h\to 0\\>}} \frac{\sin\left(h(b+h)\right)}{h(b+h)}\frac{h(b+h)}{h}\\
&= \lim_{\substack{h\to 0\\>}} \frac{h(b+h)}{h}\\
&= \lim_{\substack{h\to 0\\>}} (b+h)\\
&= b.
\end{aligned}$$

So if $b \neq 0$, then the function is not continuous, and the function is not differentiable. So the partial derivative to x does not exist.

We conclude that the alternative criterion that we propose, cannot be applied. Not all derivatives exist in every neighbourhood of $(0,0)$.

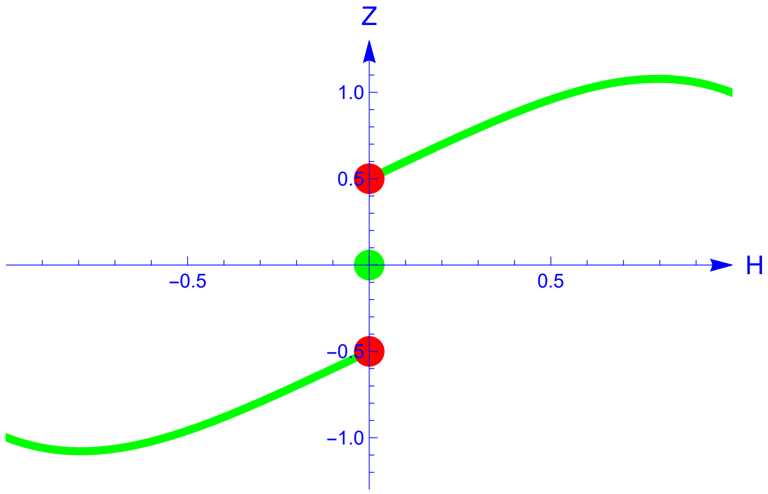

Figure 75. We see here a two dimensional figure of the graph of the function $f(h, b)$. We have drawn the function here for the value $b = 1/2$ which is exemplary for the values of b close to 0. This is not a continuous function. So it is not a differentiable function.

8.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

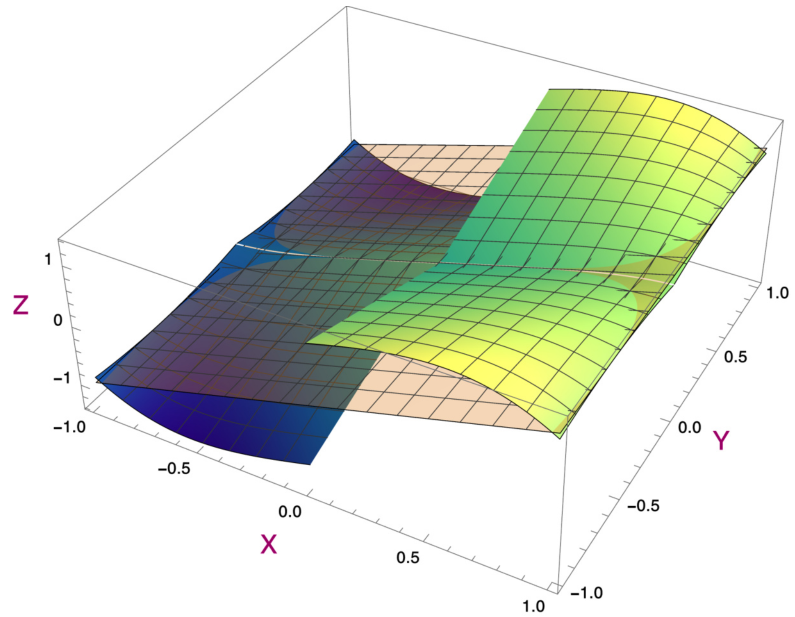

Figure 76. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane does not fit the function very nicely. It is in fact a very bad fit. It is indeed no tangent plane following our calculations later on.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

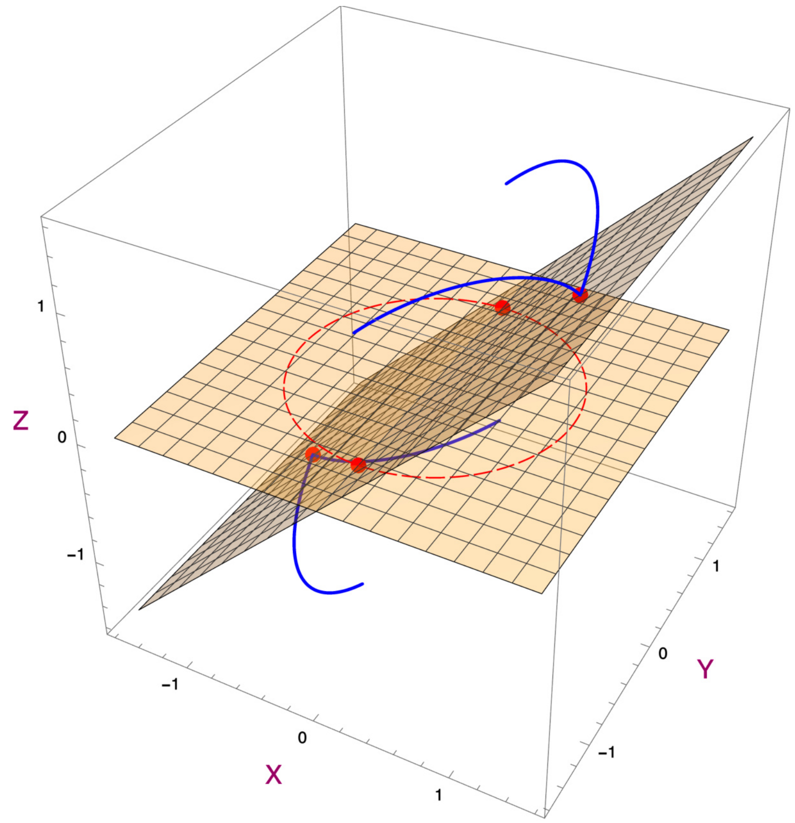

Figure 77. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ do not sweep out a nice ellipse lying in one plane! Because it is not very clear to which planes the vectors really belong, we have made an additional figure.

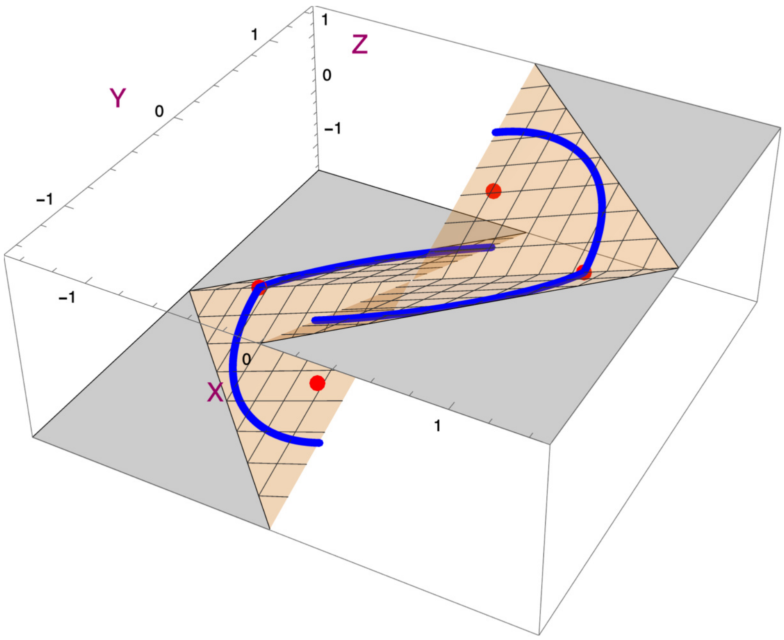

Figure 78. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$. We have made this picture to show more clearly the planes to which the vectors belong. These planes are clearly distinct from the candidate tangent plane. This is bad news for differentiability.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{\sin(h\,(h+k)) - h^2}{h\sqrt{h^2+k^2}} & \text{if } h\,k > 0 \text{ and } h \neq 0, \\ \dfrac{\sin(h\,(h-k)) - h^2}{h\sqrt{h^2+k^2}} & h\,k < 0, \\ 0 & \text{if } h = 0 \end{cases}$$

is continuous in $(0,0)$.

We restrict the function $q(h,k)$ to the continuous curves with equations $k = \lambda\,h$ with $\lambda > 0$. In that case we can focus on the definition part where $h\,k > 0$ which says that we are working in the first and third quadrant.

This part of the function is

$$q(h,k) = \frac{h^2 + \sin(h\,(h+k))}{h\sqrt{h^2+k^2}}.$$

We observe then that

$$q\big|_{k=\lambda\,h}(h,k) = \begin{cases} q(h,\lambda\,h) = \dfrac{\sin\left(h^2\,(\lambda+1)\right) + h^2}{h\,|h|\sqrt{(\lambda^2+1)}} & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

Let us try to simplify this part of this definition of $f(h,\lambda\,h)$. We have

$$\begin{aligned}\frac{\sin\left(h^2(\lambda+1)\right)+h^2}{h\sqrt{\lambda^2+1}\,|h|} &= \frac{h}{\sqrt{\lambda^2+1}|h|}+\frac{\sin\left(h^2(\lambda+1)\right)}{h\sqrt{\lambda^2+1}\,|h|}\\ &= \frac{\operatorname{sgn}(h)}{\sqrt{\lambda^2+1}}+\frac{\sin\left(h^2(\lambda+1)\right)}{h^2(\lambda+1)}\,\frac{h^2(\lambda+1)}{h\sqrt{\lambda^2+1}\,|h|}\\ &= \frac{\operatorname{sgn}(h)}{\sqrt{\lambda^2+1}}+\frac{\sin\left(h^2(\lambda+1)\right)}{h^2(\lambda+1)}\,\frac{\operatorname{sgn}(h)(\lambda+1)}{\sqrt{\lambda^2+1}}\\ &= \operatorname{sgn}(h)\left(\frac{1}{\sqrt{\lambda^2+1}}+\frac{\sin\left(h^2(\lambda+1)\right)}{h^2(\lambda+1)}\,\frac{\lambda+1}{\sqrt{\lambda^2+1}}\right).\end{aligned}$$

Let us know calculate the limit.

$$\begin{aligned}&\lim_{h\to 0}\operatorname{sgn}(h)\left(\frac{1}{\sqrt{\lambda^2+1}}+\frac{\sin\left(h^2(\lambda+1)\right)}{h^2(\lambda+1)}\,\frac{\lambda+1}{\sqrt{\lambda^2+1}}\right)\\ &=\lim_{h\to 0}\operatorname{sgn}(h)\left(\frac{1}{\sqrt{\lambda^2+1}}+\frac{\lambda+1}{\sqrt{\lambda^2+1}}\right)\\ &=\lim_{h\to 0}\operatorname{sgn}(h)\left(\frac{\lambda+2}{\sqrt{\lambda^2+1}}\right)\end{aligned}$$

So these limits do not exist.

We see that these restricted functions have no limits. But if $q(h,k)$ is continuous, all these limit values should be $q(0,0)=0$. So this function $q(h,k)$ is not continuous in $(0,0)$. The function $f(x,y)$ is not differentiable in $(0,0)$.

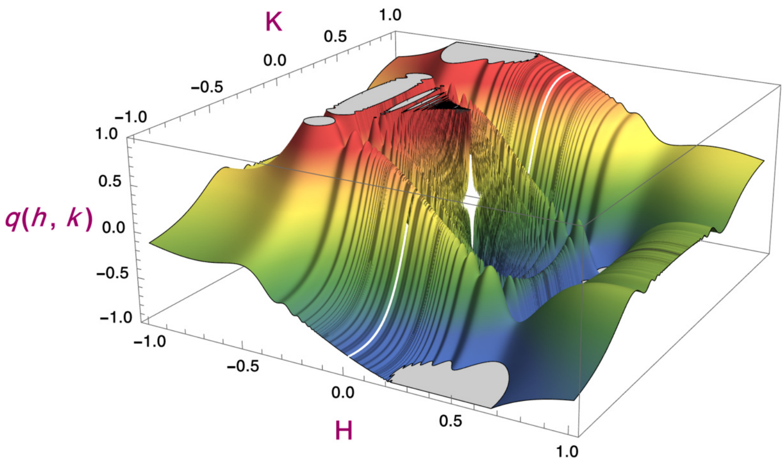

Figure 79. We see here a three dimensional figure of the graph of the function $q(h,k)$. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

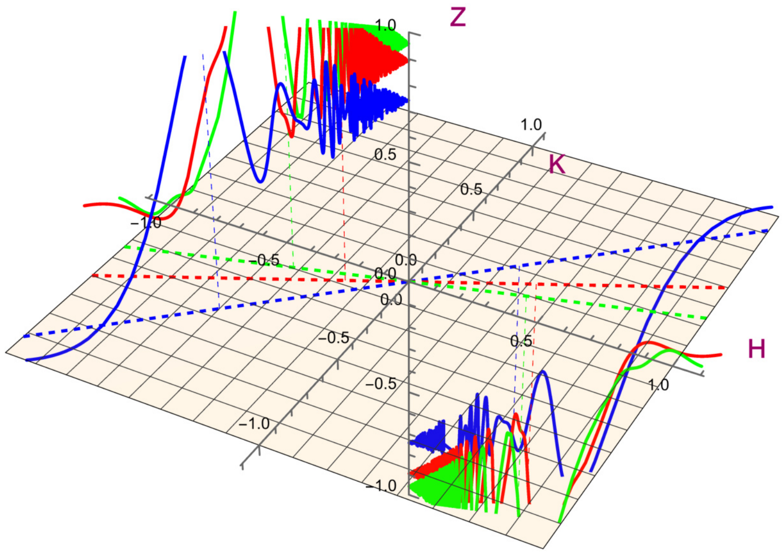

Figure 80. We have restricted the function $q(h,k)$ here to $k = 1/2\,h$ and $k = 3/10\,h$ and $k = 9/10\,h$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

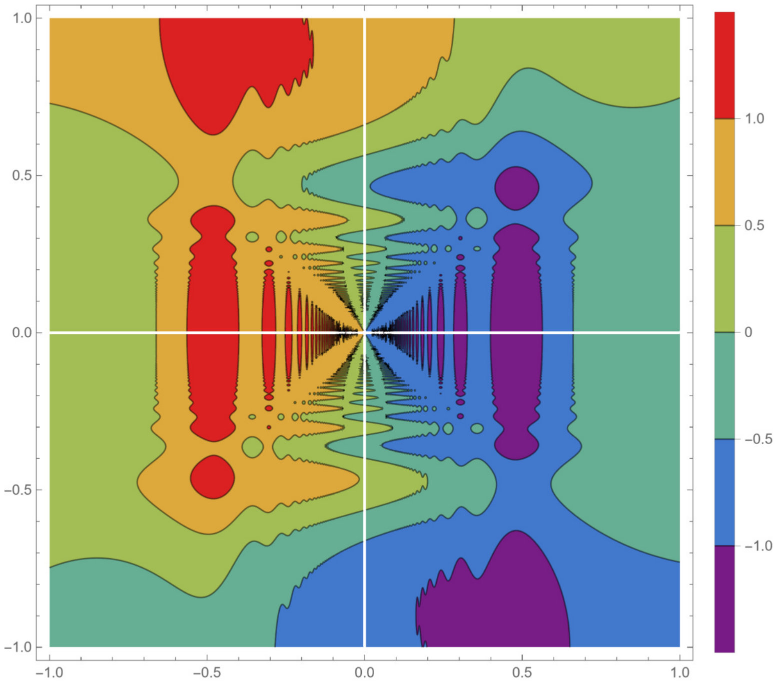

Figure 81. We see here a figure of the contour plot of the function $q(h,k)$. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

8.6 Alternative proof of differentiability (optional)

This is irrelevant. The function is not differentiable.

8.7 Continuity of the partial derivatives

This is irrelevant. The function is not differentiable.

8.8 Overview

$$f(x,y) = \begin{cases} \dfrac{\sin\left(|x\,y| + x^2\right)}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

8.9 One step further

We take a look at the magical curves $k = \lambda h$ that we used in the proof of non differentiability. We take $\lambda = 1$ and then we take the following curve in the plane: $\beta(t) = (t, t)$. This differentiable curve goes through $(0, 0)$.

We define now the curve

$$\alpha(t) = (t, t, f(t)) = (t, \lambda t, f(t, \lambda t)) .$$

This curve lies completely on the surface defined by $z = f(x, y)$. Let us take a look at this curve.

$$\alpha(t) = \begin{cases} \left(t, t, \frac{\sin(t^2+|t|^2)}{t}\right) = \left(t, t, \frac{\sin(2t^2)}{t}\right) & \text{if } t \neq 0, \\ (0,0,0) & \text{if } t = 0. \end{cases}$$

Let us check that this curve is differentiable in $t = 0$. This is necessary. By the non differentiability of $f(x, y)$, we are not guaranteed to have the differentiability of the curve. We take a look at the z-component of the curve. We have $z(t) = \frac{\sin(2t^2)}{t}$ for $t \neq 0$ and $z(0) = 0$. We calculate the derivative $z'(0)$.

$$z'(0) = \lim_{t\to 0} \frac{z(t) - z(0)}{t - 0} = \lim_{t\to 0} \frac{\sin(2t^2)}{t^2} = 2 \lim_{t\to 0} \frac{\sin(2t^2)}{2t^2} = 2.$$

We have now a tangent line of the curve in $t = 0$. Let us put all this information together in the following figure.

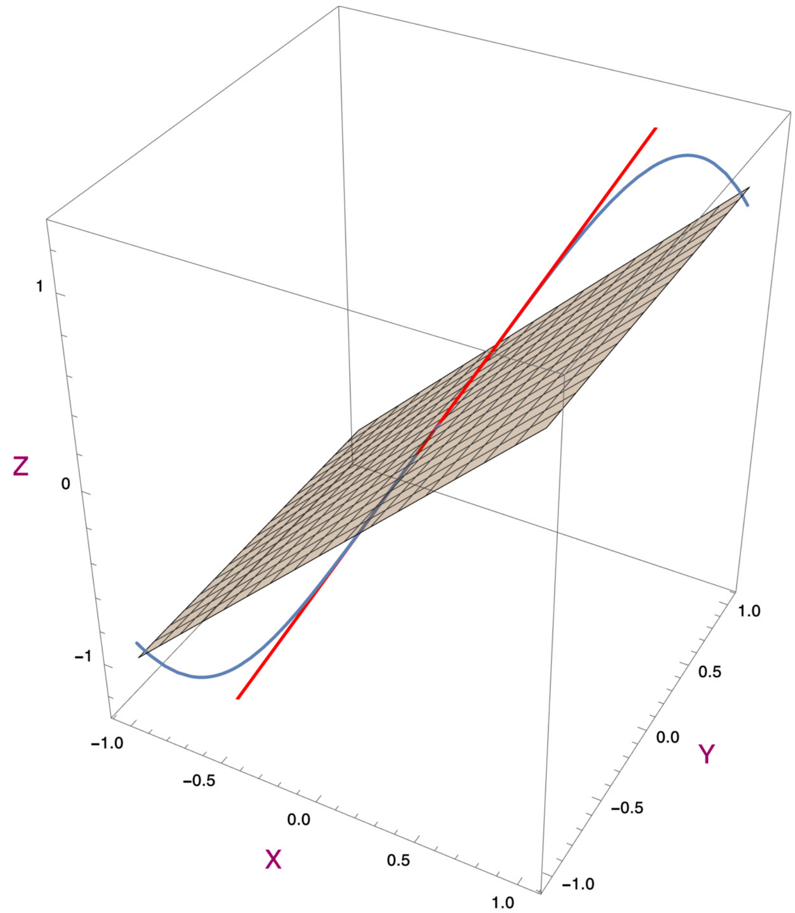

Figure 82. We see here a figure of the candidate tangent plane. The curve $\alpha(t)$ is drawn in blue. The tangent line is drawn in red. We see that the tangent line is not in the candidate tangent plane. It intersects the tangent plane transversally and not tangentially. So the candidate tangent plane is not a tangent plane.

Exercise 9.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

9.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{x^2y^2}{x^2+y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{x^2y^2}{x^2+y^2}\right| &\leq \frac{x^2y^2}{x^2+y^2} \\ &\leq \frac{\sqrt{x^2+y^2}^2\sqrt{x^2+y^2}^2}{\sqrt{x^2+y^2}^2} \\ &\leq \sqrt{x^2+y^2}^2.\end{aligned}$$

It is sufficient to take $\delta = \sqrt{\epsilon}$. We can find a δ, so we conclude that the function is continuous.

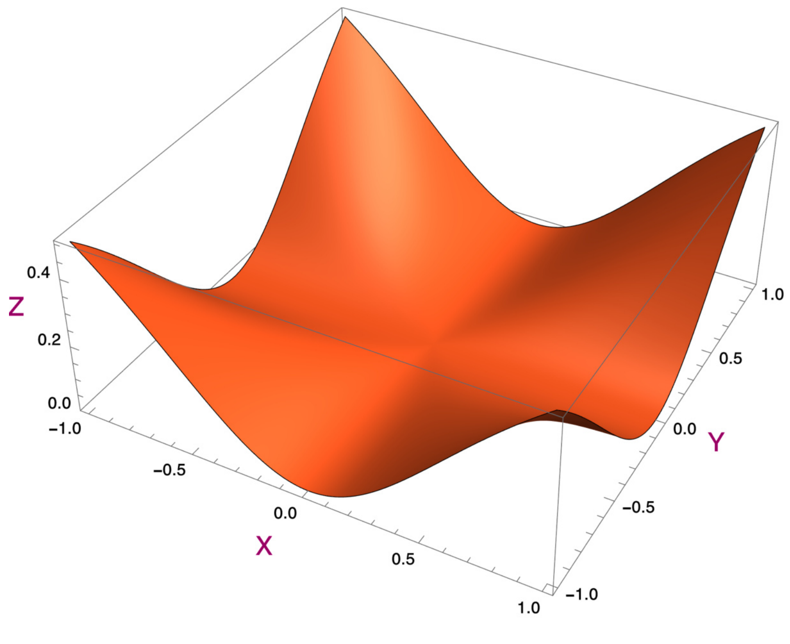

Figure 83. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

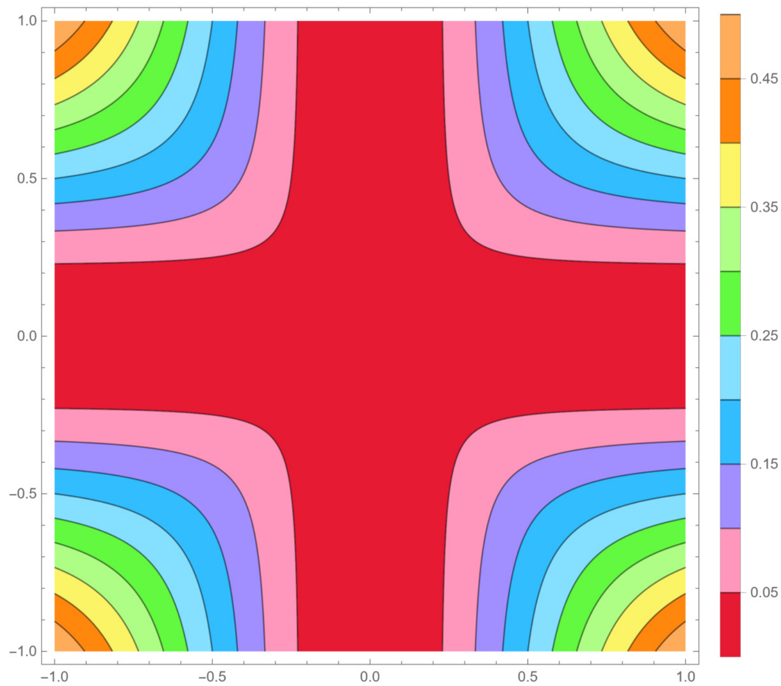

Figure 84. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

9.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).
We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

9.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$
\begin{aligned}
D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{h u^2 v^2}{u^2+v^2} \\
&= 0.
\end{aligned}
$$

So the directional derivatives do always exist.

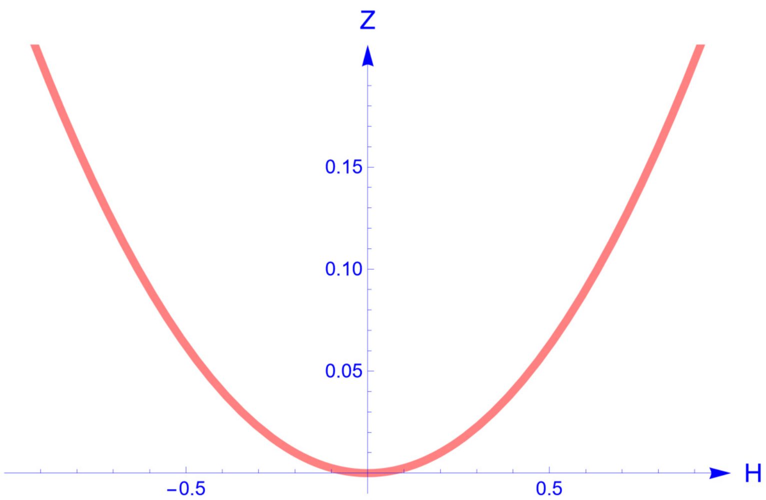

Figure 85. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$. The value of the directional derivative is 0, which is confirmed by this drawing.

9.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we

have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{2\,x\,y^4}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{2\,x^4\,y}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

$$\begin{aligned} \left|\frac{\partial f}{\partial x}(x,y)\right| &\leq \left|\frac{2\,x\,y^4}{(x^2+y^2)^2}\right| \\ &\leq \frac{2\,|x|\,y^4}{(x^2+y^2)^2} \\ &\leq \frac{2\sqrt{x^2+y^2}\,\sqrt{x^2+y^2}^{\,4}}{\sqrt{x^2+y^2}^{\,4}} \\ &\leq \sqrt{x^2+y^2} \\ &\leq 1. \end{aligned}$$

If we choose the neighbourhood $\sqrt{x^2+y^2} < 1$, then we have that the partial derivative is bounded by 1.

We do not calculate this again for $\left|\frac{\partial f}{\partial y}\right|$ by symmetry reasons.

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

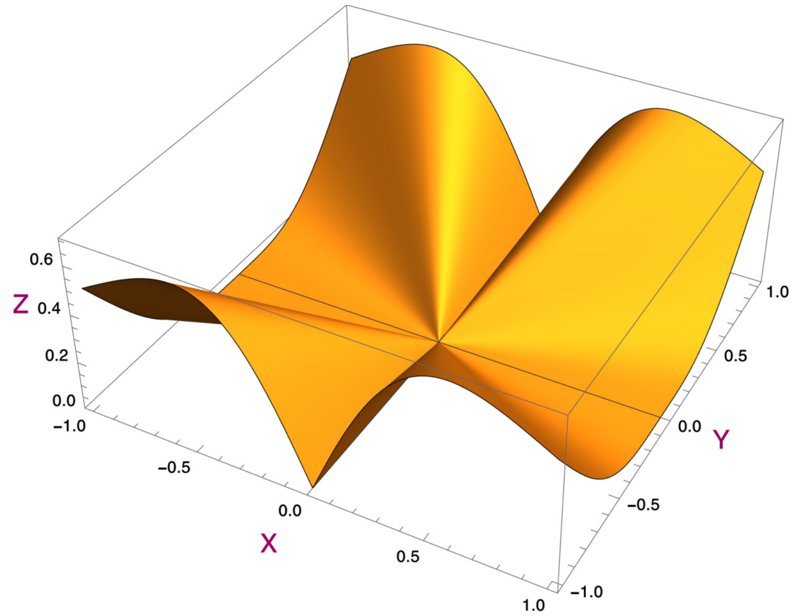

Figure 86. We see here the absolute value of the first partial derivative $\left|\dfrac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

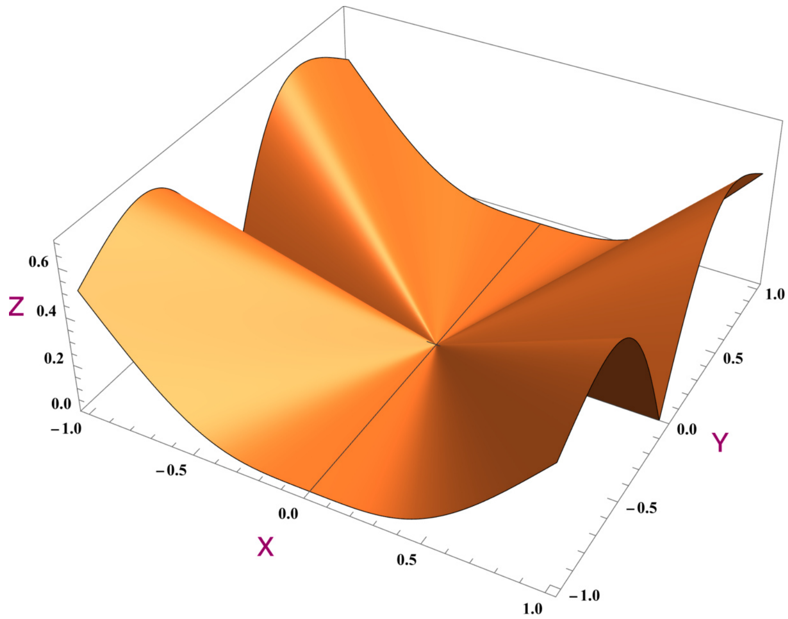

Figure 87. We see here the absolute value of the second partial derivative $\left|\frac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture.

9.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

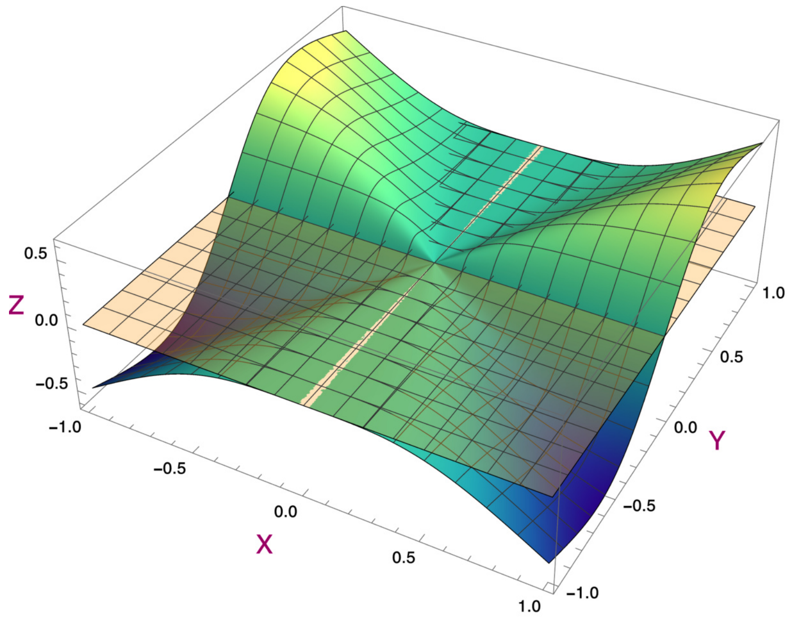

Figure 88. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function quite nicely but some doubts remain. This certainly asks for more calculation al confirmation.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

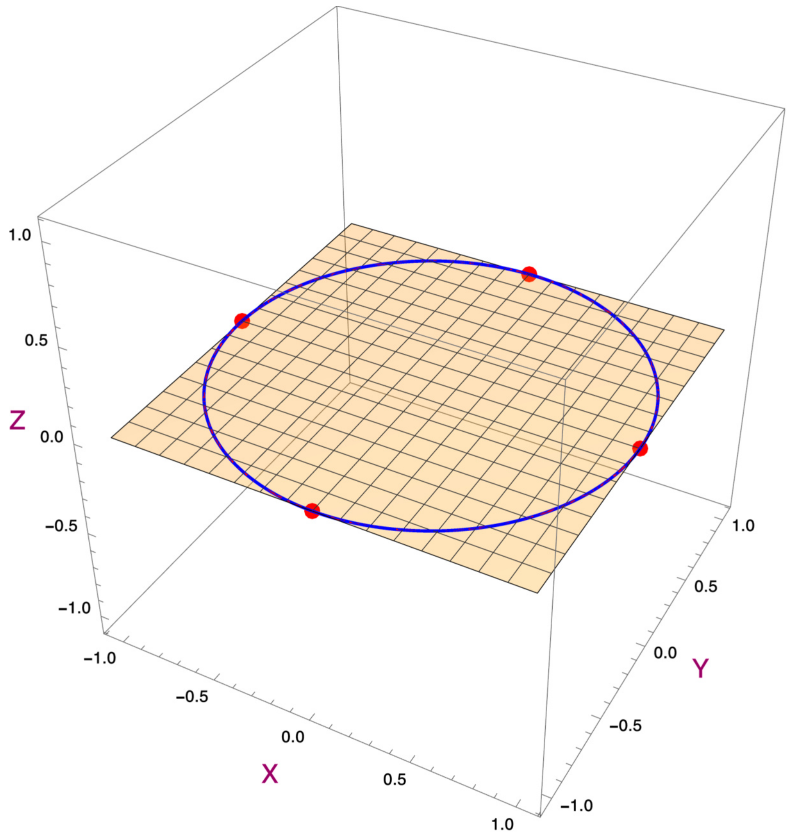

Figure 89. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h) - f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h^2\,k^2}{\left(h^2+k^2\right)^{3/2}} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left| \frac{h^2\,k^2}{(h^2+k^2)^{3/2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{h^2 k^2}{(h^2+k^2)^{3/2}}\right| &\le \frac{h^2 k^2}{(h^2+k^2)^{3/2}}\\ &\le \frac{\sqrt{h^2+k^2}^{\,2}\sqrt{h^2+k^2}^{\,2}}{(h^2+k^2)^{3/2}}\\ &\le \sqrt{h^2+k^2}^{\,5/2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon^{2/5}$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous. The function f is differentiable.

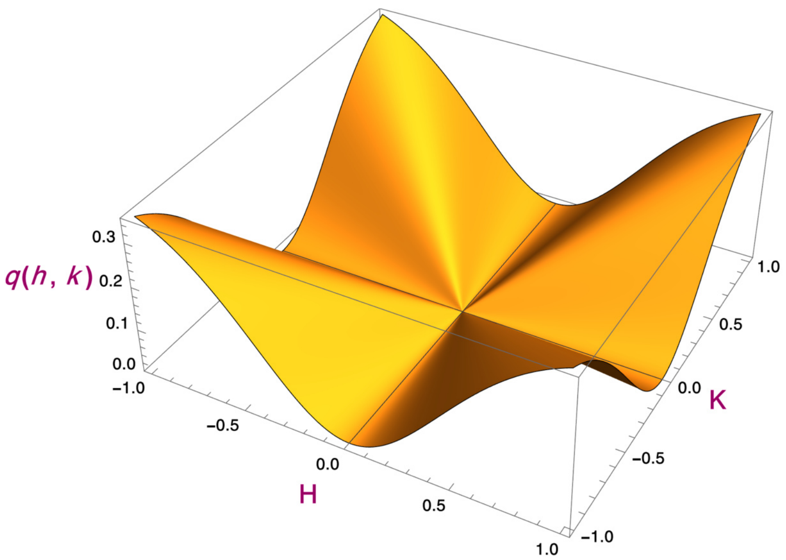

Figure 90. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

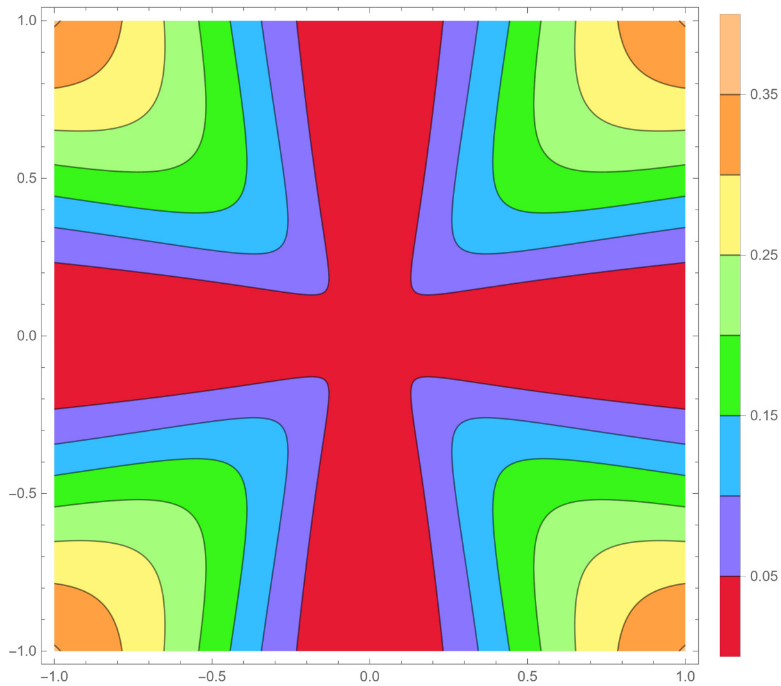

Figure 91. We see here a figure of the contour plot of the function $q(h, k)$. Only level curves of level around 0 come close to $(0, 0)$.

9.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0, 0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0, 0) = \frac{\partial f}{\partial x}(0, 0)\, u + \frac{\partial f}{\partial y}(0, 0)\, v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h, k)$. It turns out that we only have to prove now that the func-

tion $f(x, y)$ is locally Lipschitz continuous in $(0, 0)$. We cite here the criterion that we will use.

A function is differentiable in (a, b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.
2. The directional derivatives have the following form: $D_{(u,v)}(a, b) = \nabla f(a, b) \cdot (u, v) = \frac{\partial f}{\partial x}(a, b)\, u + \frac{\partial f}{\partial y}(a, b)\, v$.
3. The function is locally Lipschitz continuous in (a, b). This means that there exists at least one neighbourhood of (a, b) and a number $K > 0$ such that for all (x_1, y_1) and (x_2, y_2) in the neighbourhood, we have $|f(x_1, y_1) - f(x_2, y_2)| < K \left\|(x_1, y_1) - (x_2, y_2)\right\|$. Remark that the same K must be valid for all (x_1, y_1) and (x_2, y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be “locally flat" and thus “locally linear" which is partly expressed by this Lipschitz continuity condition.

Let us try to prove the Lipschitz condition. We are going to use a well known method.

$$\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)| \\
&\qquad = |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\qquad \leq |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)|.
\end{aligned}$$

We focus now on the first term $|f(x_1, y_1) - f(x_1, y_2)|$. We fix now x_1 and look upon $f(x_1, y)$ as a function in one variable. So it springs to mind that we can estimate this by using Lagrange’s intermediate value theorem. So we have

$$|f(x_1, y_1) - f(x_1, y_2)| = \left| \frac{\partial f}{\partial y}(x_1, \xi) \right| \, |y_1 - y_2|$$

where ξ is a number lying in the open interval (y_1, y_2) if e.g. $y_1 \leq y_2$. Remark that we already have proven that the partial derivatives are bounded in a neighbourhood, we can take that bound, say M_2 and that neighbourhood in this case also. So $\left|\frac{\partial f}{\partial y}(x_1, \xi)\right| \leq M_2$. We work in a completely analogous way for the second term $|f(x_1, y_2) - f(x_2, y_2)|$. We have in that case $\left|\frac{\partial f}{\partial x}(\xi, y_2)\right| \leq M_1$ where ξ is now a number in the open interval (x_1, x_2) if e.g. $x_1 \leq x_2$.

We take up our inequality again

$$
\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)| \\
&\qquad \leq |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\qquad \leq |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)| \\
&\qquad \leq M_2 \, |(y_1 - y_2)| + M_1 \, |(x_1 - x_2)| \\
&\qquad \leq M_2 \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + M_1 \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&\qquad \leq (M_1 + M_2) \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.
\end{aligned}
$$

So this function is locally Lipschitz continuous. We have an alternative proof for the differentiability.

9.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

We know that the partial derivative to x exists and is equal to

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{2\,x\,y^4}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We want to see if it is continuous or not.

Discussion of the continuity of the first partial derivative in $(0,0)$.
We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove the inequality $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if the inequality $||(x,y)-(0,0)|| < \delta$ holds, it follows that $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned} \left|\frac{\partial f}{\partial x}(x,y)\right| &\leq \left|\frac{2\,x\,y^4}{(x^2+y^2)^2}\right| \\ &\leq \frac{2\sqrt{x^2+y^2}\sqrt{x^2+y^2}^4}{\sqrt{x^2+y^2}^4} \\ &\leq \sqrt{x^2+y^2}. \end{aligned}$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function $\frac{\partial f}{\partial x}$ is continuous.

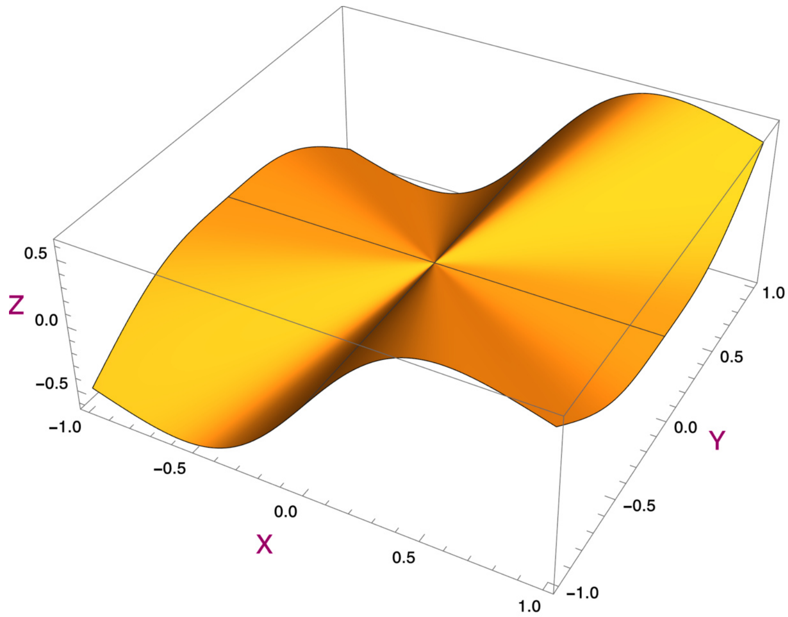

Figure 92. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial x}(x, y)$. This looks like a continuous function.

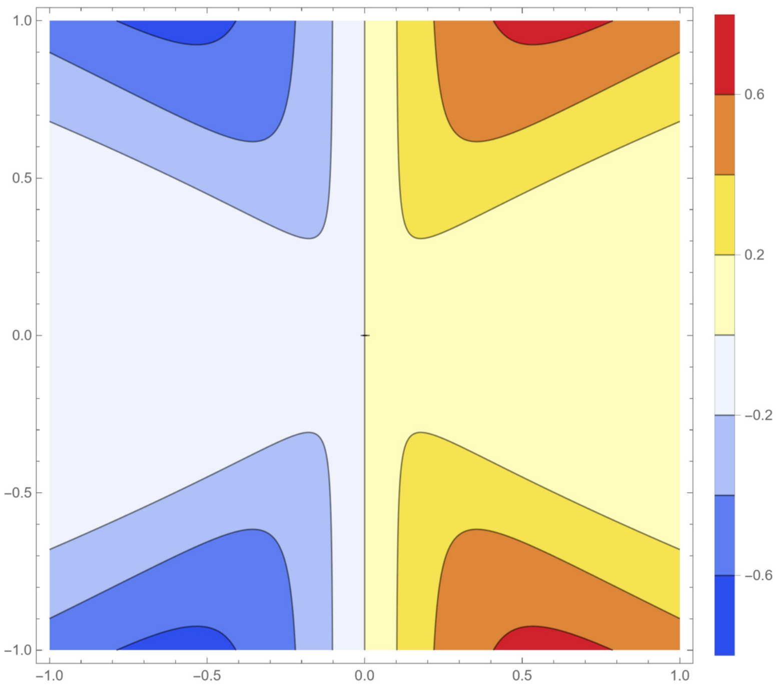

Figure 93. We see here a figure of the contour plot of the $\frac{\partial f}{\partial x}(x, y)$. Only level curves of level around 0 come close to $(0, 0)$.

Discussion of the continuity of the second partial derivative in $(0, 0)$.

We will not calculate this because by symmetry reasons, we would have an analogue calculation.

9.8 Overview

$$\begin{cases} \dfrac{x^2 y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	yes

9.9 One step further

We want to know if this function is further uneventful from the point of view of differentiability. Let us take a look at the second order partial derivative

$$\frac{\partial^2 f}{\partial x^2} = \frac{2\,y^4\,\left(y^2 - 3\,x^2\right)}{\left(x^2 + y^2\right)^3}.$$

Let us take a look of a three dimensional plot of this second order partial derivative of the function.

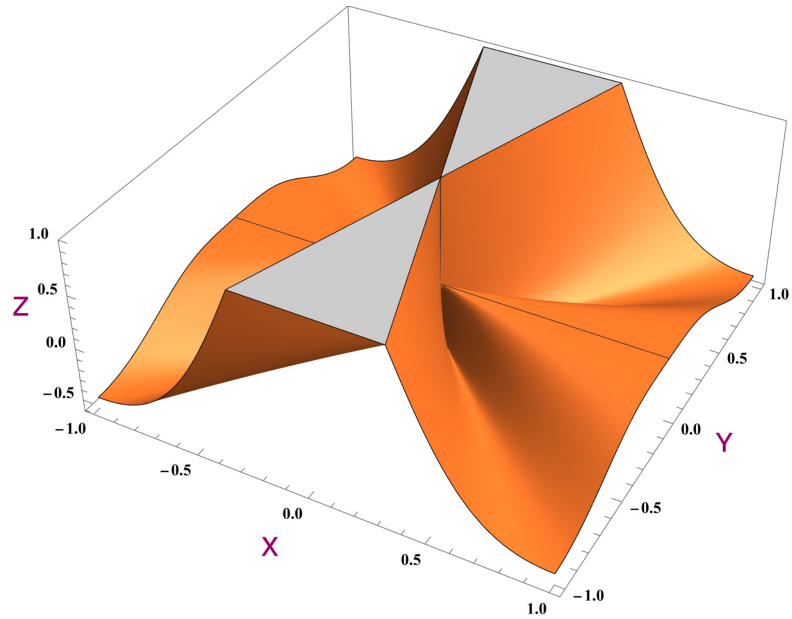

Figure 94. We see here a figure of the second order partial derivative $\frac{\partial^2 f}{\partial x^2}(x, y)$. It seems quite improbable that this second order partial derivative is continuous. We stop however our investigations here and leave this to the initiative of the interested reader.

Exercise 10.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{x\,y^4}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

10.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{x\,y^4}{x^2+y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\left|\frac{x\,y^4}{x^2+y^2}\right| \le \frac{\sqrt{x^2+y^2}\,\sqrt{x^2+y^2}^{\,4}}{\sqrt{x^2+y^2}^{\,2}}$$

$$\le \sqrt{x^2+y^2}^{\,3}.$$

It is sufficient to take $\delta = \epsilon^{1/3}$. We can find a δ, so we conclude that the function is continuous.

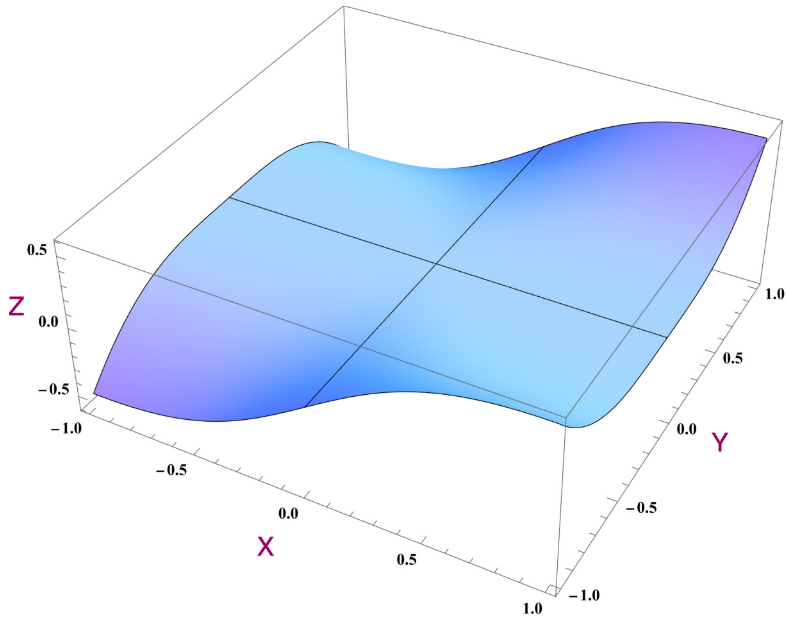

Figure 95. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

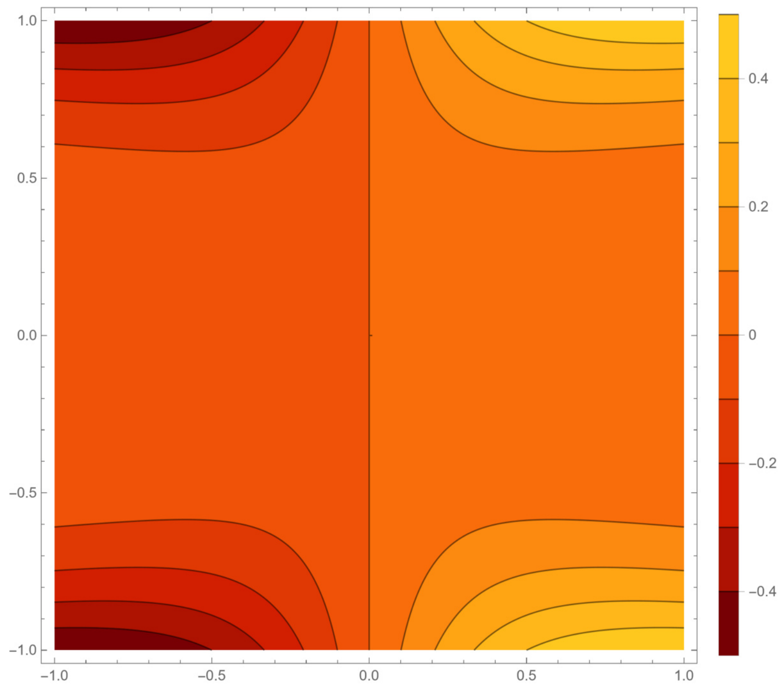

Figure 96. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

10.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).
We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

10.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$
\begin{aligned}
D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{h^2 u v^4}{u^2+v^2} \\
&= 0.
\end{aligned}
$$

So the directional derivatives do always exist.

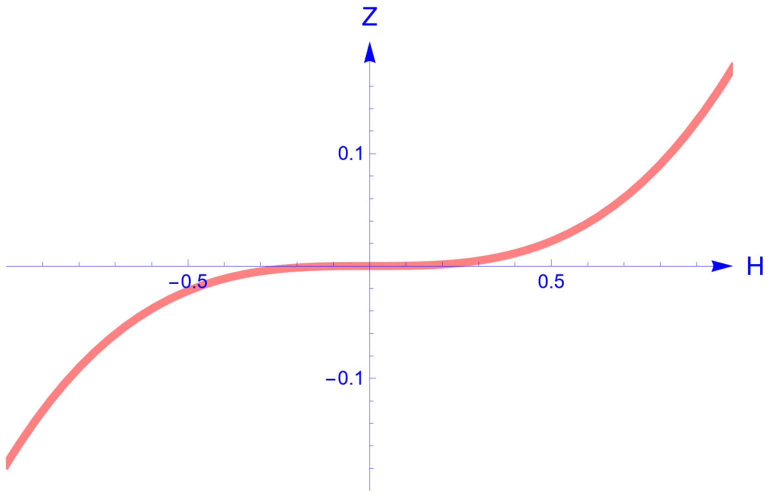

Figure 97. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$. The slope is evidently 0 and confirms the calculation.

10.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y)=\begin{cases}\dfrac{y^4\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y)=\begin{cases}\dfrac{2x\left(2x^2y^3+y^5\right)}{\left(x^2+y^2\right)^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

$$
\begin{aligned}
\left|\frac{\partial f}{\partial x}(x,y)\right| &\le \left|\frac{y^4\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}\right| \\
&\le \frac{y^4\left(y^2+x^2\right)}{\left(x^2+y^2\right)^2} \\
&\le \frac{\sqrt{x^2+y^2}^4\left(\sqrt{x^2+y^2}^2+\sqrt{x^2+y^2}^2\right)}{\sqrt{x^2+y^2}^4} \\
&\le \frac{\sqrt{x^2+y^2}^4\,2\sqrt{x^2+y^2}^2}{\sqrt{x^2+y^2}^4} \\
&\le 2\,\frac{\sqrt{x^2+y^2}^6}{\sqrt{x^2+y^2}^4} \\
&\le 2\sqrt{x^2+y^2}^2 \\
&\le 2.
\end{aligned}
$$

We have chosen here the restriction to the neighbourhood defined by $\sqrt{x^2+y^2}<1$.

Let us try to prove that $\left|\frac{\partial f}{\partial y}\right|$ is bounded.

$$\begin{aligned}
\left|\frac{\partial f}{\partial y}(x,y)\right| &\le \left|\frac{2\,x\;(2\,x^2\,y^3+y^5)}{(x^2+y^2)^2}\right| \\
&\le \frac{2\,|x|\;(2\,x^2\,|y|^3+|y|^5)}{(x^2+y^2)^2} \\
&\le \frac{2\sqrt{x^2+y^2}\,\left(2\sqrt{x^2+y^2}^{\,2}\sqrt{x^2+y^2}^{\,3}+\sqrt{x^2+y^2}^{\,5}\right)}{\sqrt{x^2+y^2}^{\,4}} \\
&\le \frac{6\sqrt{x^2+y^2}^{\,6}}{\sqrt{x^2+y^2}^{\,4}} \\
&\le 6\sqrt{x^2+y^2}^{\,2} \\
&\le 6.
\end{aligned}$$

We have chosen here again the restriction to the neighbourhood defined by $\sqrt{x^2+y^2} < 1$.

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

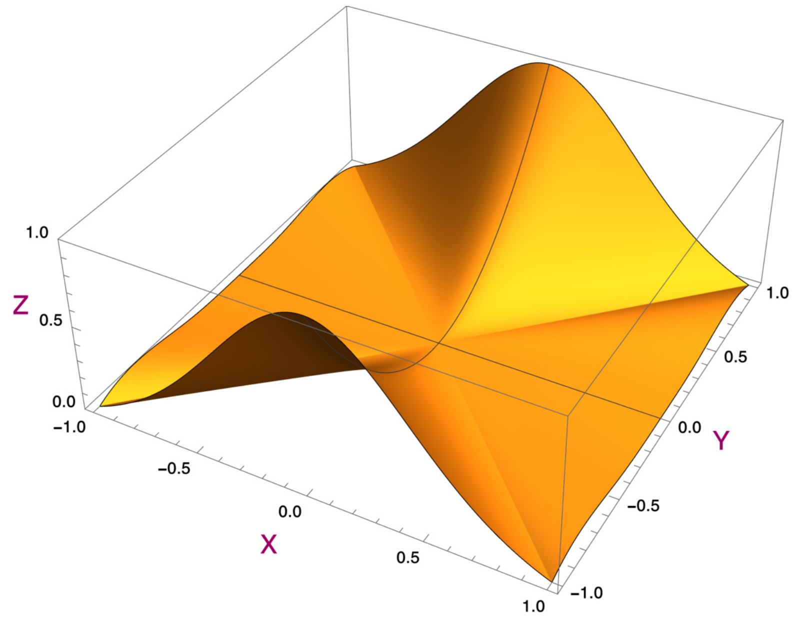

Figure 98. We see here the absolute value of the first partial derivative $\left|\dfrac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

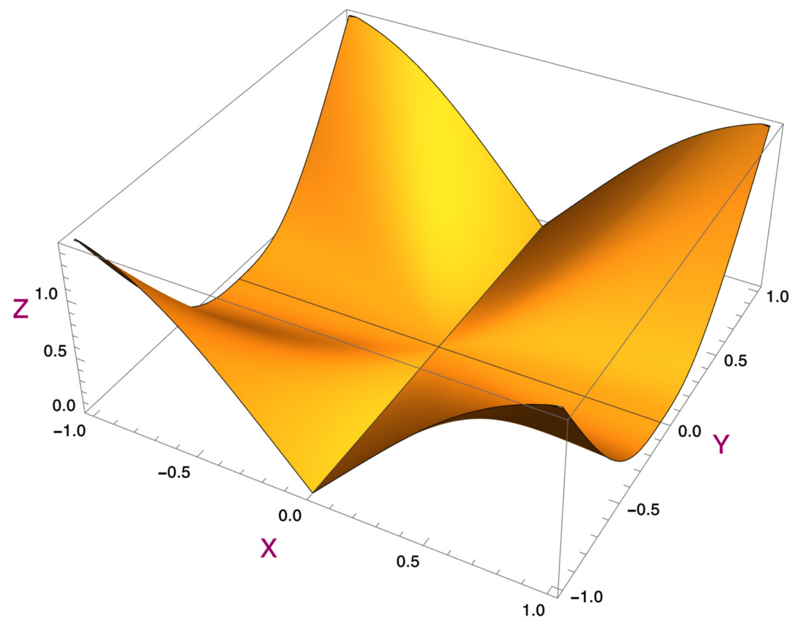

Figure 99. We see here the absolute value of the second partial derivative $\left|\dfrac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture.

10.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

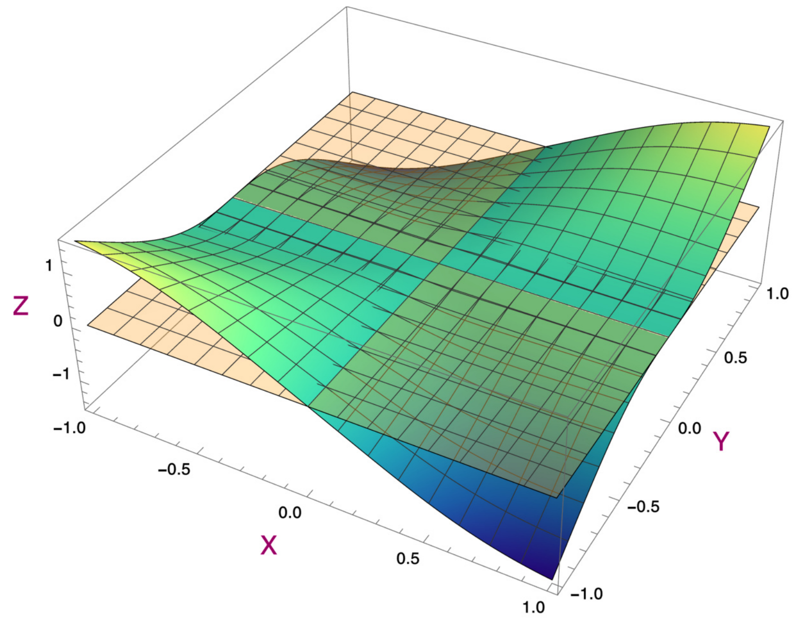

Figure 100. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0, 0)\, x + \frac{\partial f}{\partial y}(0, 0)\, y$, which is graphically the candidate tangent plane and the function $f(x, y)$. We see here that the candidate tangent plane fits the function very nicely.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we

know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

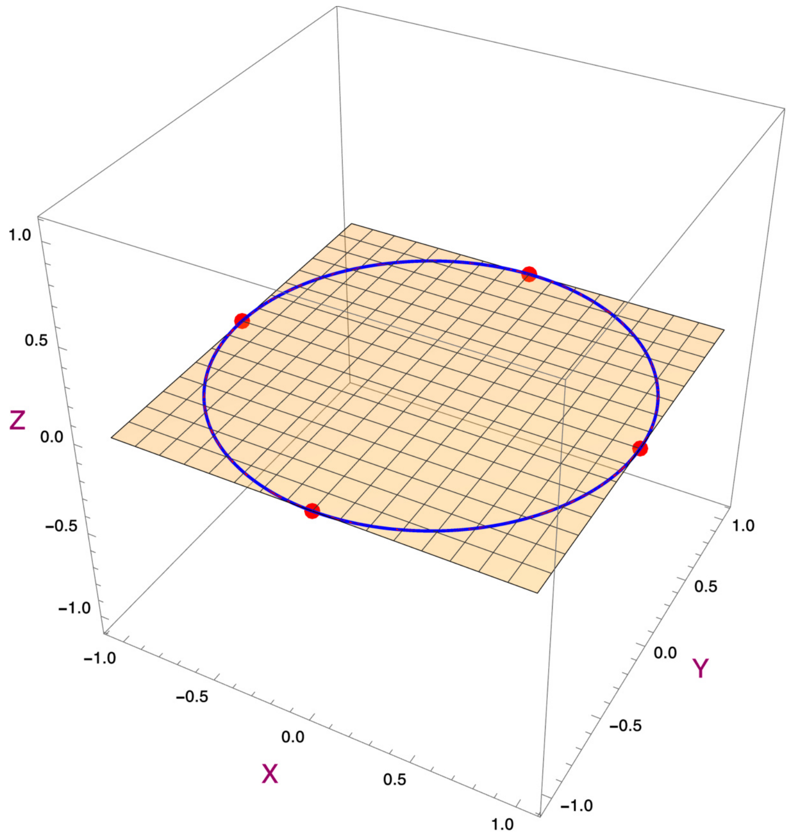

Figure 101. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h\,k^4}{\left(h^2+k^2\right)^{3/2}} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left| \frac{h\,k^4}{(h^2+k^2)^{3/2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{h\,k^4}{(h^2+k^2)^{3/2}}\right| &\le \frac{|h|\,k^4}{(h^2+k^2)^{3/2}}\\ &\le \frac{\sqrt{h^2+k^2}\,\sqrt{h^2+k^2}^4}{\sqrt{h^2+k^2}^3}\\ &\le \sqrt{h^2+k^2}^2.\end{aligned}$$

It is sufficient to take $\delta = \epsilon^{1/2}$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous. So the function f is differentiable.

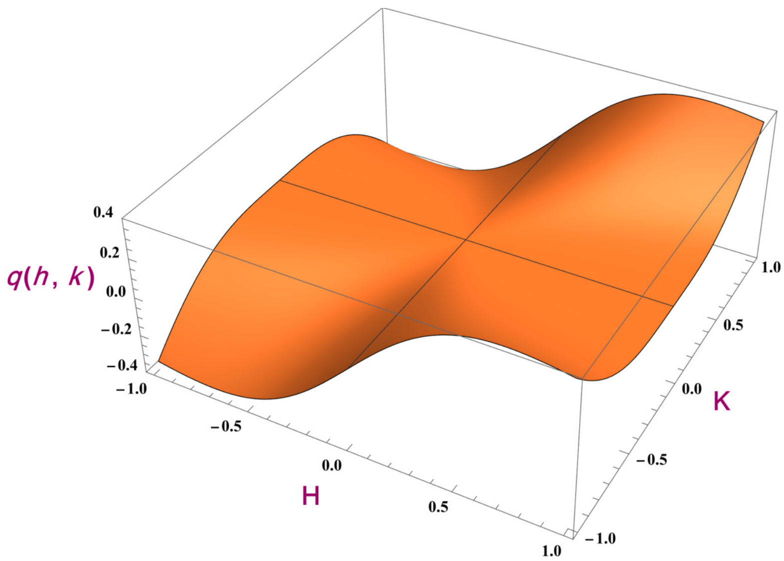

Figure 102. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

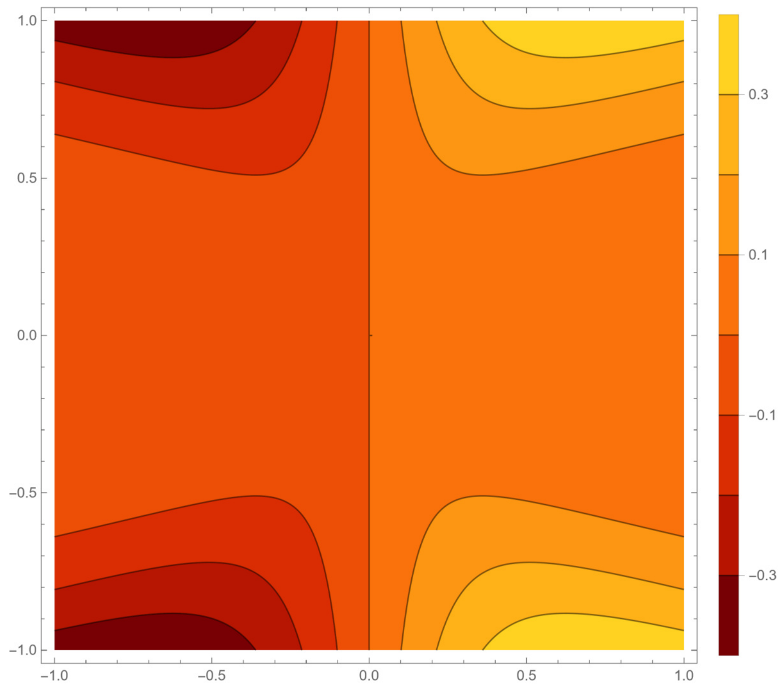

Figure 103. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

10.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\, u + \frac{\partial f}{\partial y}(0,0)\, v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the func-

tion $f(x, y)$ is locally Lipschitz continuous in $(0, 0)$. We cite here the criterion that we will use.

A function is differentiable in (a, b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.

2. The directional derivatives have the following form: $D_{(u,v)}(a, b) = \nabla f(a, b) \cdot (u, v) = \frac{\partial f}{\partial x}(a, b)\, u + \frac{\partial f}{\partial y}(a, b)\, v$.

3. The function is locally Lipschitz continuous in (a, b). This means that there exists at least one neighbourhood of (a, b) and a number $K > 0$ such that for all (x_1, y_1) and (x_2, y_2) in the neighbourhood, we have $|f(x_1, y_1) - f(x_2, y_2)| < K \left\|(x_1, y_1) - (x_2, y_2)\right\|$. Remark that the same K must be valid for all (x_1, y_1) and (x_2, y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be "locally flat" and thus "locally linear" which is partly expressed by this Lipschitz continuity condition.

Let us try to prove the Lipschitz condition. We are going to use a well known method.

$$
\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)| \\
&\qquad = |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\qquad \le |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)|.
\end{aligned}
$$

We focus now on the first term $|f(x_1, y_1) - f(x_1, y_2)|$. We fix now x_1 and look upon $f(x_1, y)$ as a function in one variable y. So it springs to mind that we can estimate this by using Lagrange's intermediate value theorem in one variable. So we have

$$
|f(x_1, y_1) - f(x_1, y_2)| = \left|\frac{\partial f}{\partial y}(x_1, \xi)\right| \; |y_1 - y_2|
$$

where ξ is a number lying in the open interval (y_1, y_2) if e.g. $y_1 \leq y_2$. Remark that we already have proven that the partial derivatives are bounded in a neighbourhood, we can take that bound found for $\frac{\partial f}{\partial y}$, say M_2 and that neighbourhood in this case also. So $\left|\frac{\partial f}{\partial y}(x_1, \xi)\right| \leq M_2$. We work in a completely analogous way for the second term $|f(x_1, y_2) - f(x_2, y_2)|$. We have in that case $|\frac{\partial f}{\partial x}(\xi, y_2)| \leq M_1|$ where ξ is now a number in the open interval (x_1, x_2) if e.g. $x_1 \leq x_2$.

We take up our inequality again

$$
\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)| \\
&\quad \leq |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\quad \leq |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)| \\
&\quad \leq M_2 \,|(y_1 - y_2)| + M_1 \,|(x_1 - x_2)| \\
&\quad \leq M_2 \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + M_1 \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&\quad \leq (M_1 + M_2) \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.
\end{aligned}
$$

So this function is locally Lipschitz continuous. We have an alternative proof for the differentiability.

10.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

We know that the partial derivative to x exists and is equal to

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{y^4\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We want to see if it is continuous or not.

Discussion of the continuity of the first partial derivative in $(0,0)$. We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove the inequality $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if the inequality $||(x,y)-(0,0)|| < \delta$ holds it follows that $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof. We repeat what we have calculated before.

$$\begin{aligned}
\left|\frac{\partial f}{\partial x}(x,y)\right| &\leq \left|\frac{y^4\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}\right| \\
&\leq \frac{y^4\left(y^2+x^2\right)}{\left(x^2+y^2\right)^2} \\
&\leq \frac{\sqrt{x^2+y^2}^4\left(\sqrt{x^2+y^2}^2+\sqrt{x^2+y^2}^2\right)}{\sqrt{x^2+y^2}^4} \\
&\leq \frac{\sqrt{x^2+y^2}^4\, 2\sqrt{x^2+y^2}^2}{\sqrt{x^2+y^2}^4} \\
&\leq 2\,\frac{\sqrt{x^2+y^2}^6}{\sqrt{x^2+y^2}^4} \\
&\leq 2\sqrt{x^2+y^2}^2.
\end{aligned}$$

It is sufficient to take $\delta = (\epsilon/2)^{1/2}$. We can find a δ, so we conclude that the function is continuous.

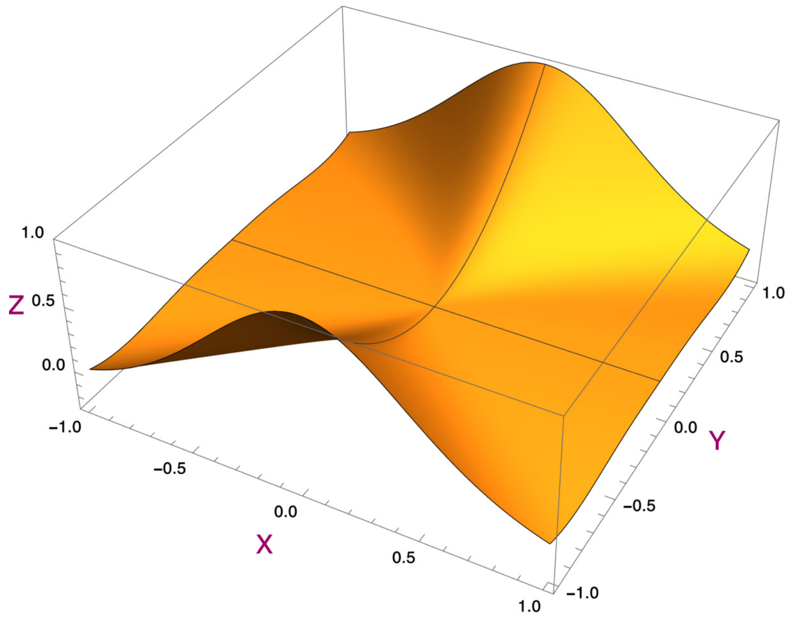

Figure 104. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial x}(x, y)$. This looks like a continuous function.

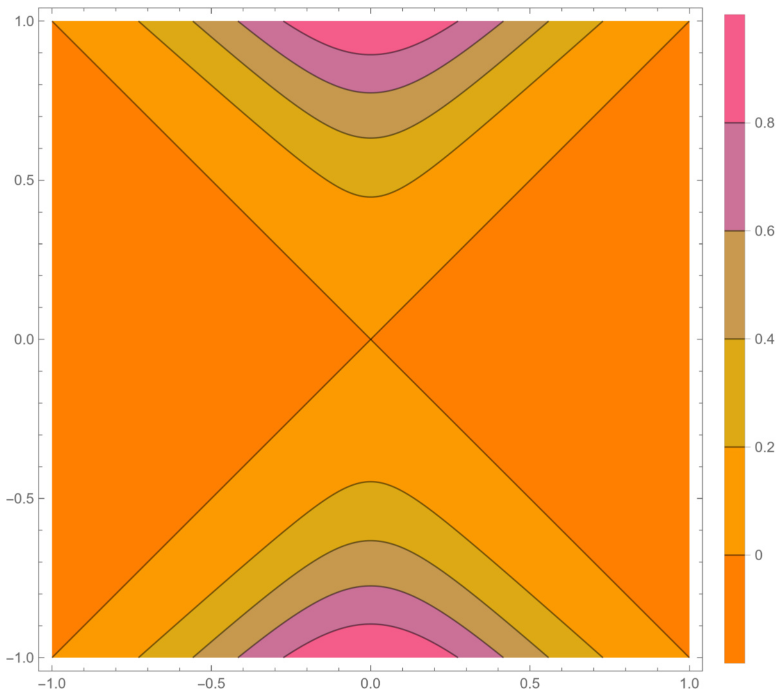

Figure 105. We see here a figure of the contour plot of the $\frac{\partial f}{\partial x}(x,y)$. Only level curves of level around 0 come close to $(0,0)$.

Discussion of the continuity of the second partial derivative in $(0,0)$.

We know that the partial derivative to y exists and is equal to

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{2x\left(2x^2y^3+y^5\right)}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We want to see if $\frac{\partial f}{\partial y}$ is continuous or not.

Discussion of the continuity in $(0,0)$.
We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove the inequality $|\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if the inequality $||(x,y)-(0,0)|| < \delta$ holds it follows that $|\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0, 0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{\partial f}{\partial y}(x, y) - \frac{\partial f}{\partial y}(0, 0) \right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{\partial f}{\partial y}(x, y) \right| &\leq \left| \frac{2x\ (2x^2y^3 + y^5)}{(x^2 + y^2)^2} \right| \\
&\leq \frac{2\,|x|\ (2x^2\,|y|^3 + |y|^5)}{(x^2 + y^2)^2} \\
&\leq \frac{2\sqrt{x^2 + y^2}\left(2\sqrt{x^2 + y^2}^2\sqrt{x^2 + y^2}^3 + \sqrt{x^2 + y^2}^5\right)}{\sqrt{x^2 + y^2}^4} \\
&\leq \frac{6\sqrt{x^2 + y^2}^6}{\sqrt{x^2 + y^2}^4} \\
&\leq 6\sqrt{x^2 + y^2}^2.
\end{aligned}$$

It is sufficient to take $\delta = (\epsilon/6)^{1/2}$. We can find a δ, so we conclude that the function is continuous.

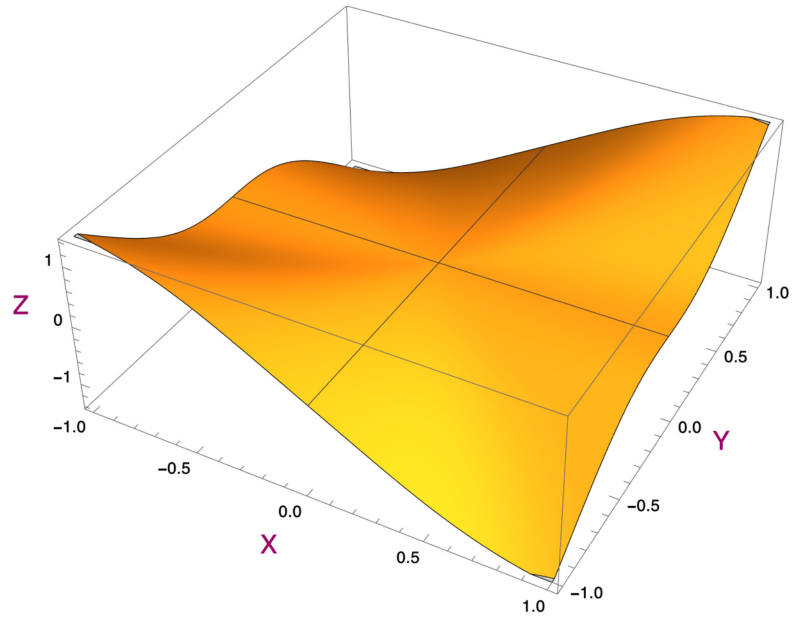

Figure 106. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial y}(x, y)$. This looks like a continuous function.

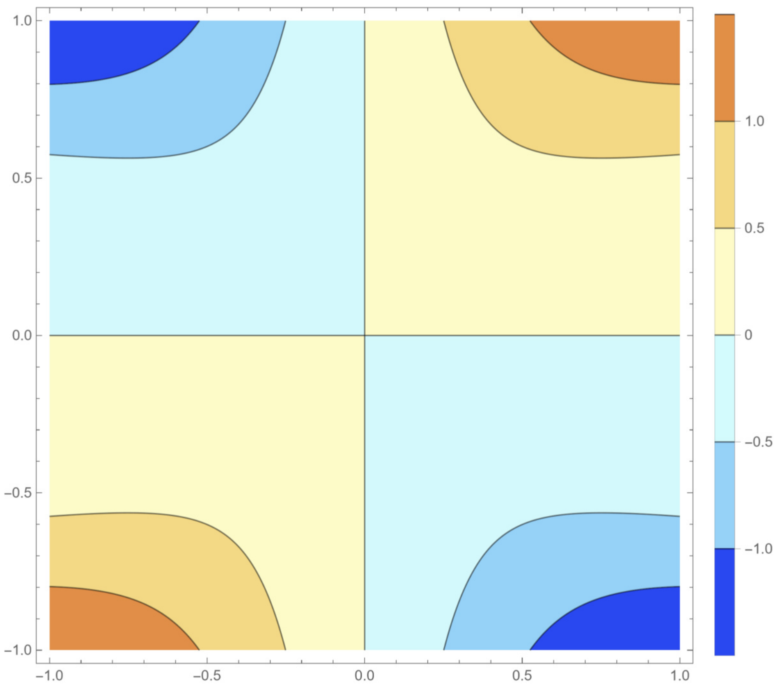

Figure 107. We see here a figure of the contour plot of the graph of the function $\frac{\partial f}{\partial y}(x, y)$. Only level curves of level around 0 come close to $(0, 0)$.

10.8 Overview

$$f(x, y) = \begin{cases} \dfrac{x\,y^4}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	yes

10.9 One step further

We want to know if this function is further uneventful from the point of view of differentiability. Let us take a look at the third order partial derivative

$$\frac{\partial^3 f}{\partial y^3} = \frac{24x^5 y\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^4}.$$

Let us take a look of a three dimensional plot of this third order partial derivative of the function.

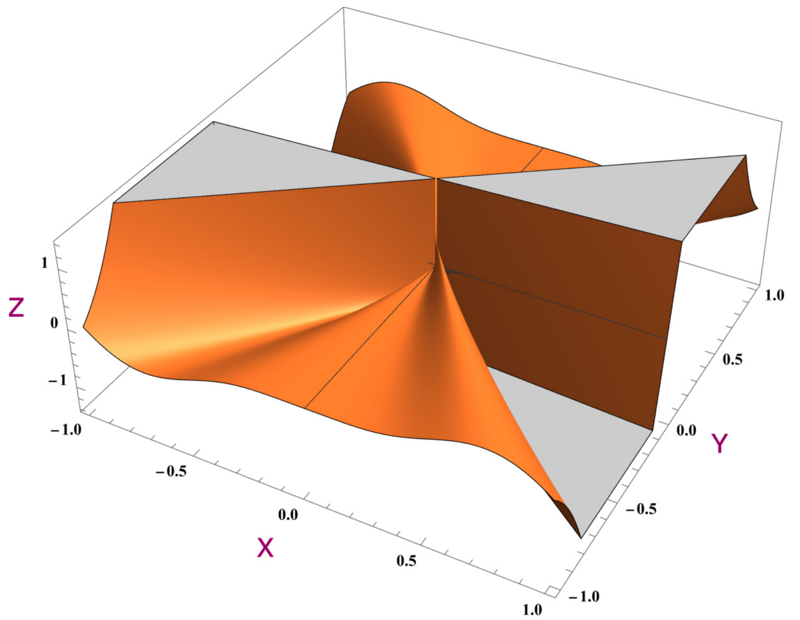

Figure 108. We see here a figure of the third order partial derivative $\frac{\partial^3 f}{\partial y^3}(x, y)$. It seems quite improbable that this third order partial derivative is continuous. We stop however our investigations here and leave this to the initiative of the interested reader.

Exercise 11.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

11.1 Continuity

We restrict the function to the continuous curves with equations $x = \lambda y^2$. We observe then that

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(\lambda x^2, x) = \dfrac{\lambda}{\lambda^2+1} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

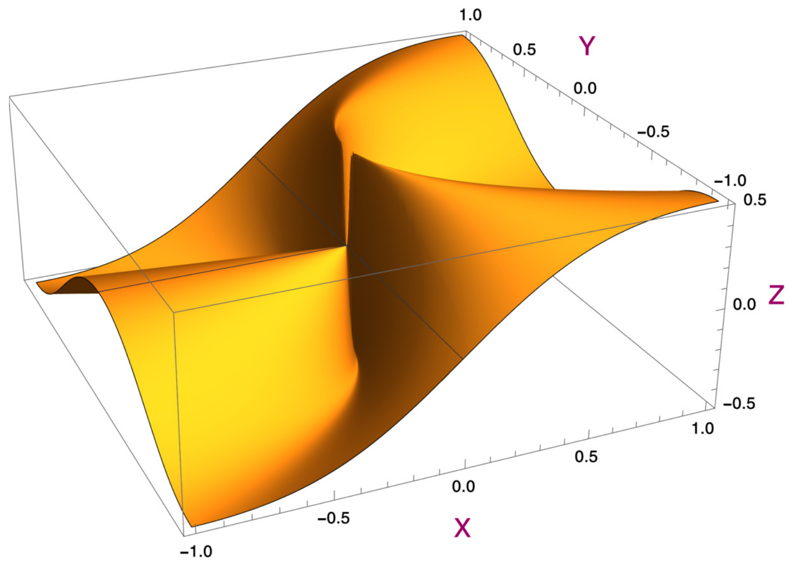

Figure 109. We see here a three dimensional figure of the graph of the function. The vertical line above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

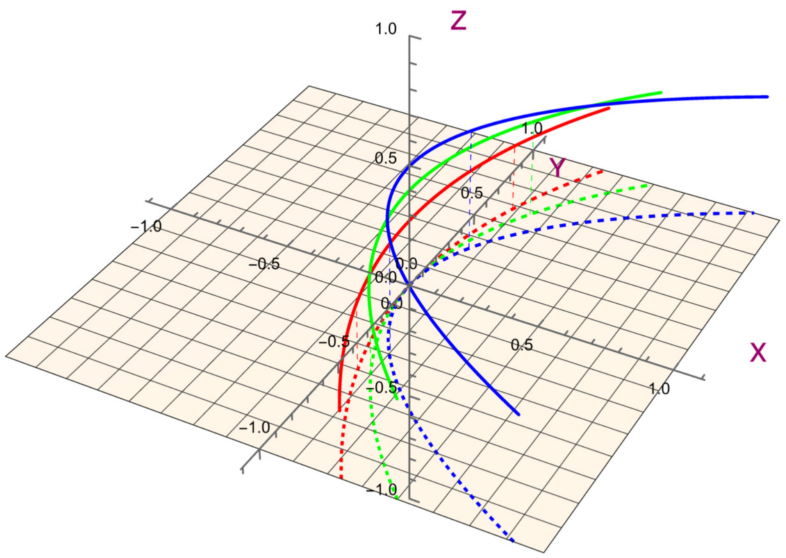

Figure 110. We have restricted the function here to $x = 3/10\,y^2$ and $x = 1/2\,y^2$ and $x = 9/10\,y^2$. We see in this figure clearly that the restrictions of the function to these continuous curves are functions that have different limits in 0.

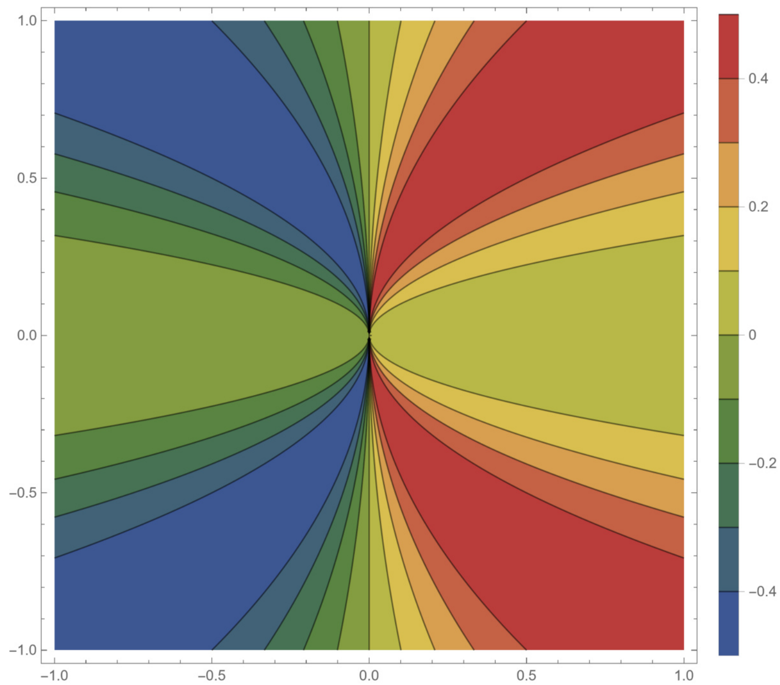

Figure 111. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

11.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0.\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

11.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{u\,v^2}{h^2\,v^4 + u^2} \\ &= \frac{u\,v^2}{u^2} \\ &= \frac{v^2}{u}. \end{aligned}$$

This calculation is only valid if $u \neq 0$. Remember that we already calculated the case $u = 0$. So the directional derivatives do always exist.

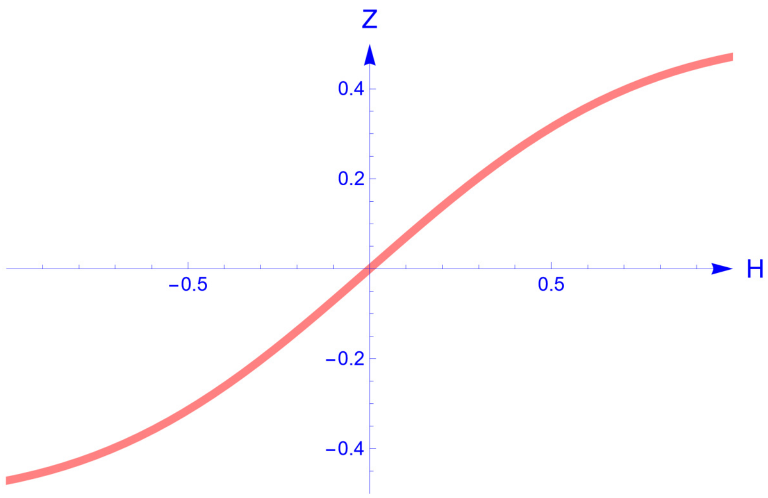

Figure 112. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

11.4 Alternative proof of continuity (optional)

Irrelevant. The function is not continuous.

11.5 Differentiability

Irrelevant. The function is not differentiable because it is not continuous.

11.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

11.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

11.8 Overview

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	no
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

Exercise 12.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y)=\begin{cases}\dfrac{y\left(x^2-y\right)^2}{x^6} & \text{if } x\neq 0;\\ 0 & \text{if } x=0.\end{cases}$$

12.1 Continuity

We restrict the function to the continuous curves with equations $y = \lambda x^2$. We observe then that

$$f|_{y=\lambda x^2}(x,y)=\begin{cases}f(x,\lambda x^2)=\dfrac{\lambda\left(x^2-\lambda x^2\right)^2}{x^4}=(\lambda-1)^2\lambda & \text{if } x\neq 0;\\ 0 & \text{if } x=0.\end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0)=0$. So this function $f(x,y)$ is not continuous.

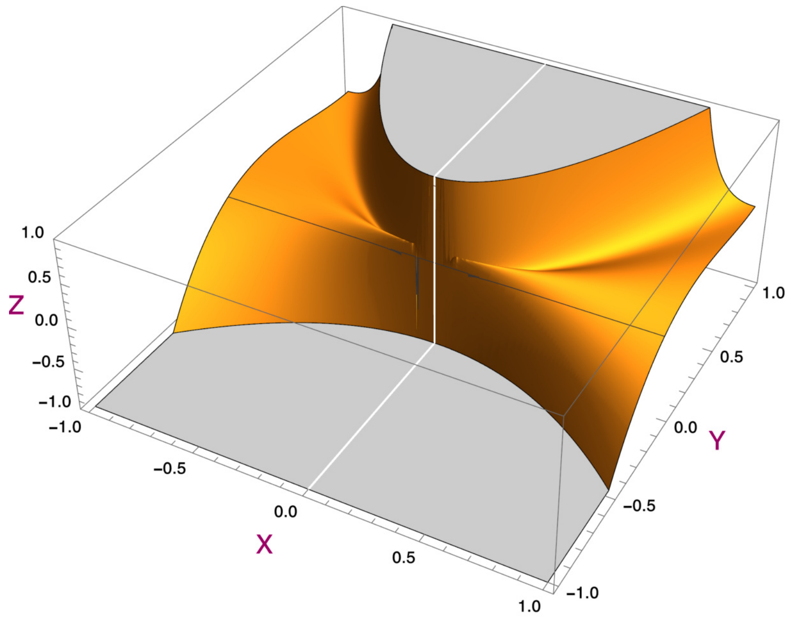

Figure 113. We see here a three dimensional figure of the graph of the function. The vertical line above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

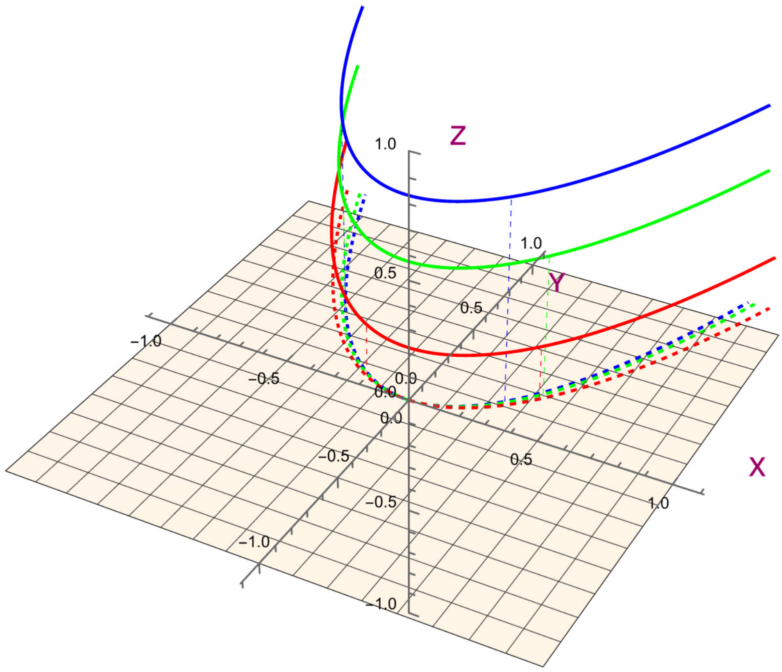

Figure 114. We have restricted the function here to $y = 1/10\,x^2$ and $y = 3/10\,x^2$ and $y = 5/10\,x^2$. We see in this figure clearly that the restrictions of the function to these continuous curves are functions that have different limits in 0.

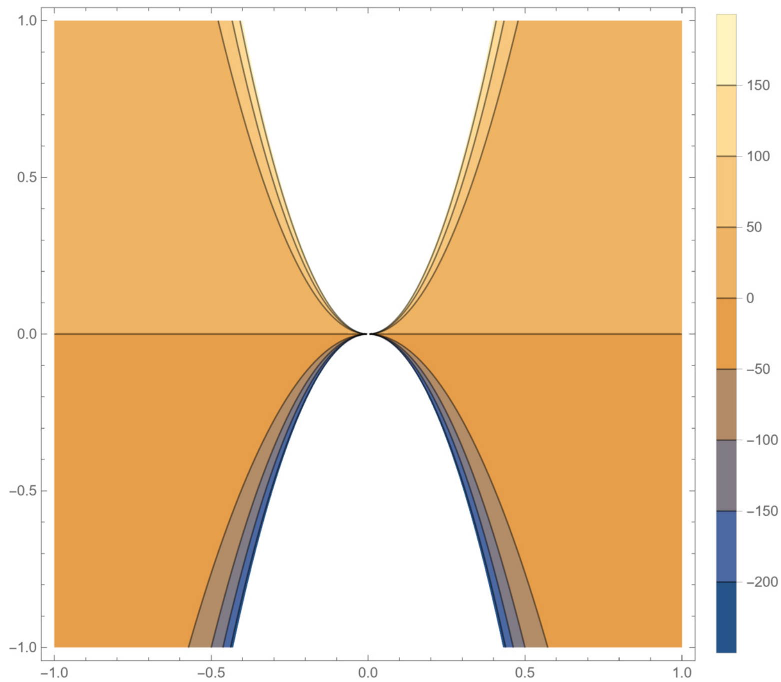

Figure 115. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

12.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

12.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following function definition if $u \neq 0$

$$f(0+h\,u, 0+h\,v) = \begin{cases} \dfrac{v\left(v - h\,u^2\right)^2}{h^4\,u^6} & \text{if } h \neq 0, \\ 0 & \text{if } h = 0. \end{cases}$$

We covered the case $u = 0$ before. We see that this function is not continuous. So it is not differentiable. And the directional derivatives do not exist if $u \neq 0$.

So the directional derivatives do not all exist.

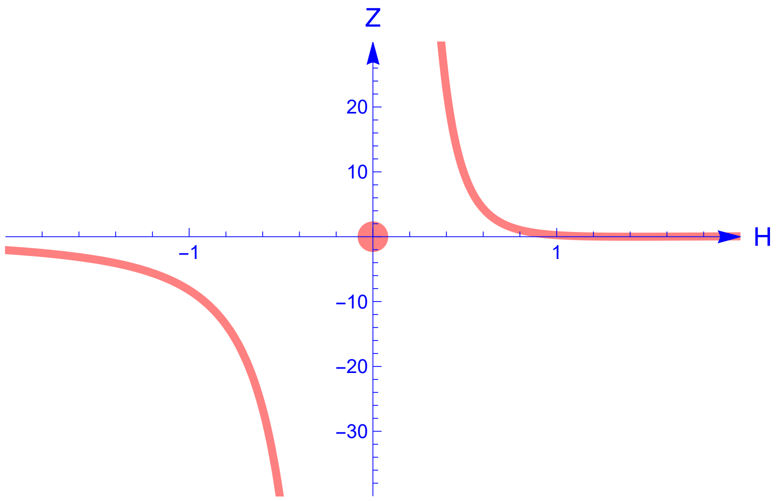

Figure 116. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$. We see the unbounded behaviour of the function. The limit is not finite.

12.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Looking for a definition of the partial derivatives.

We have to be able to define the partial derivatives in at least one neighbourhood around $(0,0)$. We have no problems with points that are in the interior of the definition domains of the classical functions. We have there the classical calculation rules for defining those functions and the partial derivatives always exist there and are even continuous.

But in our case, we have to investigate the points that are in exceptional subsets in the definition of the function.

In this case these are the points $(0,b)$.

Let us look at a point $(0,b)$ with $b \neq 0$. We are going to investigate the function in $(0,b)$ in the X-direction. This function is defined by

$$f(h,b) = \begin{cases} \dfrac{b\left(b-h^2\right)^2}{h^6} & \text{if } h \neq 0, \\ 0 & \text{if } h = 0. \end{cases}$$

We see that this function is not continuous in $h = 0$, so the derivative does not exist. The conclusion is that the partial derivative $\frac{\partial f}{\partial x}(0,b)$ does not exist for all b with $b \neq 0$.

We consult a figure for this observation.

The partial derivatives do not all exist in any neighbourhood of $(0,0)$. So the partial derivatives cannot be defined in any neighbourhood of $(0,0)$. The conclusion is that an alternative proof following the lines described at the start of this section cannot be given. Other alternative proofs can of course exist.

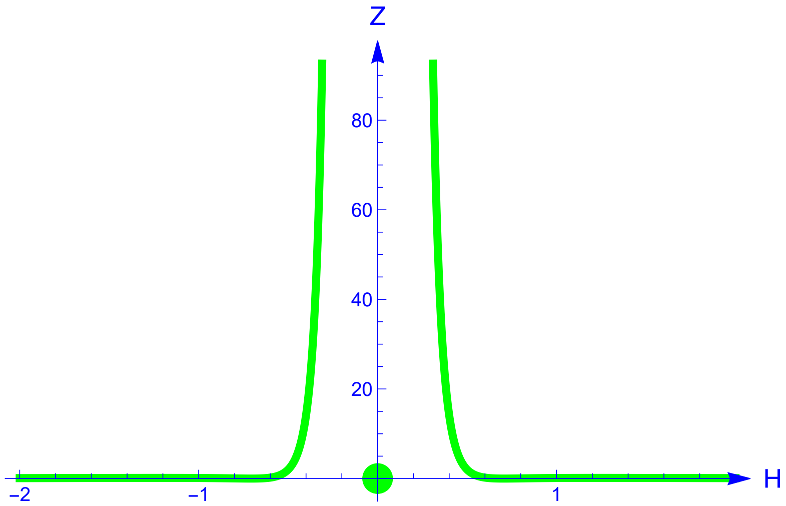

Figure 117. We see here a figure of the graph of the function restricted to the line through $(0, b)$ with direction $(1, 0)$, this is the X-direction. We have drawn the function $f(h, b)$. We see that this function is not continuous in $h = 0$. We have drawn this figure with the value $b = 1/2$.

12.5 Differentiability

This function is not differentiable because it is not continuous.

12.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

12.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

12.8 Overview

$$f(x, y) = \begin{cases} \dfrac{y\left(x^2 - y\right)^2}{x^6} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

continuous	no
partial derivatives exist	yes
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 13.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

13.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{x^3y - xy^3}{x^2+y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left|\frac{x^3 y - x y^3}{x^2+y^2}\right| &\le \frac{|x|^3\,|y| + |x|\,|y|^3}{x^2+y^2} \\
&\le \frac{\sqrt{x^2+y^2}^{\,3}\sqrt{x^2+y^2} + \sqrt{x^2+y^2}\,\sqrt{x^2+y^2}^{\,3}}{x^2+y^2} \\
&\le \frac{2\sqrt{x^2+y^2}^{\,4}}{\sqrt{x^2+y^2}^{\,2}} \\
&\le 2\sqrt{x^2+y^2}^{\,2}.
\end{aligned}$$

It is sufficient to take $\delta = (\epsilon/2)^{1/2}$. We can find a δ, so we conclude that the function is continuous.

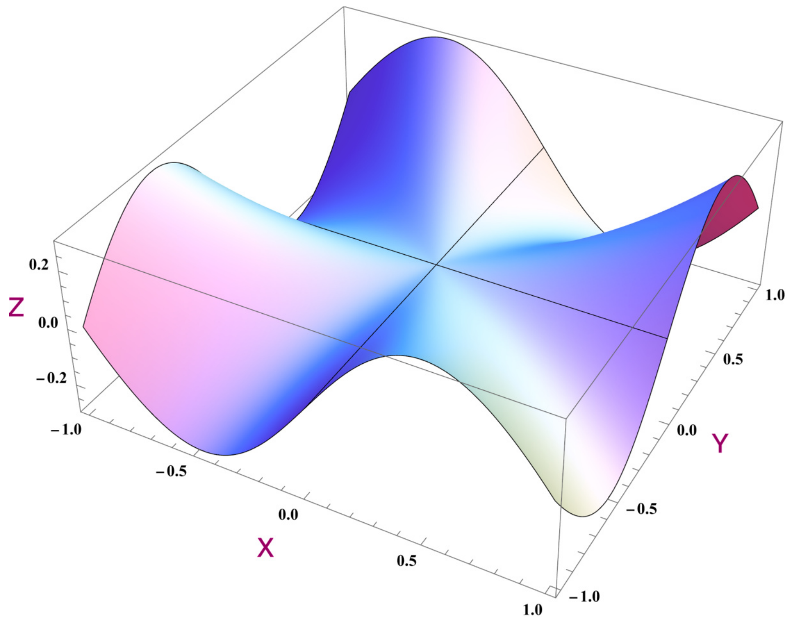

Figure 118. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

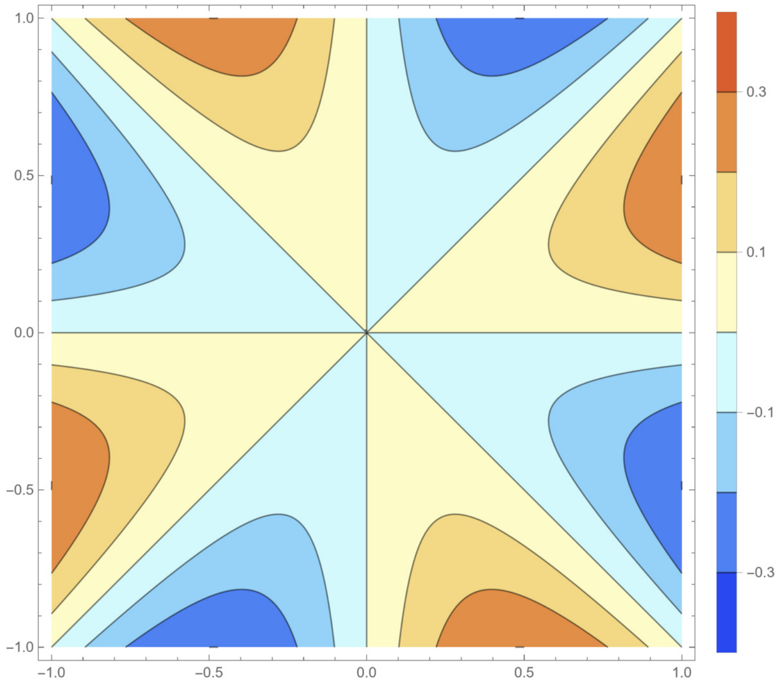

Figure 119. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

13.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).
We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} \\ &= \lim_{h \to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

13.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h \to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{h^4\,u^3\,v - h^4\,u\,v^3}{h\,(h^2\,u^2 + h^2\,v^2)} \\ &= \lim_{h\to 0} \frac{h\,u\,v\,(u^2 - v^2)}{u^2 + v^2} \\ &= 0. \end{aligned}$$

So the directional derivatives do always exist.

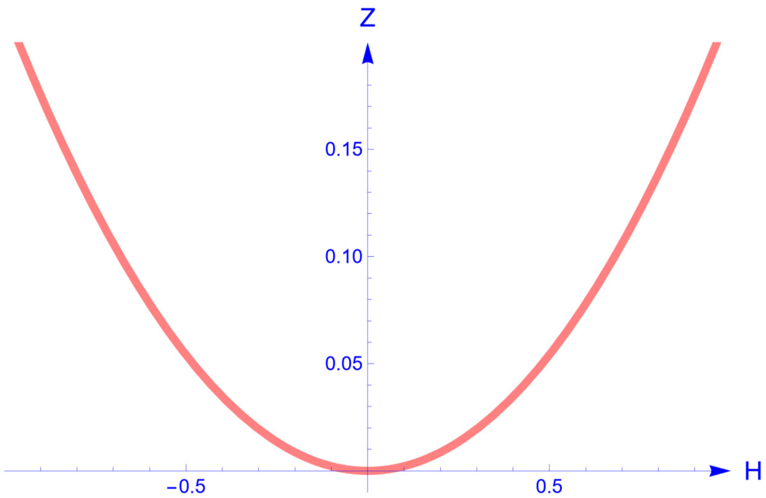

Figure 120. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = (\sqrt{3}/2, 1/2)$. The slope in 0 is 0. We have plotted here the function $f(h\,u, h\,v)$.

13.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we

have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{y\left(x^4+4x^2y^2-y^4\right)}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{x^5-4x^3y^2-xy^4}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

$$\begin{aligned}
&\left|\frac{\partial f}{\partial x}(x,y)\right| \\
&\quad \le \left|\frac{y\,(x^4+4\,x^2\,y^2-y^4)}{(x^2+y^2)^2}\right| \\
&\quad \le \frac{\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2}^4+4\sqrt{x^2+y^2}^2\sqrt{x^2+y^2}^2+\sqrt{x^2+y^2}^4\right)}{\sqrt{x^2+y^2}^4} \\
&\quad \le \frac{\sqrt{x^2+y^2}\,6\sqrt{x^2+y^2}^4}{\sqrt{x^2+y^2}^4} \\
&\quad \le \frac{6\sqrt{x^2+y^2}^5}{\sqrt{x^2+y^2}^4} \\
&\quad \le 6\sqrt{x^2+y^2} \\
&\quad \le 6.
\end{aligned}$$

We have chosen here the restriction to the neighbourhood defined by $\sqrt{x^2+y^2} < 1$.

Let us try to prove that $\left|\frac{\partial f}{\partial y}\right|$ is bounded.

$$
\begin{aligned}
\left|\frac{\partial f}{\partial y}(x,y)\right| &\leq \left|\frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}\right| \\
&\leq \frac{|x|^5 + 4|x|^3y^2 + |x|y^4}{(x^2+y^2)^2} \\
&\leq \frac{\sqrt{x^2+y^2}^5 + 4\sqrt{x^2+y^2}^3\sqrt{x^2+y^2}^2 + \sqrt{x^2+y^2}\sqrt{x^2+y^2}^4}{\sqrt{x^2+y^2}^4} \\
&\leq \frac{6\sqrt{x^2+y^2}^5}{\sqrt{x^2+y^2}^4} \\
&\leq 6\sqrt{x^2+y^2} \\
&\leq 6.
\end{aligned}
$$

We have chosen the restriction to the neighbourhood $\sqrt{x^2+y^2} < 1$.

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

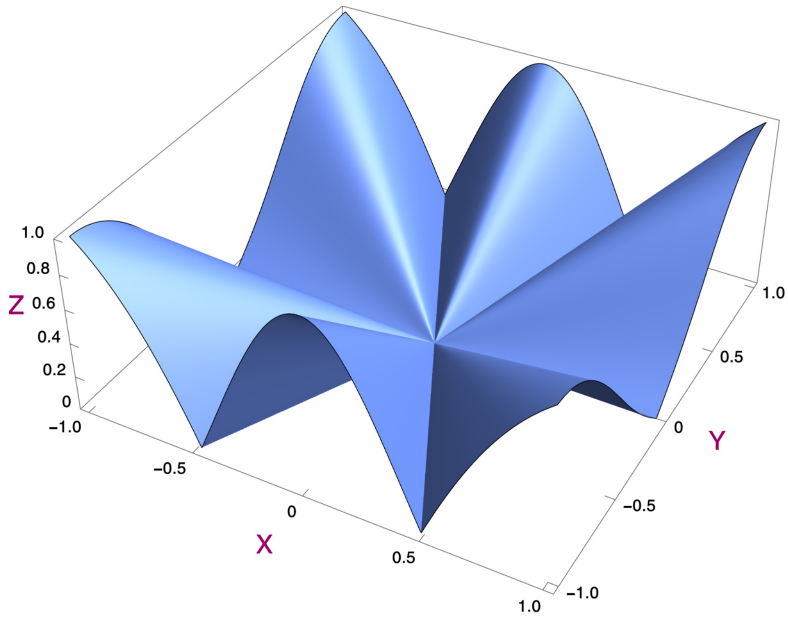

Figure 121. We see here the absolute value of the first partial derivative $\left|\dfrac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

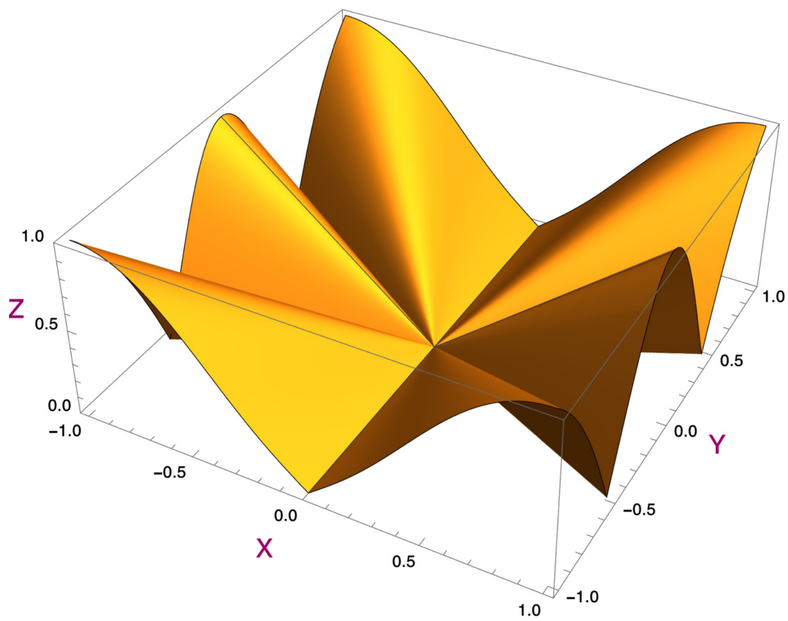

Figure 122. We see here the absolute value of the second partial derivative $\left|\dfrac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture.

13.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

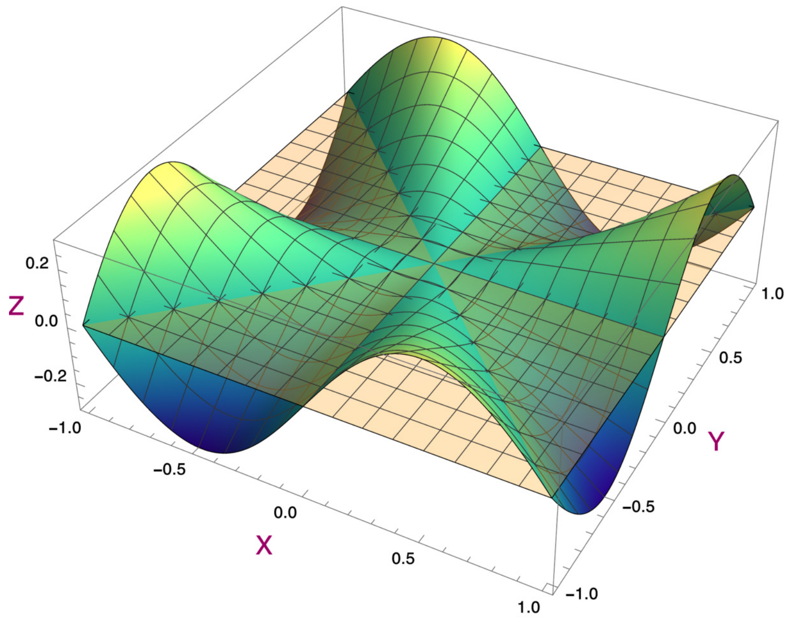

Figure 123. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function very nicely.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we

know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

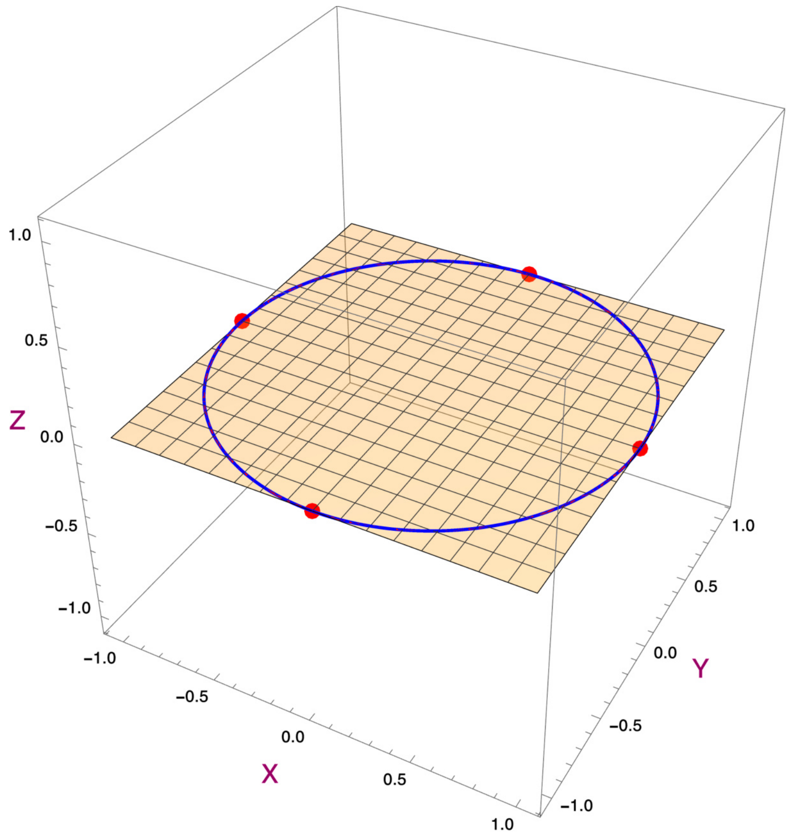

Figure 124. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h\,k\,\left(h^2 - k^2\right)}{\left(h^2 + k^2\right)^{3/2}} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2 + k^2} < \delta$, we have that

$$\left| \frac{h\,k\,(h^2 - k^2)}{(h^2 + k^2)^{3/2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{h\,k\,(h^2-k^2)}{(h^2+k^2)^{3/2}}\right| &\le \frac{\sqrt{h^2+k^2}\sqrt{h^2+k^2}\left(\sqrt{h^2+k^2}^{\,2}+\sqrt{h^2+k^2}^{\,2}\right)}{(h^2+k^2)^{3/2}}\\ &\le \frac{2\sqrt{h^2+k^2}^{\,4}}{(h^2+k^2)^{3/2}}\\ &\le 2\sqrt{h^2+k^2}^{\,5/2}.\end{aligned}$$

It is sufficient to take $\delta = (\epsilon/2)^{2/5}$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous. So f is differentiable.

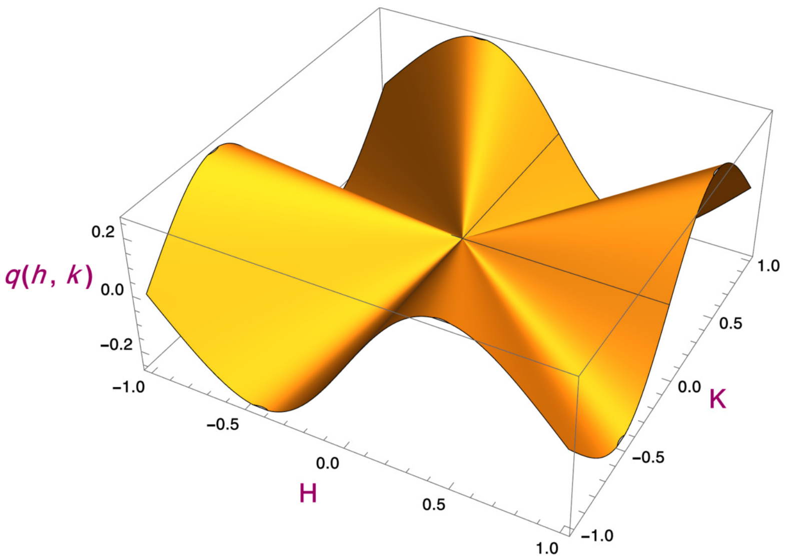

Figure 125. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

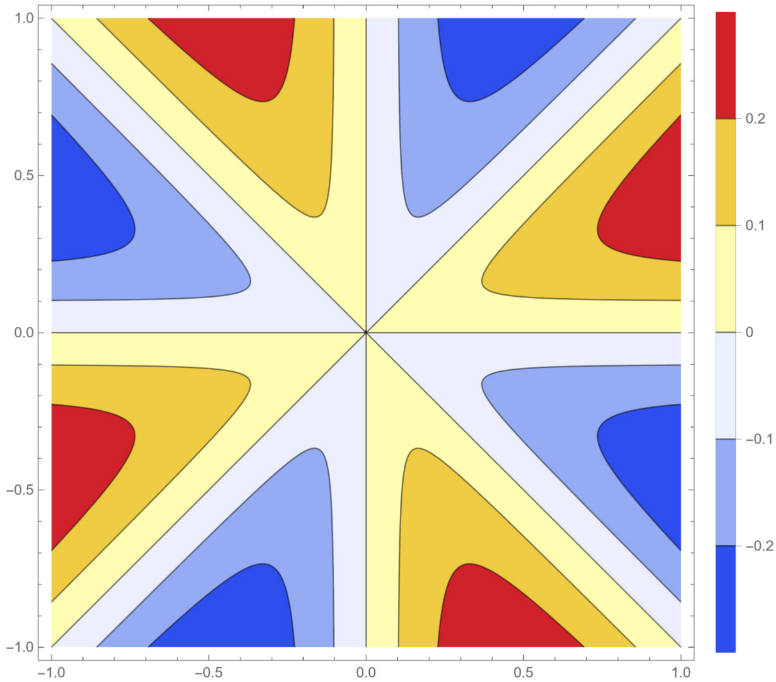

Figure 126. We see here a figure of the contour plot of the function $q(h, k)$. Only level curves of level around 0 come close to $(0, 0)$.

13.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0, 0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0, 0) = \frac{\partial f}{\partial x}(0, 0)\, u + \frac{\partial f}{\partial y}(0, 0)\, v .$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h, k)$. It turns out that we only have to prove now that the func-

tion $f(x,y)$ is locally Lipschitz continuous in $(0,0)$. We cite here the criterion that we will use.

A function is differentiable in (a,b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.
2. The directional derivatives have the following form: $D_{(u,v)}(a,b) = \nabla f(a,b)\cdot(u,v) = \frac{\partial f}{\partial x}(a,b)\,u + \frac{\partial f}{\partial y}(a,b)\,v$.
3. The function is locally Lipschitz continuous in (a,b). This means that there exists at least one neighbourhood of (a,b) and a number $K>0$ such that for all (x_1,y_1) and (x_2,y_2) in the neighbourhood, we have $|f(x_1,y_1)-f(x_2,y_2)| < K\,||(x_1,y_1)-(x_2,y_2)||$. Remark that the same K must be valid for all (x_1,y_1) and (x_2,y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be “locally flat" and thus “locally linear" which is partly expressed by this Lipschitz continuity condition.

Let us try to prove the Lipschitz condition. We are going to use a well known method.

$$\begin{aligned}
&|f(x_1,y_1)-f(x_2,y_2)| \\
&\qquad = |f(x_1,y_1)-f(x_1,y_2)+f(x_1,y_2)-f(x_2,y_2)| \\
&\qquad \le |f(x_1,y_1)-f(x_1,y_2)| + |f(x_1,y_2)-f(x_2,y_2)|.
\end{aligned}$$

We focus now on the first term $|f(x_1,y_1)-f(x_1,y_2)|$. We fix now x_1 and look upon $f(x_1,y)$ as a function in one variable y. So it springs to mind that we can estimate this by using Lagrange’s intermediate value theorem. So we have

$$|f(x_1,y_1)-f(x_1,y_2)| = \left|\frac{\partial f}{\partial y}(x_1,\xi)\right|\,|y_1-y_2|$$

where ξ is a number lying in the open interval (y_1, y_2) if e.g. $y_1 \leq y_2$. Remark that we already have proven that the partial derivatives are bounded in a neighbourhood, we can take that bound found for $\frac{\partial f}{\partial y}$, say M_2 and that neighbourhood in this case also. So $\left|\frac{\partial f}{\partial y}(x_1, \xi)\right| \leq M_2$. We work in a completely analogous way for the second term $|f(x_1, y_2) - f(x_2, y_2)|$. We have in that case $\left|\frac{\partial f}{\partial x}(\xi, y_2)\right| \leq M_1$ where ξ is now a number in the open interval (x_1, x_2) if e.g. $x_1 \leq x_2$.

We take up our inequality again

$$\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)| \\
&\quad \leq |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\quad \leq |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)| \\
&\quad \leq M_2\,|(y_1 - y_2)| + M_1\,|(x_1 - x_2)| \\
&\quad \leq M_2\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + M_1\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&\quad \leq (M_1 + M_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.
\end{aligned}$$

So this function is locally Lipschitz continuous. We have an alternative proof for the differentiability.

13.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

We know that the first partial derivative exists and is equal to

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{y\left(x^4 + 4x^2y^2 - y^4\right)}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We want to see if this partial derivative to x is continuous or not.

Discussion of the continuity of the first partial derivative in $(0,0)$.
We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove the inequality $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if the inequality $||(x,y) - (0,0)|| < \delta$ holds, it follows that $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

We calculate

$$\begin{aligned}
&\left|\frac{\partial f}{\partial x}(x,y)\right| \\
&\quad\le \left|\frac{y\,(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}\right| \\
&\quad\le \frac{|y|\,(x^4+4x^2y^2+y^4)}{(x^2+y^2)^2} \\
&\quad\le \frac{\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2}^4+4\sqrt{x^2+y^2}^2\sqrt{x^2+y^2}^2+\sqrt{x^2+y^2}^4\right)}{\sqrt{x^2+y^2}^4} \\
&\quad\le \frac{\sqrt{x^2+y^2}\,6\sqrt{x^2+y^2}^4}{\sqrt{x^2+y^2}^4} \\
&\quad\le \frac{6\sqrt{x^2+y^2}^5}{\sqrt{x^2+y^2}^4} \\
&\quad\le 6\sqrt{x^2+y^2}.
\end{aligned}$$

It is sufficient to take $\delta = \epsilon/6$. We can find a δ, so we conclude that the function is continuous.

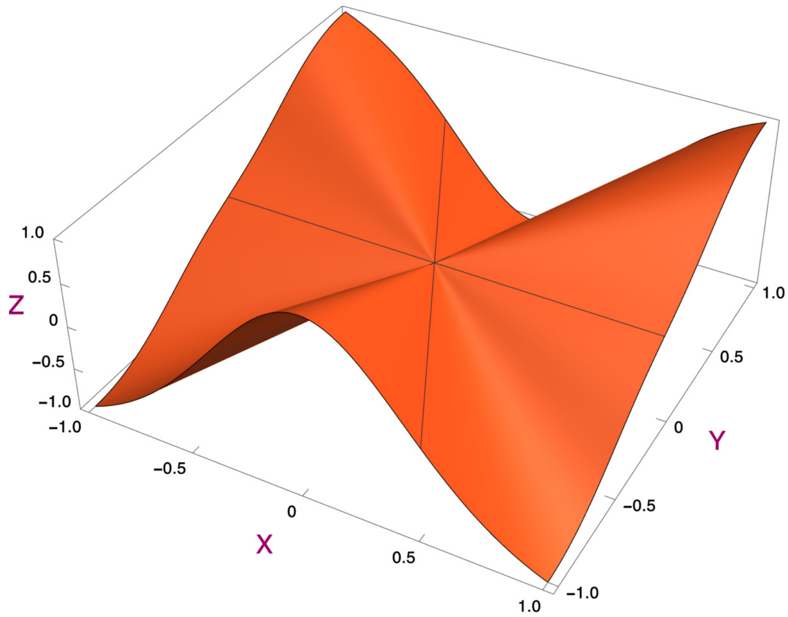

Figure 127. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial x}(x, y)$. This looks like a continuous function.

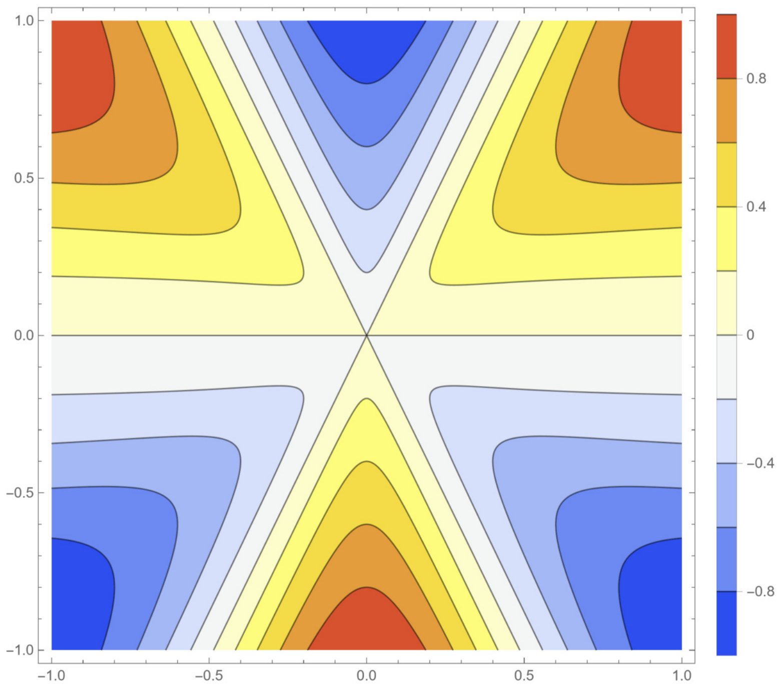

Figure 128. We see here a figure of the contour plot of the $\frac{\partial f}{\partial x}(x, y)$. Only level curves of level around 0 come close to $(0, 0)$.

Discussion of the continuity of the second partial derivative in $(0, 0)$.

We know that the second first order partial derivative exists and is equal to

$$\frac{\partial f}{\partial y}(x, y) = \begin{cases} \dfrac{x^5 - 4x^3y^2 - xy^4}{\left(x^2 + y^2\right)^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

We want to see if it is continuous or not.

Discussion of the continuity in $(0, 0)$.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove the inequality $|\frac{\partial f}{\partial y}(x, y) - \frac{\partial f}{\partial y}(0, 0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if the inequality $||(x, y) - (0, 0)|| < \delta$ holds, it follows that $|\frac{\partial f}{\partial y}(x, y) - \frac{\partial f}{\partial y}(0, 0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(0,0) \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{\partial f}{\partial y}(x,y) \right| &\leq \left| \frac{x^5 - 4x^3y^2 - xy^4}{\left(x^2+y^2\right)^2} \right| \\
&\leq \frac{|x|^5 + 4|x|^3y^2 + |x|y^4}{\left(x^2+y^2\right)^2} \\
&\leq \frac{\sqrt{x^2+y^2}^5 + 4\sqrt{x^2+y^2}^3\sqrt{x^2+y^2}^2 + \sqrt{x^2+y^2}\sqrt{x^2+y^2}^4}{\sqrt{x^2+y^2}^4} \\
&\leq \frac{6\sqrt{x^2+y^2}^5}{\sqrt{x^2+y^2}^4} \\
&\leq 6\sqrt{x^2+y^2}.
\end{aligned}$$

It is sufficient to take $\delta = \epsilon/6$. We can find a δ, so we conclude that the function is continuous.

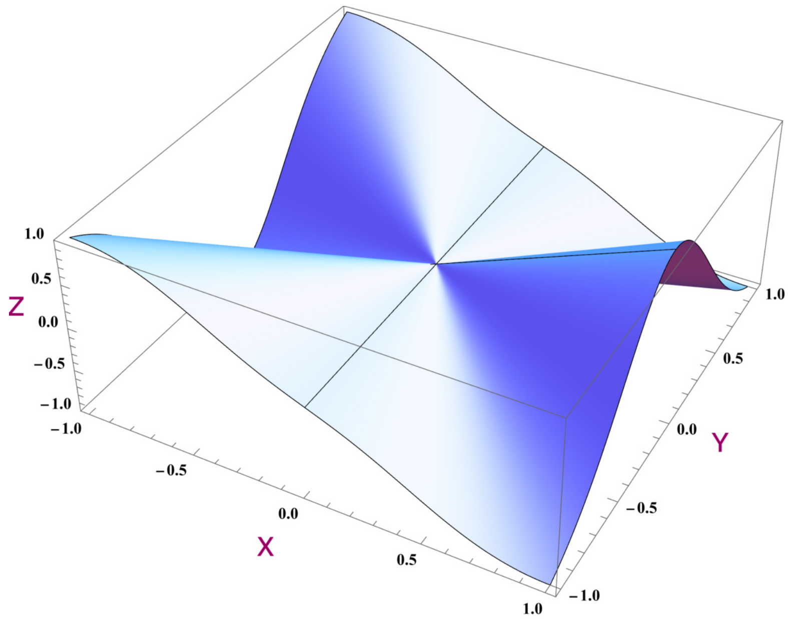

Figure 129. We see here a three dimensional figure of the graph of the second partial derivative $\frac{\partial f}{\partial y}(x, y)$. This looks like a continuous function.

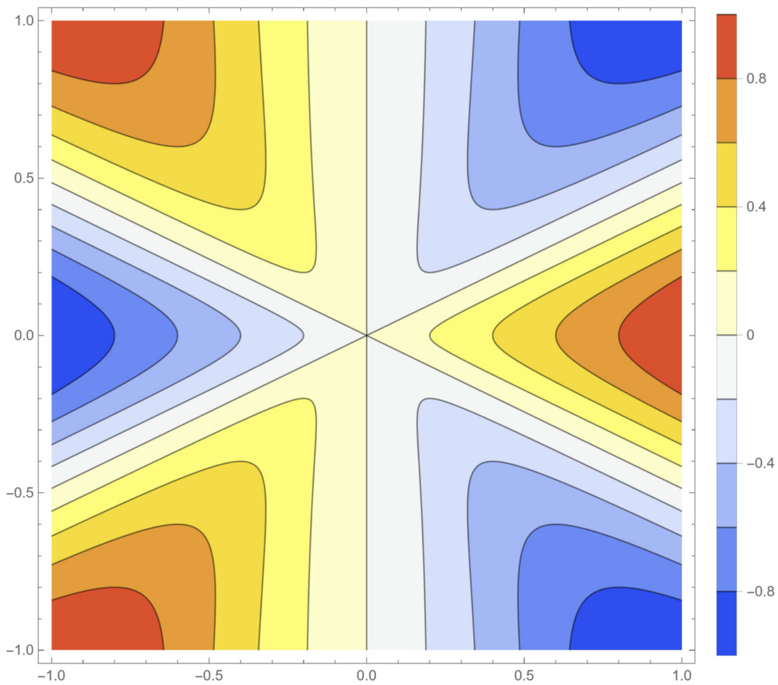

Figure 130. We see here a figure of the contour plot of the partial derivative $\frac{\partial f}{\partial y}(x,y)$. Only level curves of level around 0 come close to $(0,0)$.

13.8 Overview

$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	yes

13.9 One step further

We want to know if this function is further uneventful from the point of view of differentiability. Let us take a look at the second partial derivative

$$\frac{\partial^2 f}{\partial x^2} = -\frac{4\,x\,y^3\,(x^2 - 3\,y^2)}{(x^2 + y^2)^3}.$$

Let us take a look of a three dimensional plot of this partial derivative to y of the function.

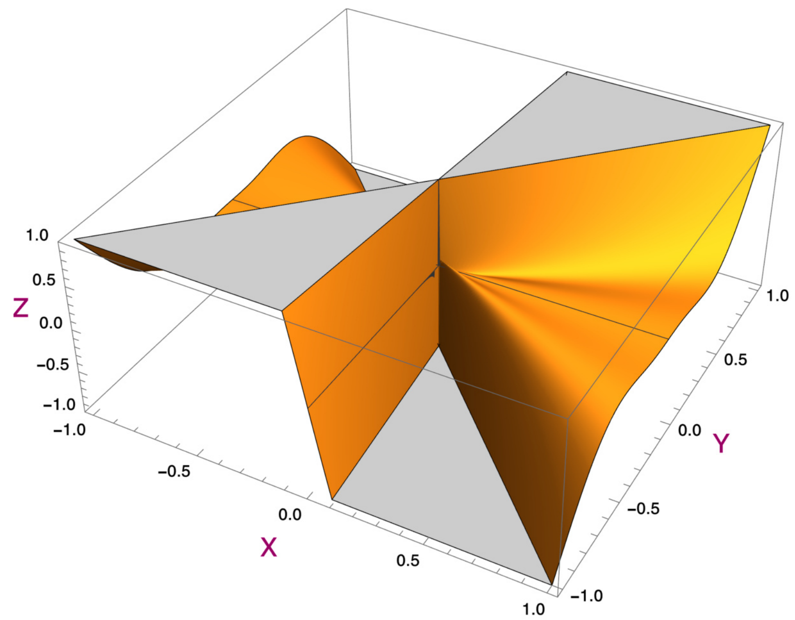

Figure 131. We see here a figure of the second order partial derivative $\frac{\partial^2 f}{\partial x^2}(x, y)$. It seems quite improbable that this second order partial derivative is continuous. We stop however our investigations here and leave this to the initiative of the interested reader.

Exercise 14.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^6 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

14.1 Continuity

We restrict the function to the continuous curves with equations $y = \lambda x^2$. We observe then that

$$f|_{y=\lambda x^2}(x,y) = \begin{cases} f(x,\lambda x^2) = \dfrac{\lambda}{\lambda^2 + x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

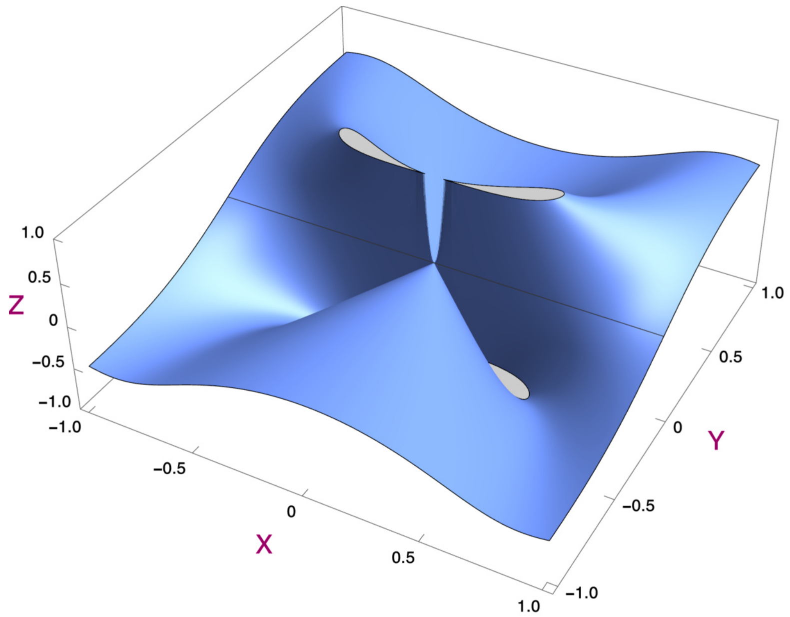

Figure 132. We see here a three dimensional figure of the graph of the function. The vertical line above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

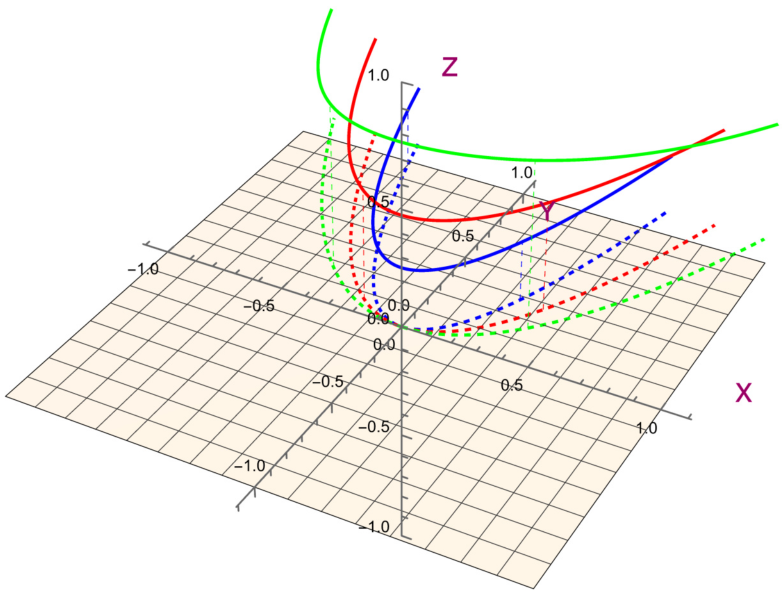

Figure 133. We have restricted the function here to $y = 21/10\,x^2$ and $y = 13/10\,x^2$ and $y = 39/10\,x^2$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

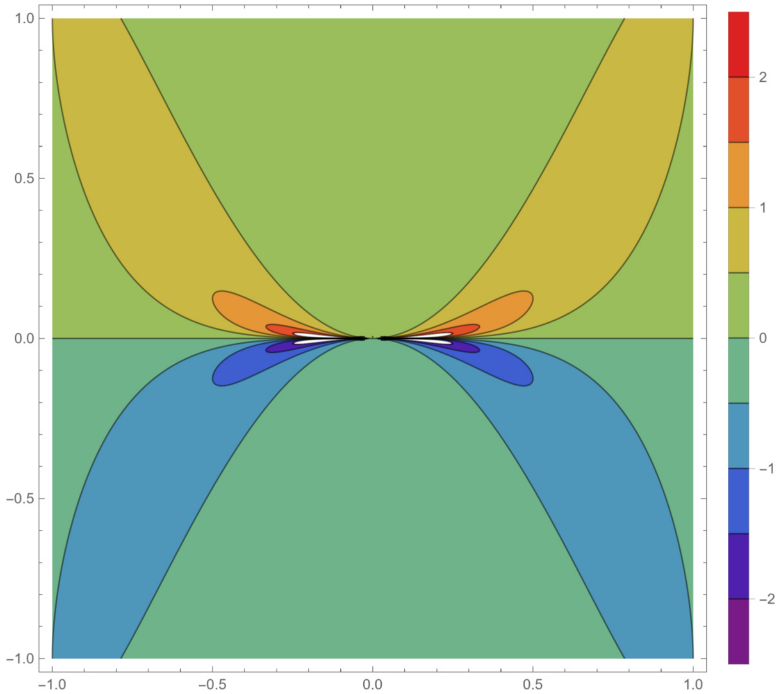

Figure 134. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

14.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

14.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit for $v \neq 0$

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{u^2 v}{h^4 u^6 + v^2} \\ &= \frac{u^2}{v}. \end{aligned}$$

This calculation is valid if $v \neq 0$. Remember that we already did the $v = 0$ case. So the directional derivatives do always exist.

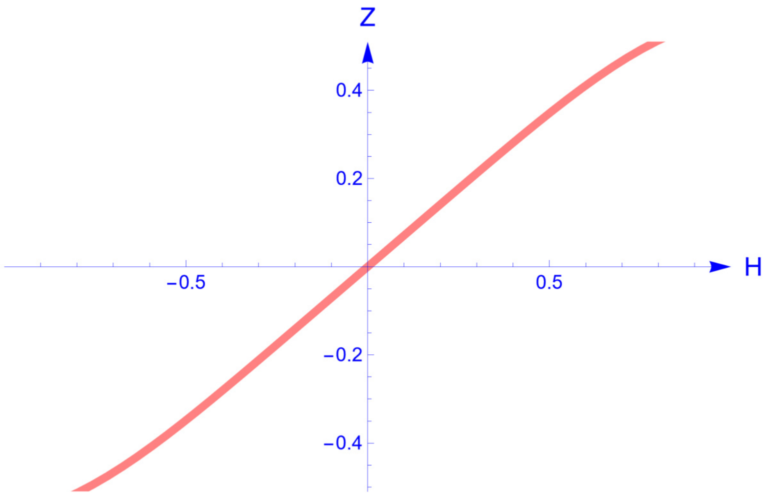

Figure 135. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$.

14.4 Alternative proof of continuity (optional)

This is irrelevant. The function is not continuous.

14.5 Differentiability

Irrelevant. The function is not differentiable because it is not continuous.

14.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

14.7 Continuity of the partial derivatives

The function is not differentiable. This is irrelevant.

14.8 Overview

$$\begin{cases} \dfrac{x^2 y}{x^6 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

continuous	no
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

Exercise 15.

Discuss the
1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \frac{x^2\,|y|^{5/4}}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

15.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{x^2\,|y|^{5/4}}{x^4+y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned} \left| \frac{x^2\,|y|^{5/4}}{x^4+y^2} \right| &\leq \frac{1}{2}\,|y|^{1/4} \\ &\leq \frac{1}{2}\sqrt{x^2+y^2}^{\,1/4}. \end{aligned}$$

Because $x^2\,|y|/(x^4+y^2) \le 1/2$, we have the first step.

It is sufficient to take $\delta = (2\,\epsilon)^4$. We can find a δ, so we conclude that the function is continuous.

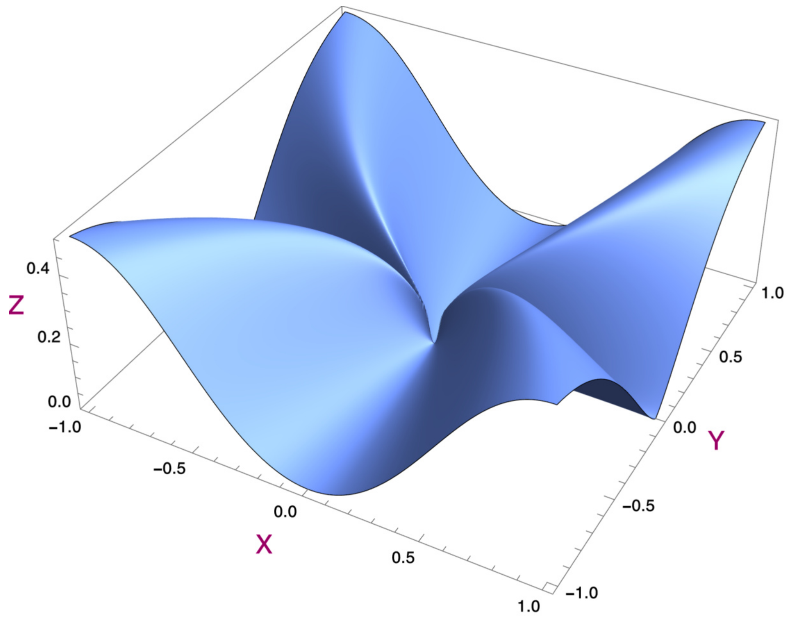

Figure 136. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

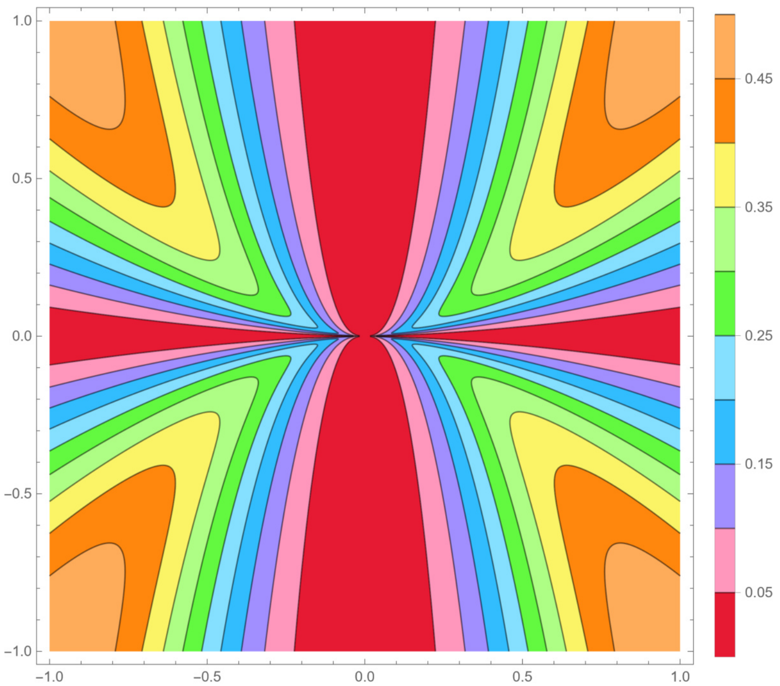

Figure 137. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

15.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

15.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{u^2\,|h\,v|^{5/4}}{h^3\,u^4 + h\,v^2} \\ &= \lim_{h\to 0} \frac{u^2\,|h|^{5/4}\,|v|^{5/4}}{h\,(h^2\,u^4 + v^2)} \\ &= \lim_{h\to 0} \frac{\operatorname{sgn}(h)\,u^2\,|h|^{1/4}\,|v|^{5/4}}{(h^2\,u^4 + v^2)} \\ &= 0. \end{aligned}$$

This calculation is only valid if $v \neq 0$. But we covered that case before. So the directional derivatives do always exist.

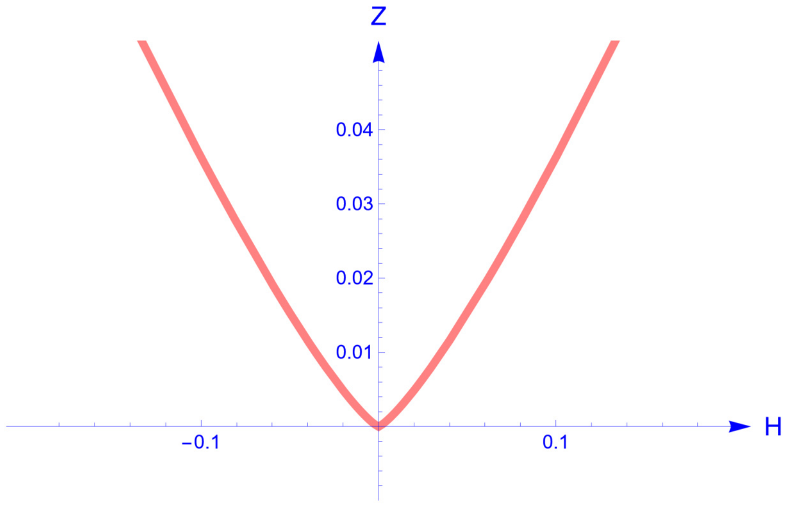

Figure 138. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

15.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

In order to be able to calculate the partial derivatives we first rewrite the function f.

$$f(x,y) = \begin{cases} \dfrac{x^2\,(-y)^{5/4}}{x^4+y^2} & y < 0; \\ \dfrac{x^2\,y^{5/4}}{x^4+y^2} & y \geq 0 \text{ and } (x,y) \neq (0,0); \\ 0 & (x,y) = (0,0). \end{cases}$$

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{2\,(-y)^{5/4}\left(x\,y^2 - x^5\right)}{\left(x^4+y^2\right)^2} & y < 0; \\ -\dfrac{2\,y^{5/4}\left(x^5 - x\,y^2\right)}{\left(x^4+y^2\right)^2} & y \geq 0 \text{ and } (x,y) \neq (0,0); \\ 0 & (x,y) = (0,0). \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{x^2\,\sqrt[4]{-y}\,\left(3\,y^2 - 5\,x^4\right)}{4\left(x^4+y^2\right)^2} & y < 0; \\ \dfrac{x^2\,\sqrt[4]{y}\,\left(5\,x^4 - 3\,y^2\right)}{4\left(x^4+y^2\right)^2} & y \geq 0. \text{ and } (x,y) \neq (0,0); \\ 0 & (x,y) = (0,0). \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

We do not investigate the partial derivative to x any further and concentrate us on the unboundedness of the partial derivative to y. Let us try to prove that $\left|\frac{\partial f}{\partial y}\right|$ is unbounded.

We are going to substitute $y = x^2$ in the following defining part of $\frac{\partial f}{\partial y}$ if $y > 0$.

$$\frac{\partial f}{\partial y}(x,y) = \frac{x^2\,\sqrt[4]{y}\,(5\,x^4 - 3\,y^2)}{4\,(x^4+y^2)^2} \qquad \text{with} \qquad y > 0.$$

So by substituting $y = x^2$:

$$\frac{\partial f}{\partial y}(x,x^2) = \frac{1}{8\,(x^2)^{3/4}}.$$

We see now the unbounded behaviour of $\frac{\partial f}{\partial y}$.

It is clear that the partial derivative $\frac{\partial f}{\partial y}$ is not bounded in any neighbourhood of $(0,0)$. So we do not have an alternative proof for the continuity.

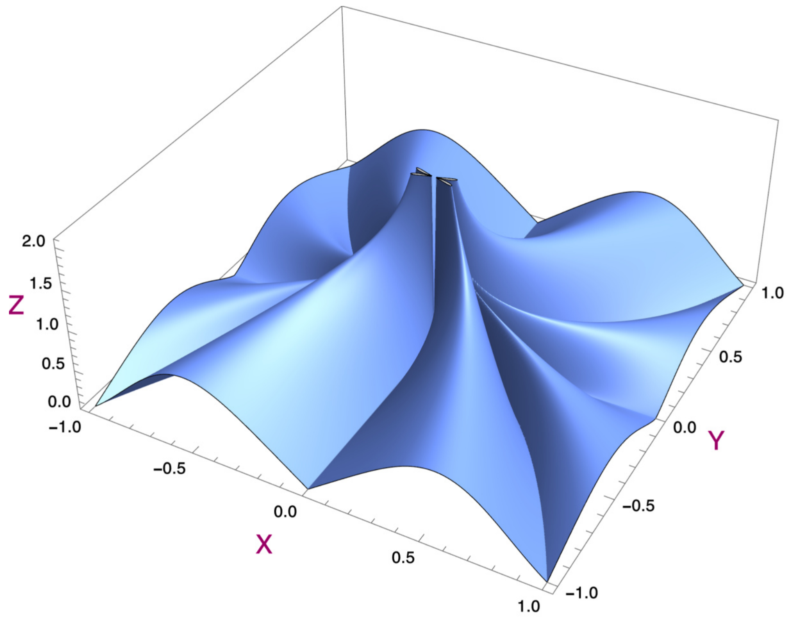

Figure 139. We see here the absolute value of the first partial derivative $\left|\dfrac{\partial f}{\partial x}\right|$. We can observe possible boundedness from this picture. There remains some doubts though. We can never be sure relying upon a visual basis alone. This asks for a proof, but we are not going to do this calculation. We turn our attention to the partial derivative to y because there is something wrong there.

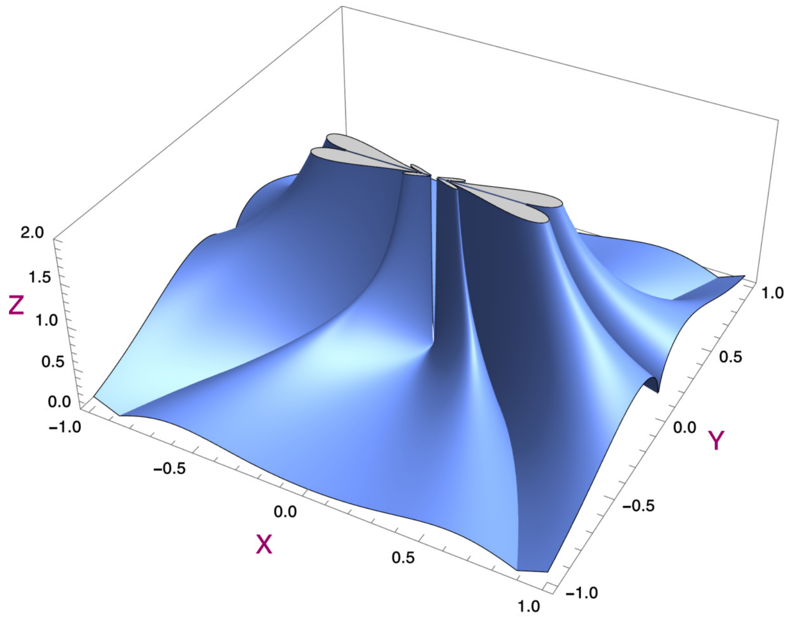

Figure 140. We see here the absolute value of the second partial derivative $\left|\dfrac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture.

15.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

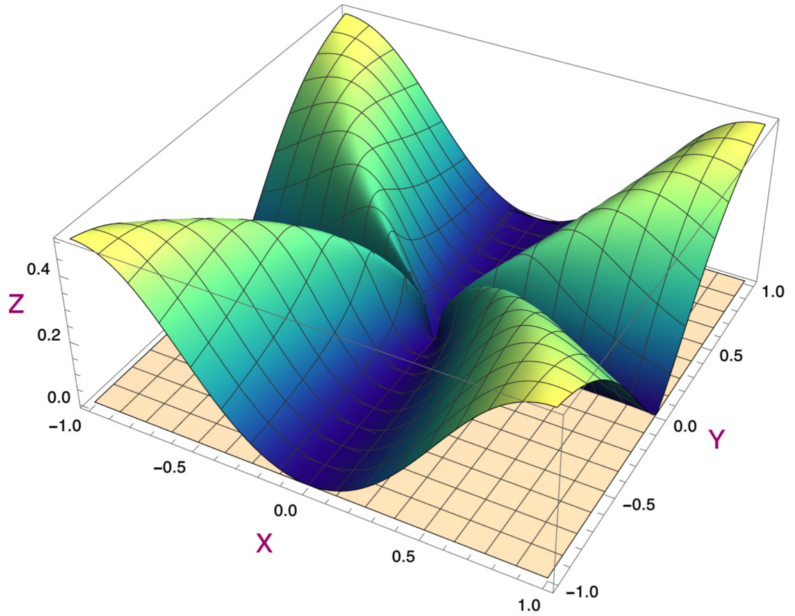

Figure 141. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane does not fit the function very nicely. It is indeed no tangent plane following our calculations. It is difficult to imagine a worse situation then this. But remark anyway that the candidate tangent plane sticks very well to the X-axis direction and the Y-axis direction. This is caused by the term $(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)) \cdot (h,k)$ in the definition of the quotient. This minimal good behaviour will be always the case.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

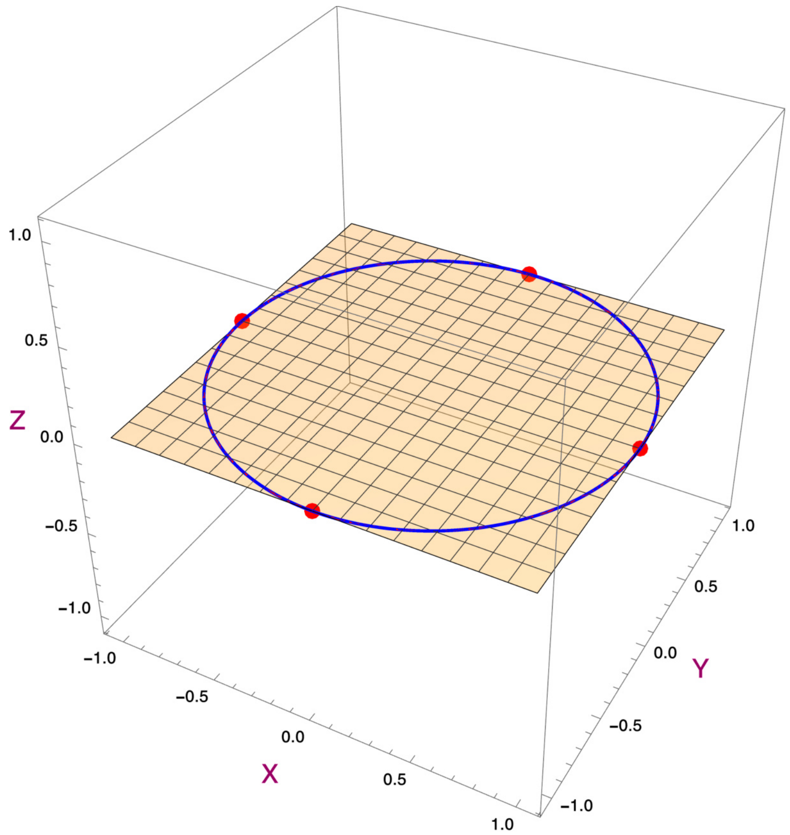

Figure 142. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h, k)$ in $(0, 0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h^2\,|k|^{5/4}}{\sqrt{h^2+k^2}\,\left(h^4+k^2\right)} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We restrict the function $q(h,k)$ to the continuous curves with equations $k = \lambda\,h^2$ with $\lambda = 1$. We observe then that

$$q\,|_{k=h^2}(h,k) = \begin{cases} q(h,h^2) = \dfrac{|h|^{5/2}}{2\,h^2\,\sqrt{h^4+h^2}} = \dfrac{h^2\,|h|^{1/2}}{2\,h^2\,|h|\,\sqrt{h^2+1}} & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

We see that this function is unbounded in any neighbourhood of $(0,0)$. So the limit does not exist. The function $f(x,y)$ is not differentiable in $(0,0)$.

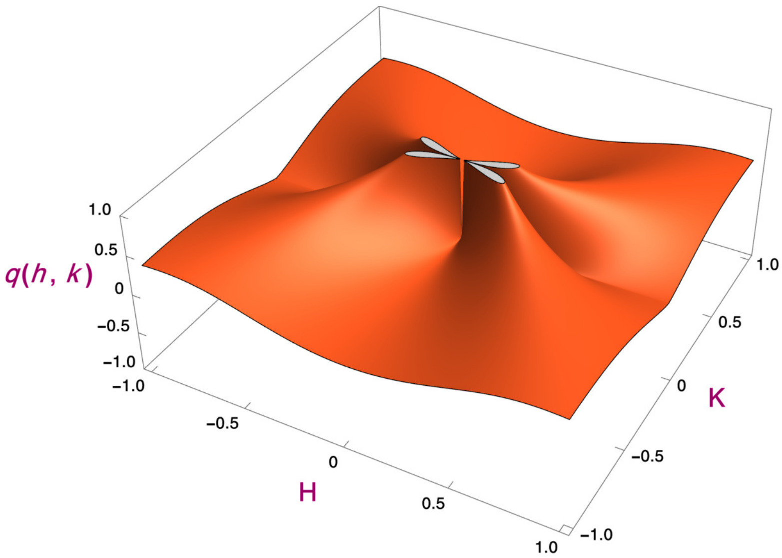

Figure 143. We see here a three dimensional figure of the graph of the function $q(h, k)$. The unboundedness above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

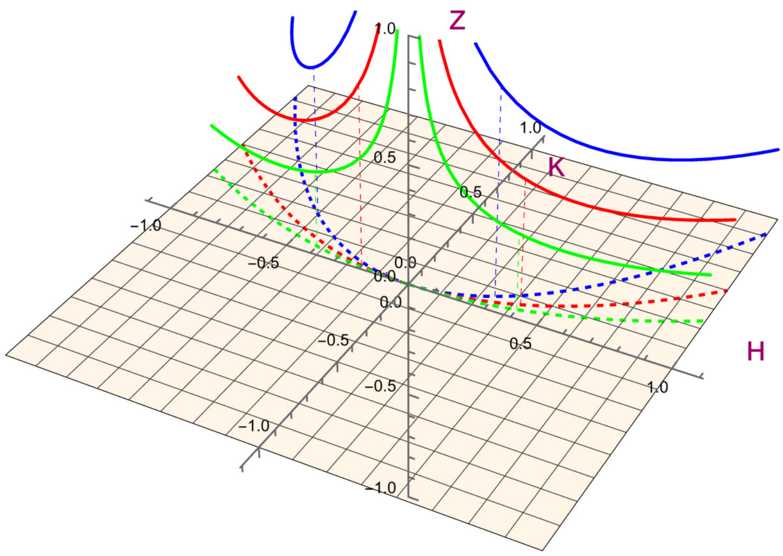

Figure 144. We have restricted the function $q(h, k)$ here to $k = 1/2\,h^2$ and $k = 3/10\,h^2$ and $k = 9/10\,h^2$. We see in this figure clearly that the restrictions of the function to these curves are functions that have different limits in 0.

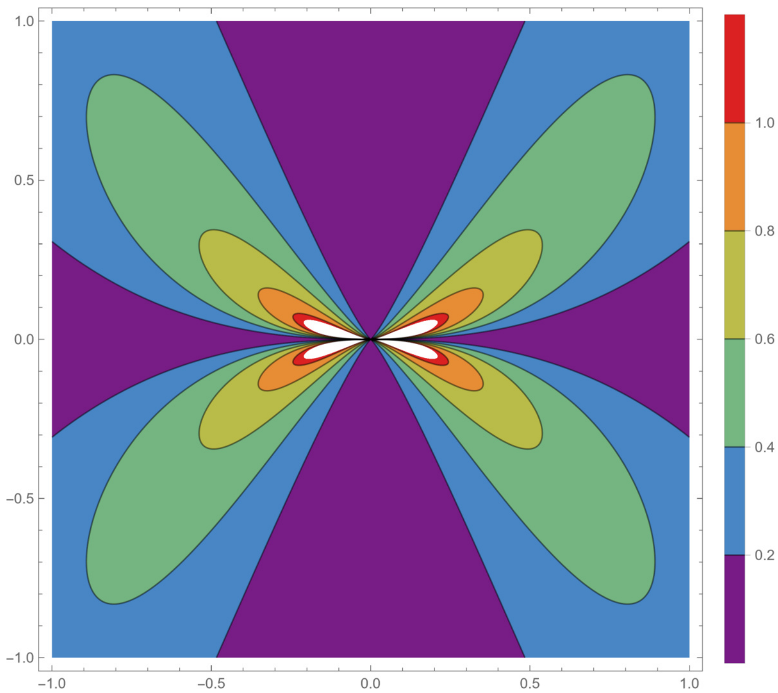

Figure 145. We see here a figure of the contour plot of the function $q(h,k)$. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

15.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

15.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

15.8 Overview

$$f(x,y) = \begin{cases} \dfrac{x^2\,|y|^{5/4}}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

15.9 One step further

We have used in the calculations for differentiability that we had some magical curves $k = \lambda h^2$ which behaved very strangely when mapped by $q(h, k)$. We want to see what is going on with these curves. Let us define the 3-dimensional curve in parametric form that projects in the (h, k)-plane to our curve $k = \lambda h^2$ where we use $\lambda = 1$: $(x(t), y(t), z(t)) = (t, \lambda t^2, f(t, \lambda t^2)) = \left(t, \lambda t^2, \frac{|\lambda t^2|^{5/4}}{(\lambda^2+1)\, t^2}\right)$.

This curve lies completely in the surface defined by the function. It is clear that the tangent vector lies in the tangent plane if the function is differentiable. Now we have a candidate tangent plane, we draw that and draw also the curve.

We remark that the composition $f(x(t), y(t))$ must be differentiable if f itself is differentiable because $x(t)$ and $y(t)$ are differentiable. We want to draw this curve to see if it is differentiable. So we must have that the curve

$$(x(t), y(t), z(t)) = \left(t, \lambda t^2, \frac{\lambda \sqrt[4]{\lambda t^2}}{\lambda^2 + 1}\right)$$

is differentiable. We take a look at the following figure.

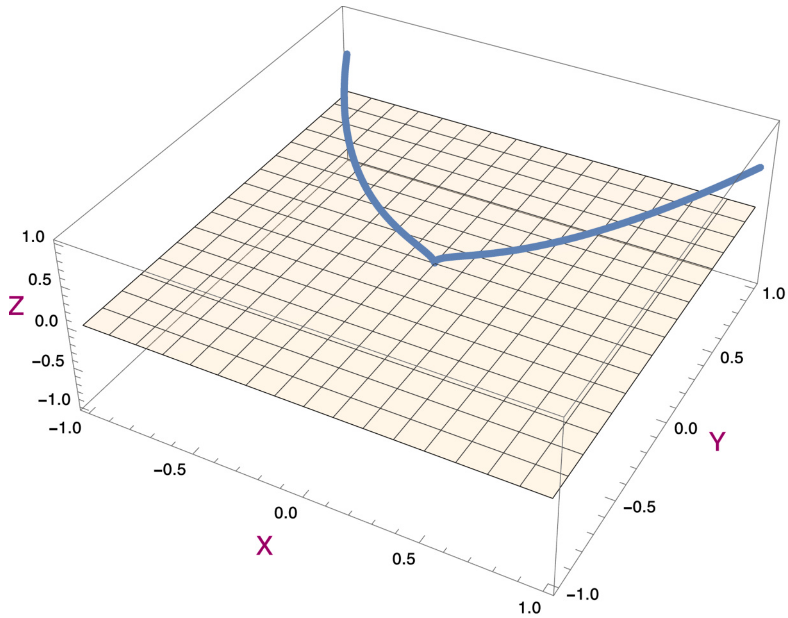

Figure 146. The curve is not even differentiable in $t = 0$. So we cannot even talk about tangency. There is no tangent line to draw. We have used $\lambda = 1$ in this figure.

Exercise 16.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y)=\sqrt[3]{x^2y}.$$

16.1 Continuity

We investigate continuity. This function is composed from classical functions that are already proven in the theory of being continuous. So we do not have to prove anything more. The function is continuous.

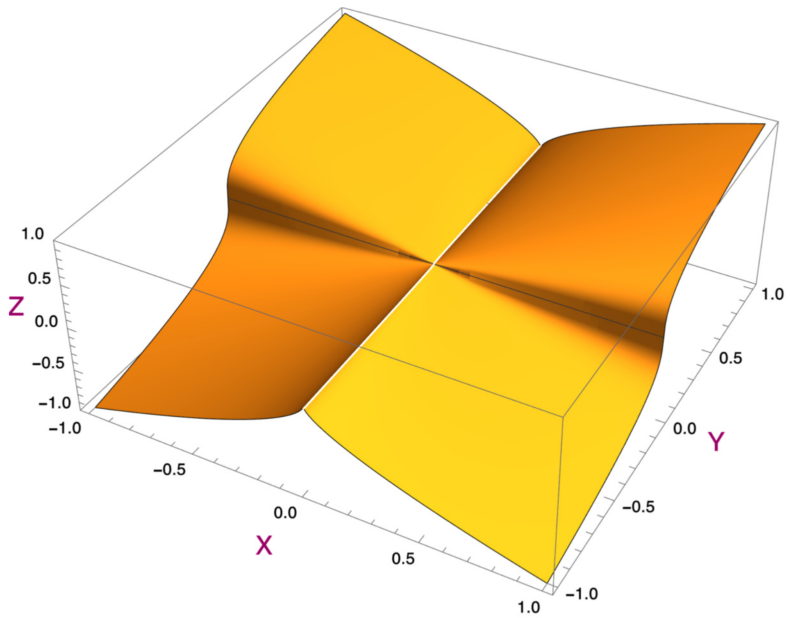

Figure 147. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

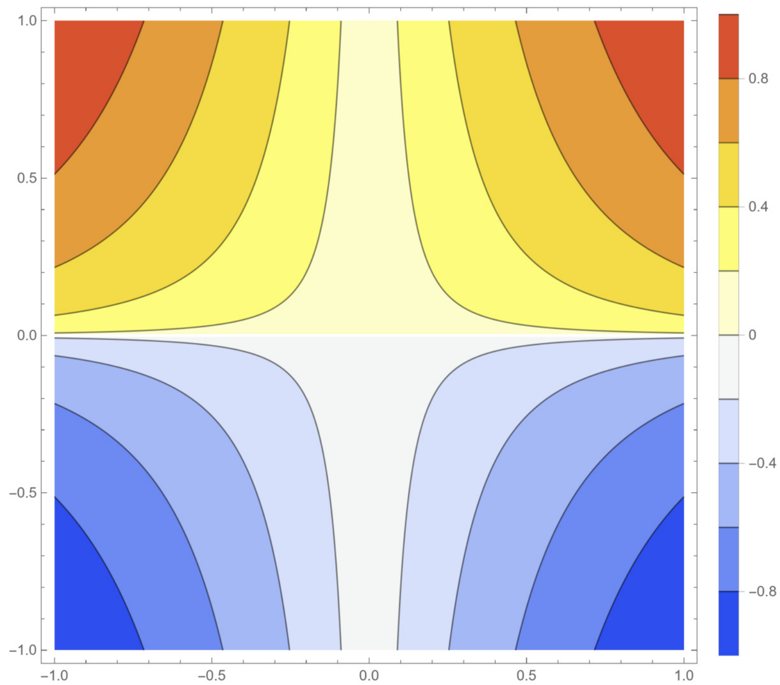

Figure 148. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0, 0)$.

16.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f\,|_{y=0}(x, y) = 0.$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = 0.$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

16.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0}\frac{f(0+hu,0+hv)-f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt[3]{h^2\,u^2}\,\sqrt[3]{h\,v}}{h} \\ &= \sqrt[3]{u^2}\,\sqrt[3]{v}. \end{aligned}$$

So the directional derivatives do always exist.

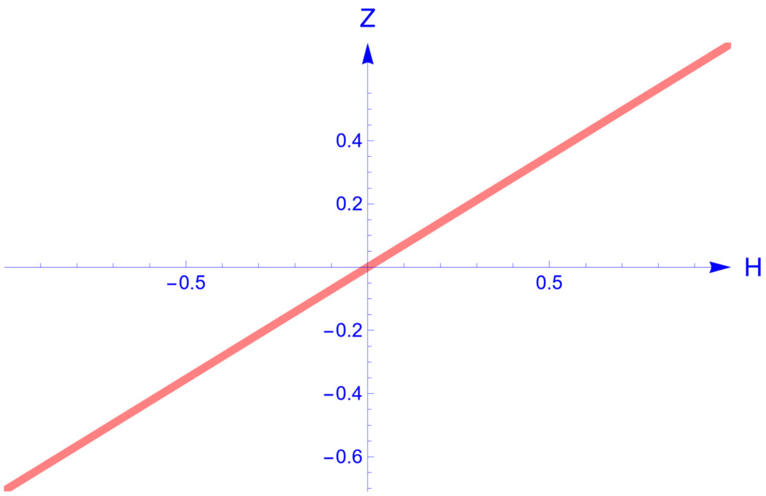

Figure 149. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = (1/\sqrt{2}, 1/\sqrt{2})$. We have plotted here the function $f(h\,u, h\,v)$.

16.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{2\sqrt[3]{y}}{3\sqrt[3]{x}} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{x^{2/3}}{3\,y^{2/3}} & y \neq 0 \\ 0 & y = 0. \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

It is clear that the partial derivatives $|\frac{\partial f}{\partial x}|$ and $|\frac{\partial f}{\partial y}|$ are not bounded in any neighbourhood of $(0,0)$.

Because at least one and actually in this case two partial derivatives are unbounded in any neighbourhood of $(0,0)$, we do not have an alternative proof for the continuity.

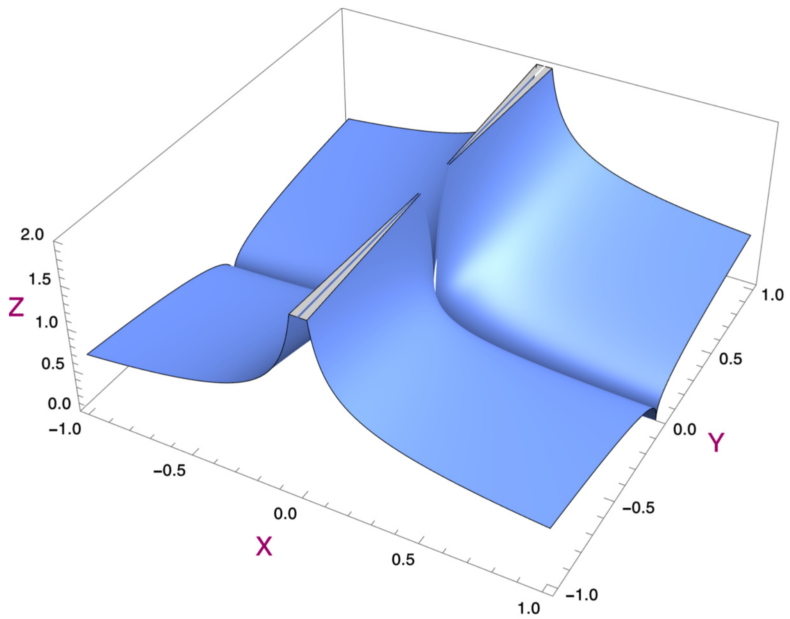

Figure 150. We see here the absolute value of the first partial derivative $\left|\dfrac{\partial f}{\partial x}\right|$. We can observe the unboundedness from this picture.

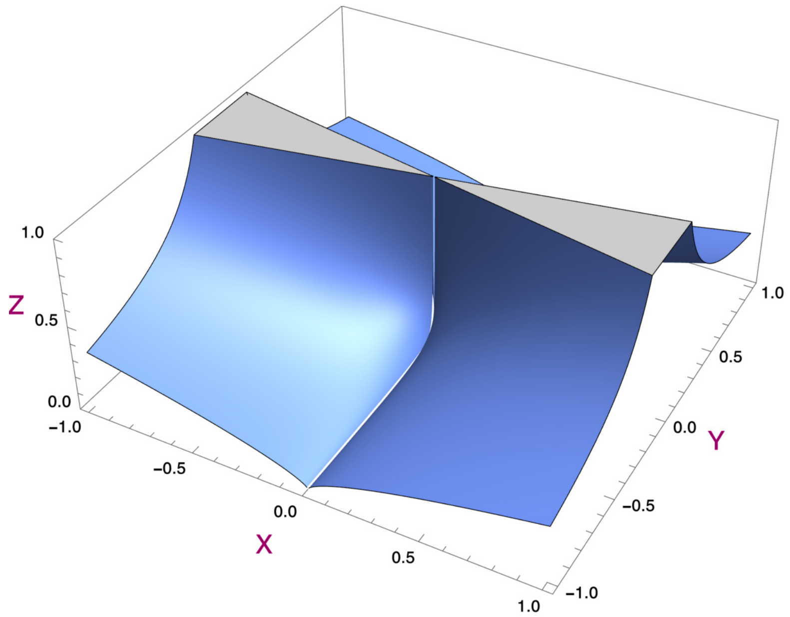

Figure 151. We see here the absolute value of the second partial derivative $\left|\dfrac{\partial f}{\partial y}\right|$. We can observe the unboundedness from this picture.

16.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

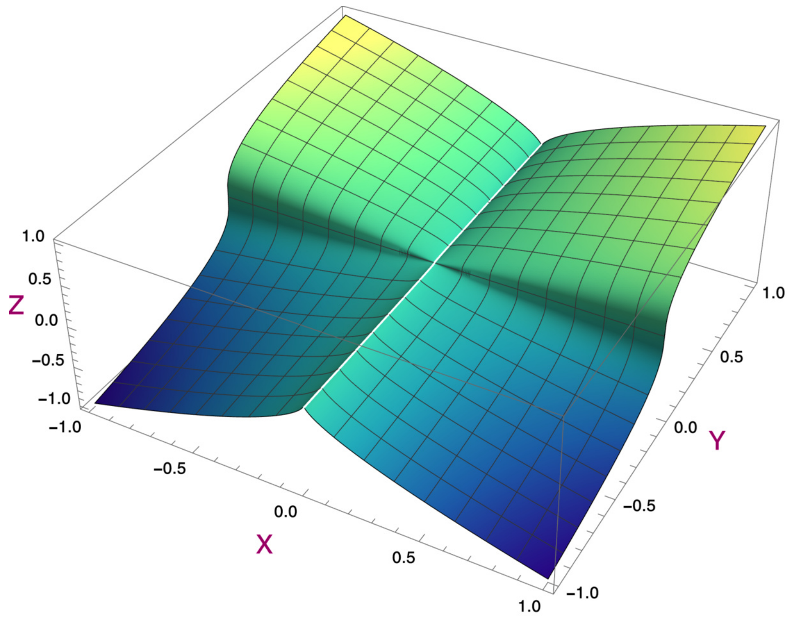

Figure 152. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane does not fit the function very nicely. It is indeed no tangent plane following our calculations. It fits nicely the coordinate axes but that is always the case. But on these axes, the function behaviour is vertical to the tangent plane. We will have to rely on the further calculations.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

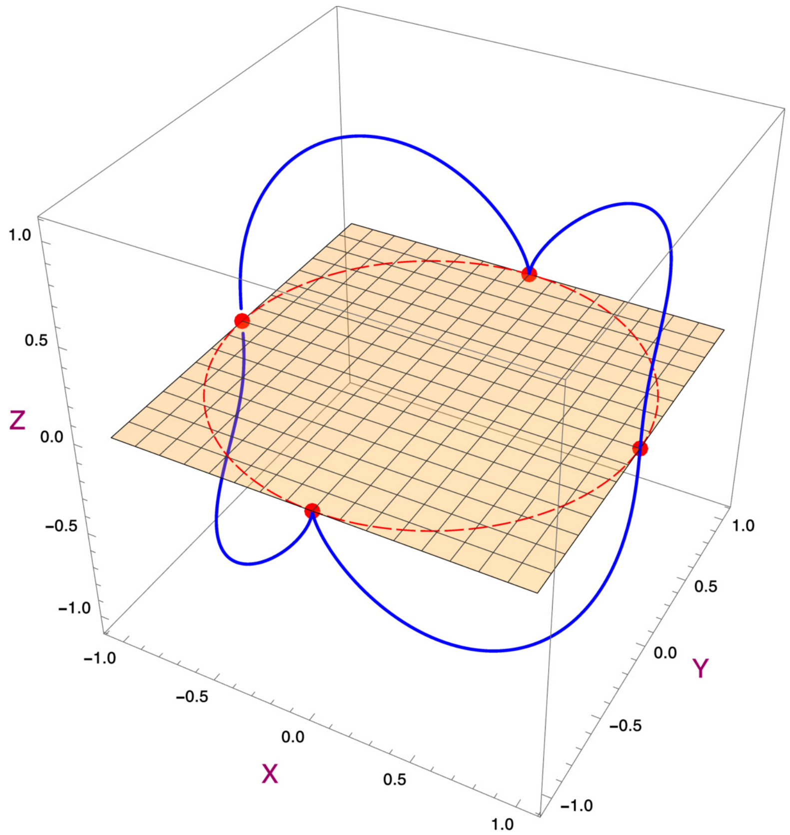

Figure 153. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ do not sweep out a nice ellipse lying in one plane! This is really bad news for differentiability.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h, k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{\sqrt[3]{h^2}\,\sqrt[3]{k}}{\sqrt{h^2+k^2}} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We restrict the function $q(h,k)$ to the continuous curves with equations $k = \lambda\,h$. We observe then that

$$q\,|_{k=\lambda\,h}(h,k) = \begin{cases} q(h,\lambda\,h) = \dfrac{\sqrt[3]{h^2}\,\sqrt[3]{h\lambda}}{\sqrt{h^2\lambda^2+h^2}} = \dfrac{\operatorname{sgn}(h)\,\sqrt[3]{\lambda}}{\sqrt{\lambda^2+1}} & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

We see that these restricted functions have no limits. But if $q(h,k)$ is continuous, all these limit values should be $q(0,0) = 0$. So this function $q(h,k)$ is not continuous in $(0,0)$. The function $f(x,y)$ is not differentiable in $(0,0)$.

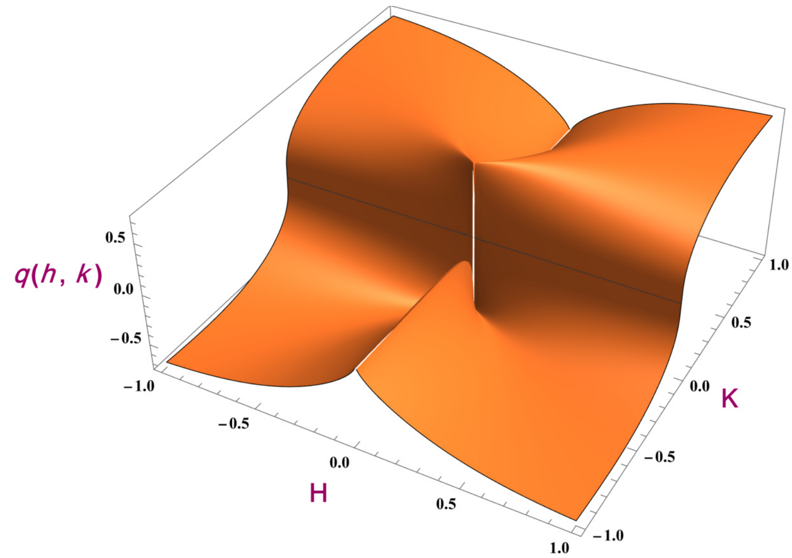

Figure 154. We see here a three dimensional figure of the graph of the function $q(h,k)$. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

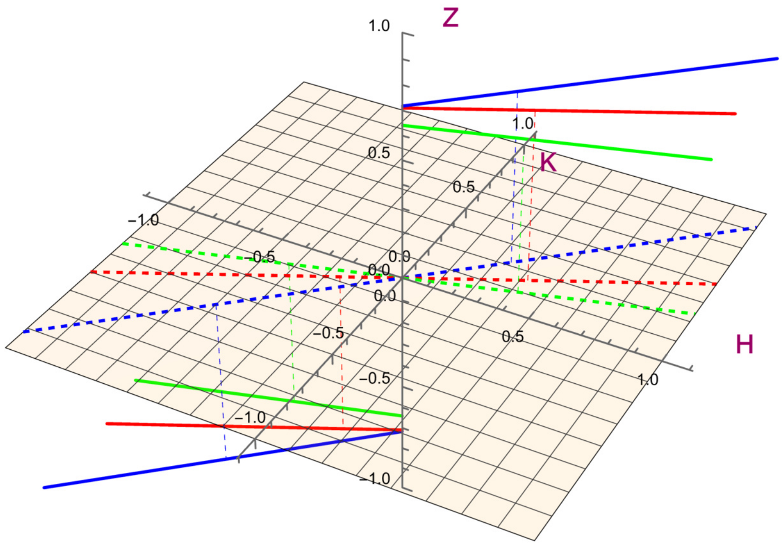

Figure 155. We have restricted the function $q(h,k)$ here to $k = 1/2\,h$ and $k = 3/10\,h$ and $k = 9/10\,h$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

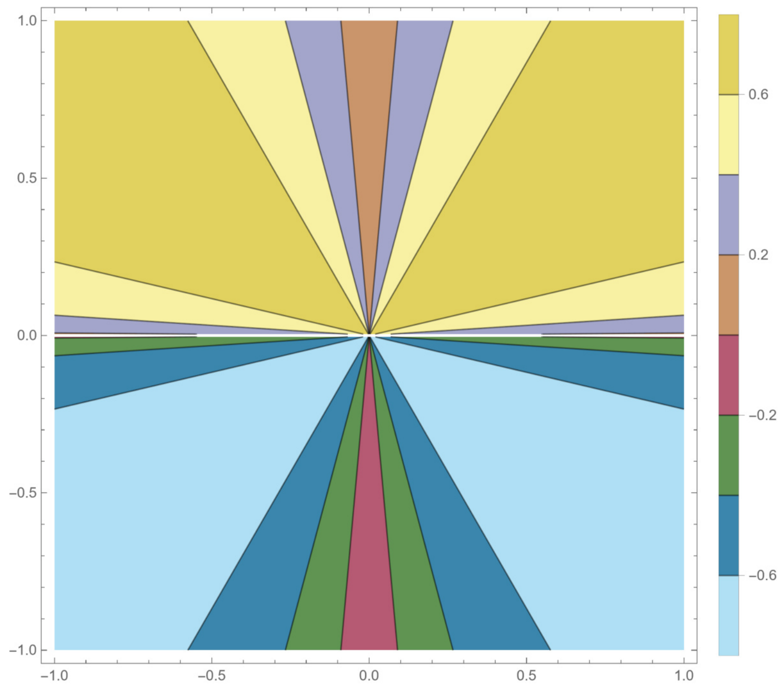

Figure 156. We see here a figure of the contour plot of the function $q(h,k)$. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

16.6 Alternative proof of differentiability (optional)

The function is not differentiable, so this is irrelevant. We have an alternative proof of the non differentiability.

Suppose that we already met in the course the differentiation rule of the composition of two differentiable functions. This is also called the chain rule. Then we have proven the following. If the function is differentiable in (a,b), then the directional derivative can be calculated is follows.

$$D_{(u,v)}f(a,b) = \frac{\partial f}{\partial x}(a,b)\,u + \frac{\partial f}{\partial y}(a,b)\,v.$$

Important remark. This formula is only valid if the function is differentiable. One of the most common mistakes is that one uses this formula in the case of non differentiability. It seems to be very easy to calculate quickly the partial derivatives if they exist and then use this formula *without checking differentiability* but it is alas wrong.

We have calculated the directional derivatives and we saw that

$$D_{u,v} f(0,0) = u^{2/3} v^{1/3}.$$

and this is certainly not the linear function in u and v which we should have in the case of differentiability. So we conclude that the function is not differentiable.

16.7 Continuity of the partial derivatives

This is irrelevant. The function is not differentiable.

16.8 Overview

$$f(x,y) = \sqrt[3]{x^2 y}.$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

16.9 One step further

We have met the magical curves $k = \lambda h$ when investigating the differentiability. We want to see what is really going on there. Let us take a look at the curve

$$(x(t), y(t), z(t)) = (t, \lambda t, f(t, \lambda t)) = \left(t, \lambda t, t \sqrt[3]{\lambda}\right).$$

This curve projects to the curve $k = \lambda h$ on the (h,k)-plane. Let us calculate the tangent vector in $t = 0$ of this curve. This is $\left(1, \lambda, \sqrt[3]{\lambda}\right)$. We will draw this situation. The tangent vector will be a vector in the tangent plane if the function is differentiable.

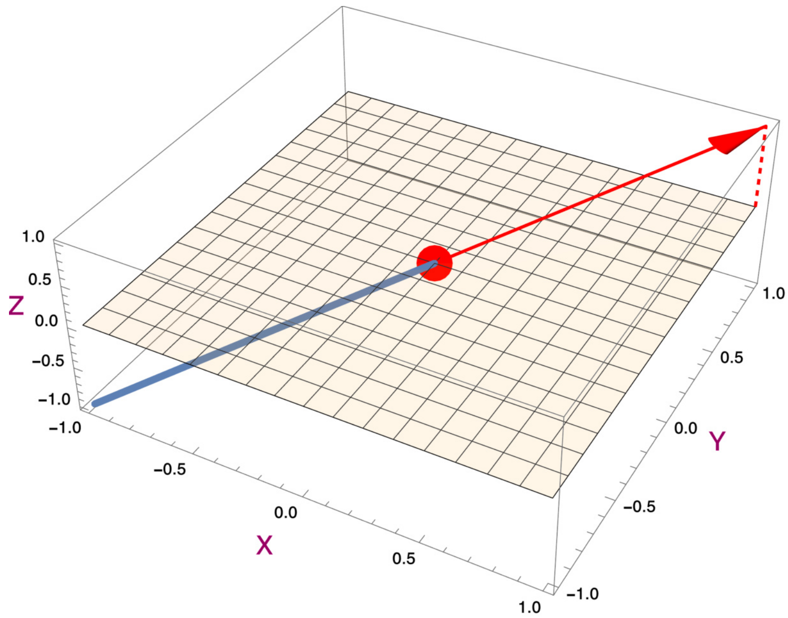

Figure 157. We see here a figure of the candidate tangent plane and the curve with equation $(x(t), y(t), z(t)) = (t, \lambda t, f(t, \lambda t))$. The tangent vector is on this line. We see that this line intersects the candidate tangent plane transversally and not tangentially.

Exercise 17.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} x^2 y \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

17.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| x^2 y \sin\left(\frac{1}{x}\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left| x^2\, y \,\sin\left(\frac{1}{x}\right)\right| &\leq x^2\,|y|\,\left|\sin\left(\frac{1}{x}\right)\right| \\ &\leq \sqrt{x^2+y^2}^{\,2}\sqrt{x^2+y^2} \\ &\leq \sqrt{x^2+y^2}^{\,3}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon^{1/3}$. We can find a δ, so we conclude that the function is continuous.

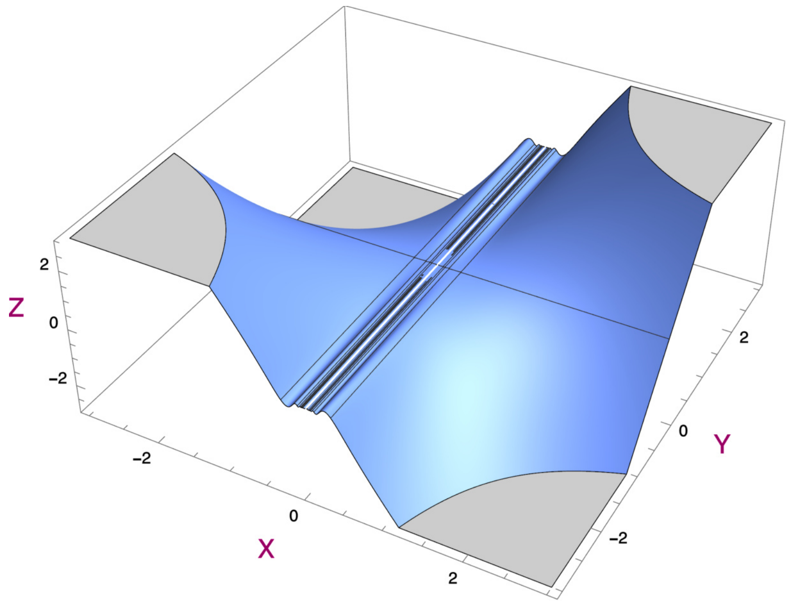

Figure 158. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

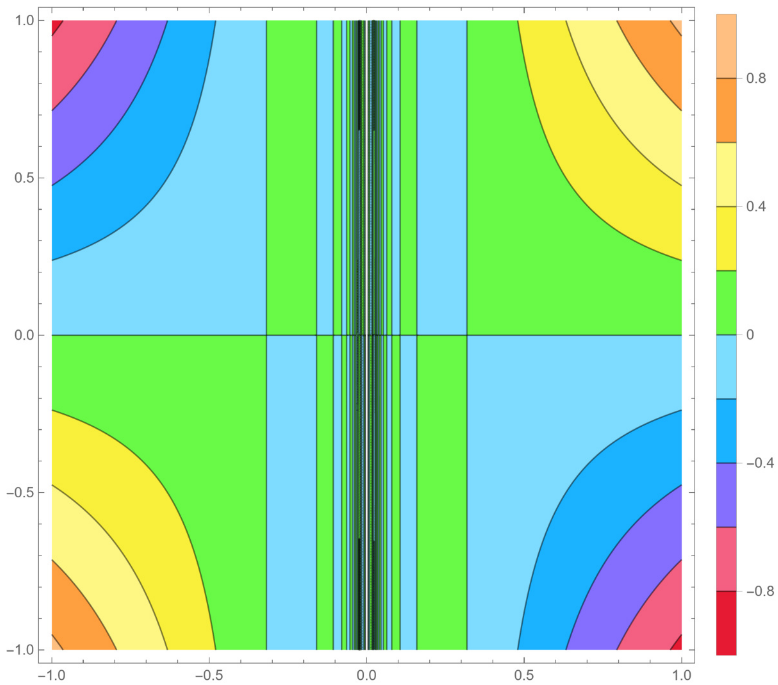

Figure 159. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

17.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).
We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

17.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$
\begin{aligned}
D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\
&= \lim_{h\to 0} h^2\,u^2\,v \sin\left(\frac{1}{h\,u}\right) \\
&= 0.
\end{aligned}
$$

We have supposed in the last limit that $u \neq 0$. We have already calculated that case in the previous section on partial derivatives.

So the directional derivatives do always exist.

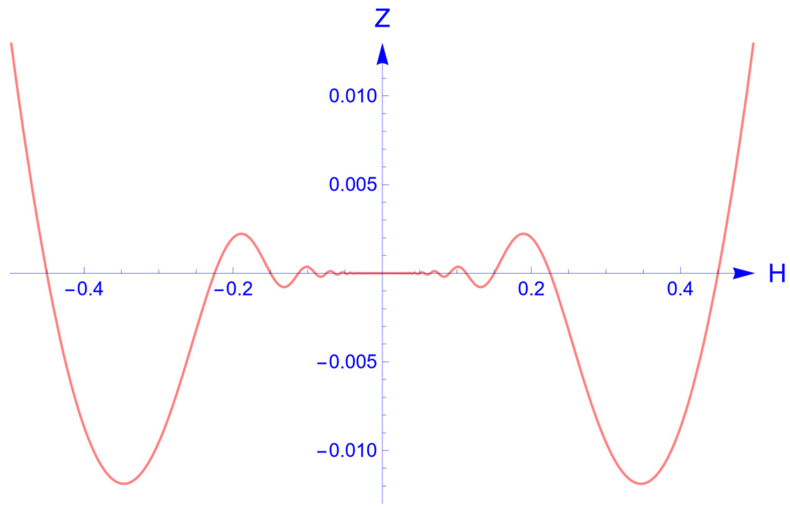

Figure 160. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

17.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we

have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

We are going to take a look at the partial derivative to x. We investigate first the existence of the partial derivatives in a point $(0, b)$ where b is close to 0. We consider the following function in $(0, b)$.

$$f(h, b) = \begin{cases} b\,h^2 \sin\left(\frac{1}{h}\right) & \text{if } h \neq 0, \\ 0 & \text{if } h = 0. \end{cases}$$

Because $\lim_{h\to 0} b\,h^2 \sin\left(\frac{1}{h}\right) = 0 = f(0, b)$, we have that the function is continuous. We have

$$\begin{aligned} \frac{\partial f}{\partial x}(0, b) &= \lim_{h\to 0} \frac{f(h, b) - f(0, b)}{h} \\ &= \lim_{h\to 0} b\,h \sin\left(\frac{1}{h}\right) \\ &= 0. \end{aligned}$$

We see that the derivative to x exists in $(0, b)$. The derivative in the Y-direction is immediately calculated because of the definition of this function. It is 0.

Remember that we already calculated the partial derivatives in $(0, 0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x, y) = \begin{cases} y\left(2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x, y) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

$$\begin{aligned}
\left|\frac{\partial f}{\partial x}(x,y)\right| &\leq \left|y\left(2\,x\,\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)\right)\right| \\
&\leq |y|\,\left(2\,|x|\,\left|\sin\left(\frac{1}{x}\right)\right| + \left|\cos\left(\frac{1}{x}\right)\right|\right) \\
&\leq |y|\,\left(2\,|x|+1\right) \\
&\leq \sqrt{x^2+y^2}\,\left(2\sqrt{x^2+y^2}+1\right) \\
&\leq 3\sqrt{x^2+y^2} \\
&\leq 3.
\end{aligned}$$

We have chosen the restriction to the neighbourhood $\sqrt{x^2+y^2} < 1$ and that justifies the last step in the inequalities.

Let us try to prove that $\left|\frac{\partial f}{\partial y}(x,y)\right|$ is bounded.

$$\begin{aligned}
\left|\frac{\partial f}{\partial y}(x,y)\right| &\leq \left|x^2\,\sin\left(\frac{1}{x}\right)\right| \\
&\leq x^2\left|\sin\left(\frac{1}{x}\right)\right| \\
&\leq x^2 \\
&\leq \sqrt{x^2+y^2}^{\,2} \\
&\leq 1.
\end{aligned}$$

We have chosen here the restriction to the neighbourhood defined by $\sqrt{x^2+y^2} < 1$ and that justifies the last step in the inequalities.

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

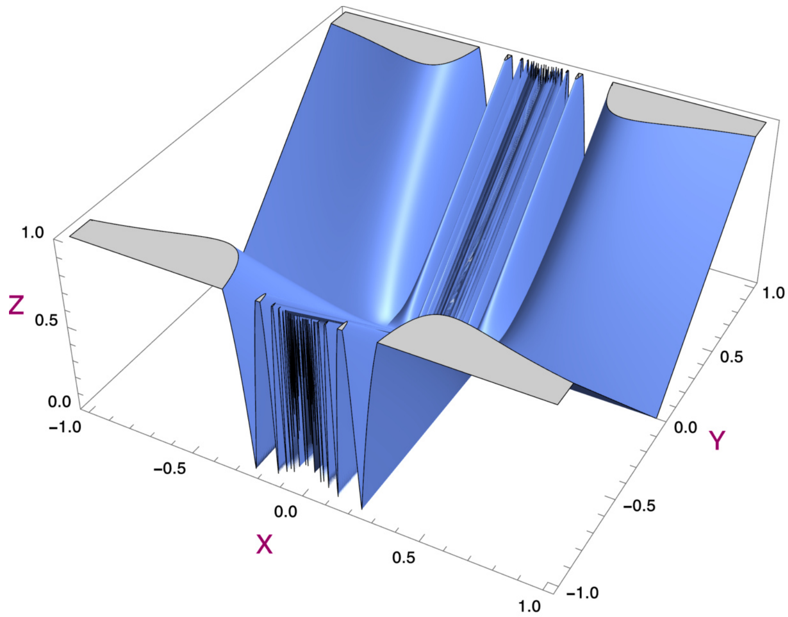

Figure 161. We see here the absolute value of the first partial derivative $\left|\dfrac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

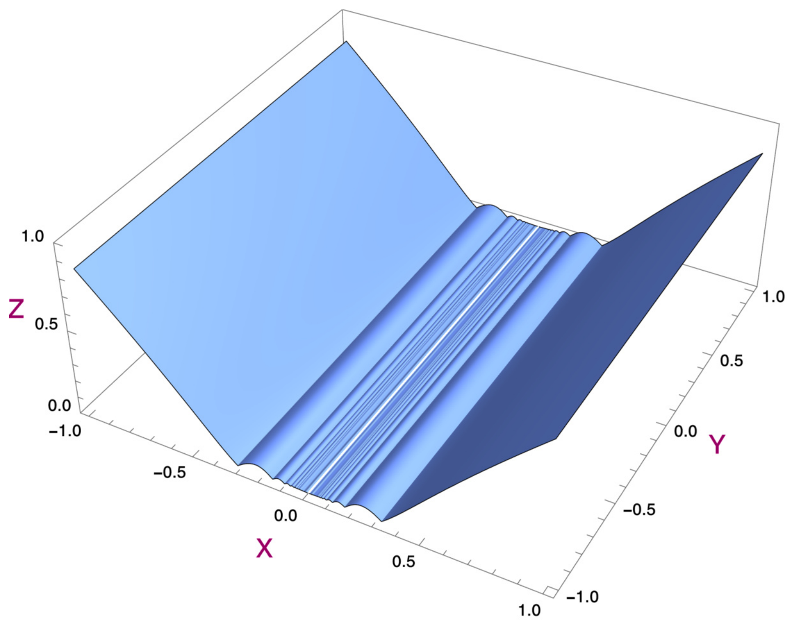

Figure 162. We see here the absolute value of the second partial derivative $\left|\dfrac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture.

17.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

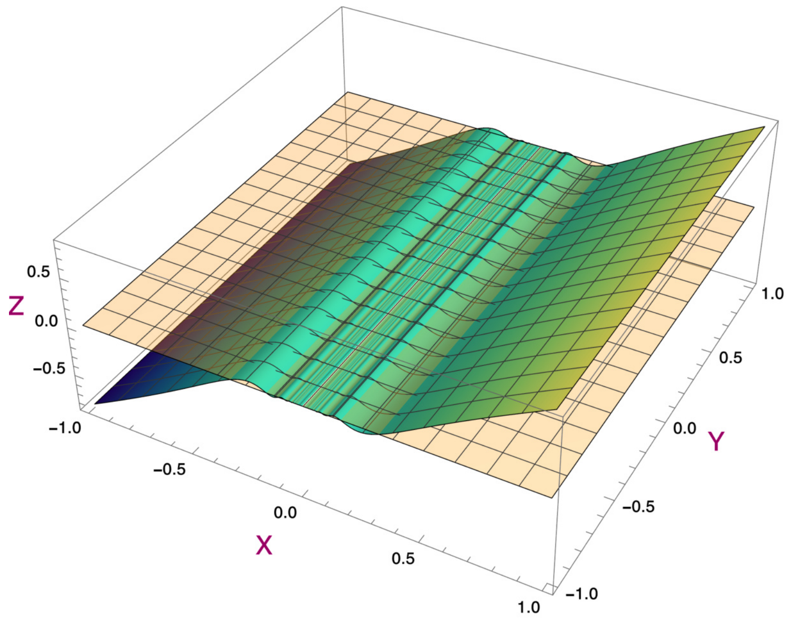

Figure 163. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x, y)$. We see here that the candidate tangent plane fits the function very nicely.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we

know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

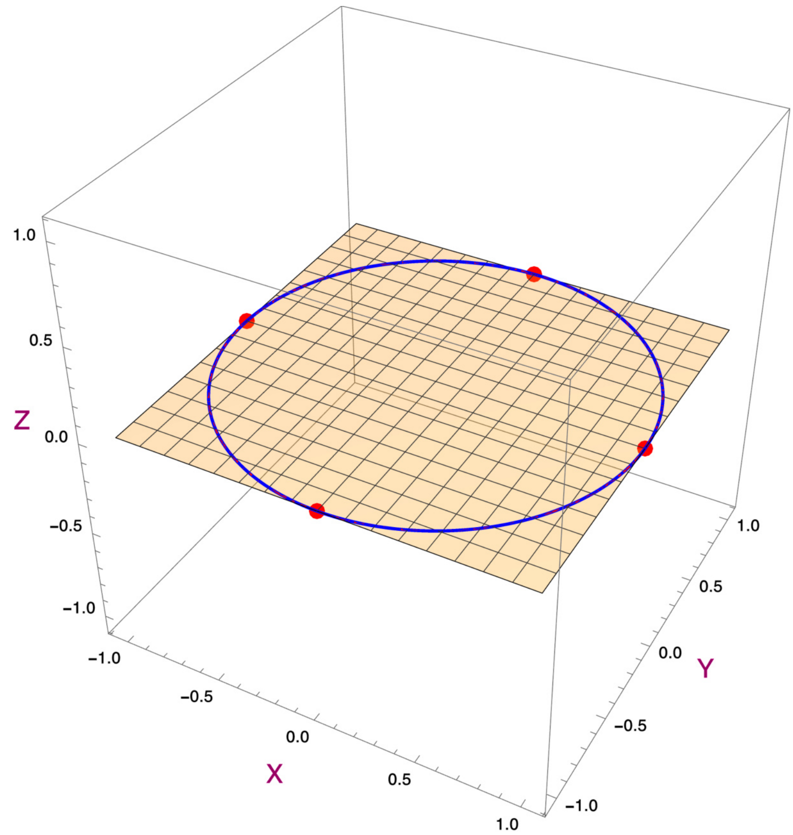

Figure 164. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h^2\,k\,\sin\left(\dfrac{1}{h}\right)}{\sqrt{h^2+k^2}} & \text{if } (h,k) \neq (0,0) \text{ and } h \neq 0; \\ 0 & \text{if } (h,k) = (0,0) \text{ or } h = 0 \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left| \frac{h^2\,k\,\sin\left(\frac{1}{h}\right)}{\sqrt{h^2+k^2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\left| \frac{h^2 k \sin\left(\frac{1}{h}\right)}{\sqrt{h^2+k^2}} \right| \leq \frac{h^2 |k| \left|\sin\left(\frac{1}{h}\right)\right|}{\sqrt{h^2+k^2}}$$

$$\leq \frac{h^2 |k|}{\sqrt{h^2+k^2}}$$

$$\leq \frac{\sqrt{h^2+k^2}^2 \sqrt{h^2+k^2}}{\sqrt{h^2+k^2}}$$

$$\leq \sqrt{h^2+k^2}^2.$$

It is sufficient to take $\delta = \epsilon^{1/2}$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous.

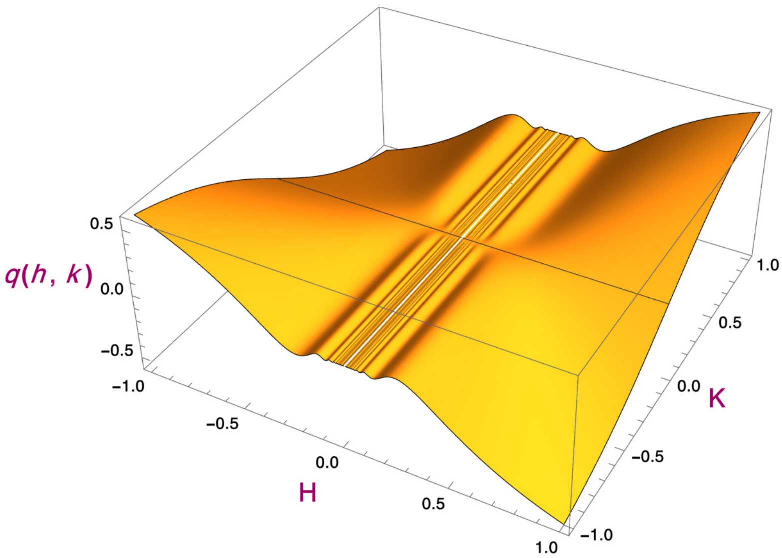

Figure 165. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

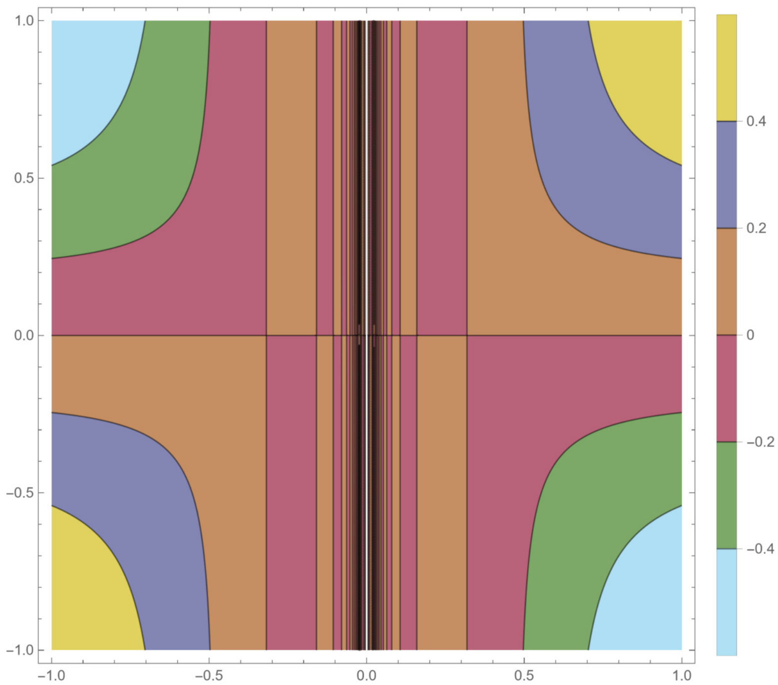

Figure 166. We see here a figure of the contour plot of the function $q(h, k)$. Only level curves of level around 0 come close to $(0, 0)$.

17.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0, 0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0, 0) = \frac{\partial f}{\partial x}(0, 0)\, u + \frac{\partial f}{\partial y}(0, 0)\, v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h, k)$. It turns out that we only have to prove now that the func-

tion $f(x, y)$ is locally Lipschitz continuous in $(0, 0)$. We cite here the criterion that we will use.

A function is differentiable in (a, b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.

2. The directional derivatives have the following form: $D_{(u,v)}(a, b) = \nabla f(a, b) \cdot (u, v) = \frac{\partial f}{\partial x}(a, b)\, u + \frac{\partial f}{\partial y}(a, b)\, v$.

3. The function is locally Lipschitz continuous in (a, b). This means that there exists at least one neighbourhood of (a, b) and a number $K > 0$ such that for all (x_1, y_1) and (x_2, y_2) in the neighbourhood, we have $|f(x_1, y_1) - f(x_2, y_2)| < K \|(x_1, y_1) - (x_2, y_2)\|$. Remark that the same K must be valid for all (x_1, y_1) and (x_2, y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be "locally flat" and thus "locally linear" which is partly expressed by this Lipschitz continuity condition.

Let us try to prove the Lipschitz condition. We are going to use a well known method.

$$
\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)| \\
&\qquad = |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\qquad \le |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)|.
\end{aligned}
$$

We focus now on the first term $|f(x_1, y_1) - f(x_1, y_2)|$. We fix now x_1 and look upon $f(x_1, y)$ as a function in one variable y. So it springs to mind that we can estimate this by using Lagrange's intermediate value theorem. So we have

$$
|f(x_1, y_1) - f(x_1, y_2)| = \left| \frac{\partial f}{\partial y}(x_1, \xi) \right| \, |y_1 - y_2|
$$

where ξ is a number in the open interval (y_1, y_2) if e.g. $y_1 \le y_2$. Remark that we already have proven that the partial derivatives are bounded in a neighbourhood, we can take that bound found for $\frac{\partial f}{\partial y}$, say M_2 and that neighbourhood in this case also. So $\left|\frac{\partial f}{\partial y}(x_1, \xi)\right| \le M_2$. We work in a completely analogous way for the second term $|f(x_1, y_2) - f(x_2, y_2)|$. We have in that case $\left|\frac{\partial f}{\partial x}(\xi, y_2)\right| \le M_1$ where ξ is now a number in the open interval (x_1, x_2) if e.g. $x_1 \le x_2$.

We take up our inequality again

$$\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)| \\
&\quad \le |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\quad \le |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)| \\
&\quad \le M_2\, |(y_1 - y_2)| + M_1\, |(x_1 - x_2)| \\
&\quad \le M_2 \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + M_1 \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&\quad \le (M_1 + M_2) \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.
\end{aligned}$$

So this function is locally Lipschitz continuous. We have an alternative proof for the differentiability.

17.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

We know that the first partial derivative exists and is equal to

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} y\left(2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We want to see if it is continuous or not.

We observe **three** cases. The **first** case is $x \neq 0$. The **second** one is the point $(0,0)$. The **last one** is the points $(0,b)$ where $b \neq 0$.

For the **first** case $x \neq 0$ we remark that we work there with classical functions that are infinitely differentiable. There is no problem in that case.

Let us start investigating the **second** case.

Discussion of the continuity of the first partial derivative in $(0,0)$. We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove the inequality $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if the inequality $||(x,y) - (0,0)|| < \delta$ holds, it follows that $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{\partial f}{\partial x}(x,y)\right| &\leq \left|y\left(2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)\right)\right| \\ &\leq |y|\left(2|x|\left|\sin\left(\frac{1}{x}\right)\right|+\left|\cos\left(\frac{1}{x}\right)\right|\right) \\ &\leq |y|\ (2|x|+1) \\ &\leq \sqrt{x^2+y^2}\left(2\sqrt{x^2+y^2}+1\right) \\ &\leq 3\sqrt{x^2+y^2}.\end{aligned}$$

We have chosen here the restriction to the neighbourhood defined by $\sqrt{x^2+y^2}<1$.

It is sufficient to take $\delta=\epsilon/3$. We can find a δ, so we conclude that the function is continuous in $(0,0)$.

We investigate now the **third** case. Let us define the following function in $(0,b)$, $b\neq 0$ in the X-direction. This function equals

$$\frac{\partial f}{\partial x}(h,b)=\begin{cases}2\,b\,h\sin\left(\frac{1}{h}\right)-b\cos\left(\frac{1}{h}\right) & \text{if } h\neq 0,\\ 0 & \text{if } h=0.\end{cases}$$

We see that the first term in the main definition is certainly continuous in $h=0$. We doubt however the continuity of the second term. This term equals

$$-b\cos\left(\frac{1}{h}\right).$$

We drop the non zero constant coefficient of the term and give it a name

$$g(h)=\cos\left(\frac{1}{h}\right).$$

This is a standard example of a non continuous function. We can be more explicit though. Let us define a sequence $h_n=\frac{1}{2\pi n}$, $n\in\mathbf{N}_0$, that converges to zero. We see that $g(h_n)=1$ and $\lim_{n\to\infty}g(h_n)=1$. We define also another sequence $h_n=\frac{2}{\pi(4n+1)}$, $n\in\mathbf{N}_0$, that converges to zero. We see that $g(h_n) = 0$ and $\lim_{n\to\infty} g(h_n) = 0$. This is impossible if g is continuous.

We conclude that the partial derivative to x is not continuous in any neighbourhood of $(0,0)$. We cannot apply this criterion for proving differentiability.

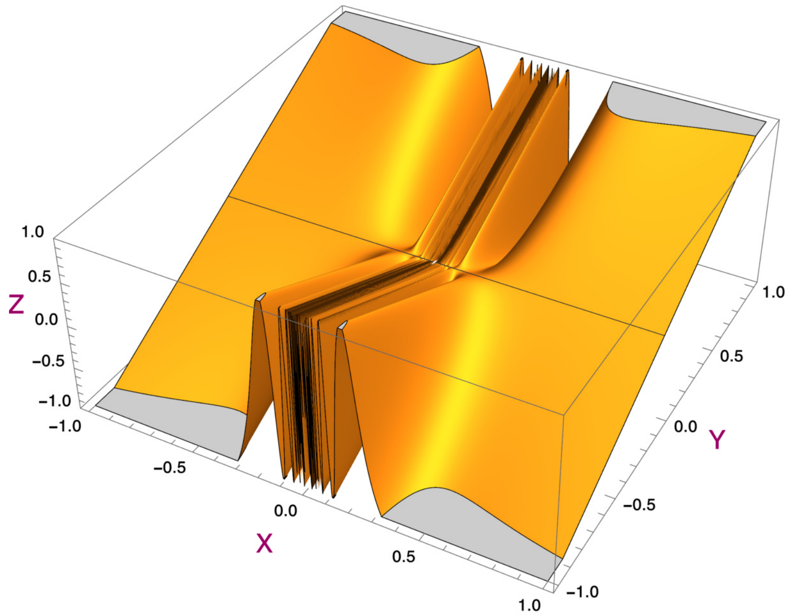

Figure 167. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial x}(x,y)$. This looks like a continuous function in $(0,0)$. There are grave doubts about the continuity in the points $(0,b)$.

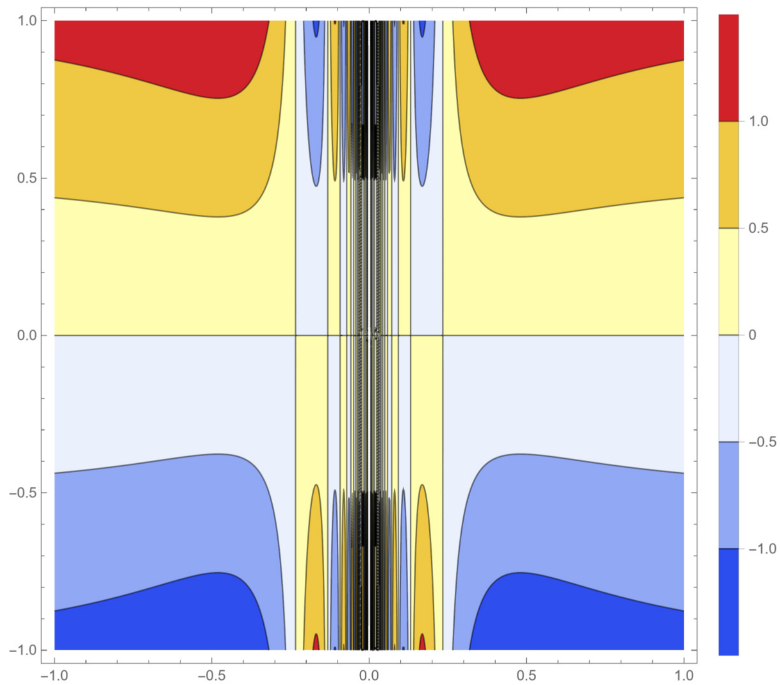

Figure 168. We see here a figure of the contour plot of the $\frac{\partial f}{\partial x}(x, y)$. Only level curves of level around 0 come close to $(0,0)$. This is an indication of continuity in $(0,0)$ for this partial derivative. There are grave doubts about the continuity in the points $(0, b)$.

17.8 Overview

$$f(x,y) = \begin{cases} x^2 y \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

17.9 One step further

We want to know if this function is further uneventful from the point of view of differentiability. Let us take a look at the second order partial derivative

$$\frac{\partial^2 f}{\partial x^2}(x, y) = \frac{y\left(\left(2x^2 - 1\right)\sin\left(\frac{1}{x}\right) - 2x\cos\left(\frac{1}{x}\right)\right)}{x^2}.$$

Let us take a look of a three dimensional plot of this second order partial derivative of the function.

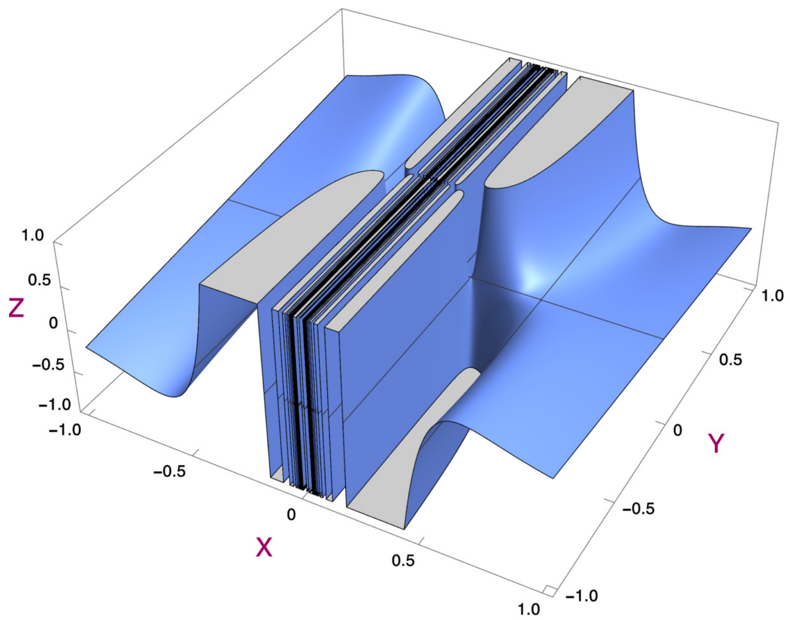

Figure 169. We see here a figure of the second order partial derivative $\frac{\partial^2 f}{\partial x^2}(x, y)$. It seems quite improbable that this second order partial derivative is continuous. We stop however our investigations here and leave this to the initiative of the interested reader.

Exercise 18.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

18.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{xy}{\sqrt{x^2+y^2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{x\,y}{\sqrt{x^2+y^2}}\right| &\leq \frac{|x|\,|y|}{\sqrt{x^2+y^2}}\\ &\leq \frac{\sqrt{x^2+y^2}\,\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\\ &\leq \sqrt{x^2+y^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function is continuous.

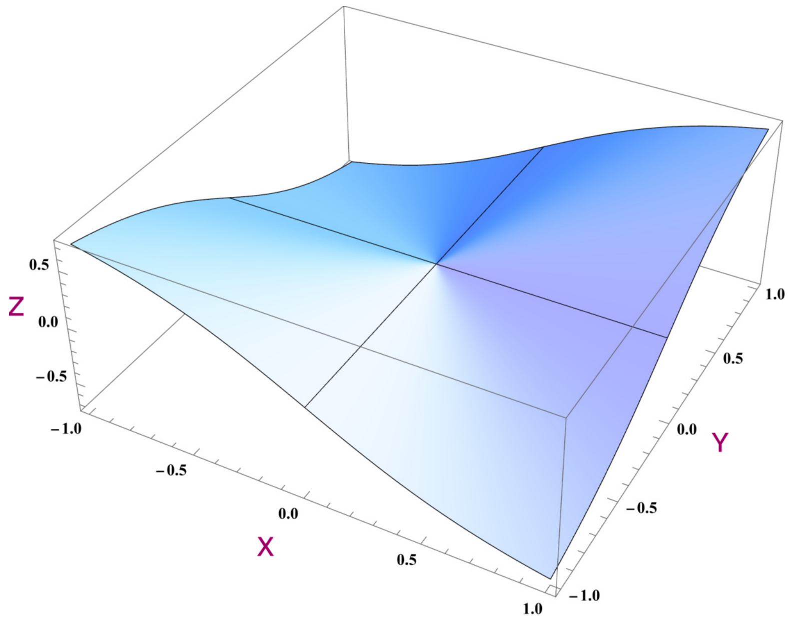

Figure 170. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

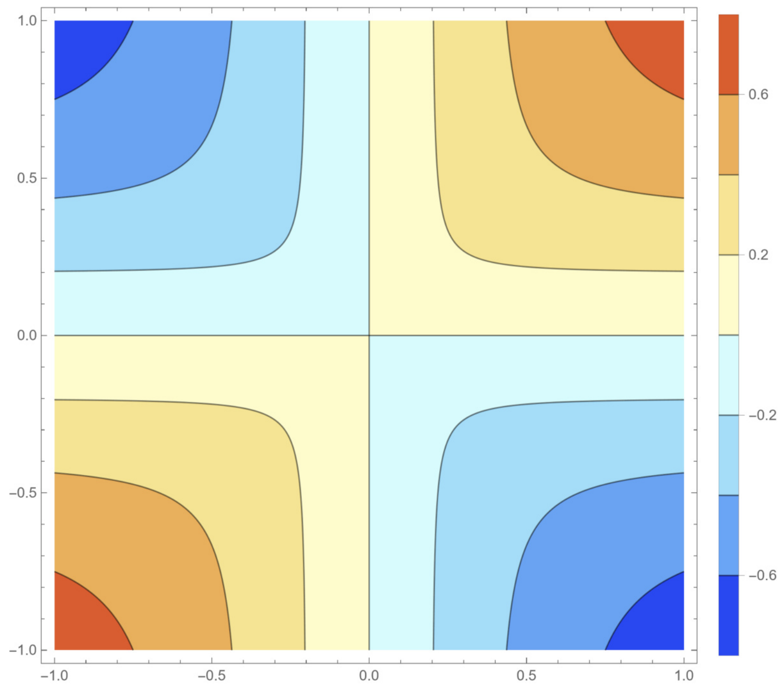

Figure 171. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

18.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).
We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} \\ &= \lim_{h \to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

18.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h \to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$
\begin{aligned}
D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{h\,u\,v}{\sqrt{h^2\,(u^2+v^2)}} \\
&= \lim_{h\to 0} \operatorname{sgn}(h)\,\frac{u\,v}{\sqrt{u^2+v^2}}.
\end{aligned}
$$

So the directional derivatives do not exist if $u\,v \neq 0$. They only exist in the X-direction and the Y-direction.

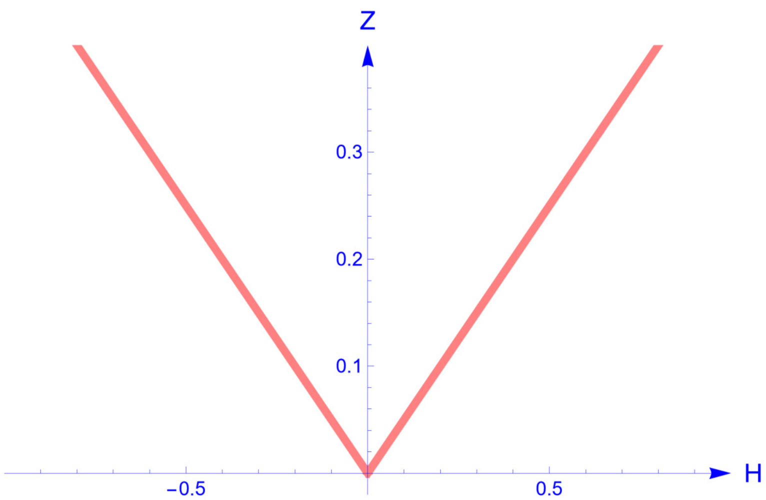

Figure 172. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$. We see that there is no tangent line in 0.

18.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{y^3}{\left(x^2+y^2\right)^{3/2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{x^3}{\left(x^2+y^2\right)^{3/2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

$$\begin{aligned} \left|\frac{\partial f}{\partial x}(x,y)\right| &\leq \left|\frac{y^3}{(x^2+y^2)^{3/2}}\right| \\ &\leq \frac{|y|^3}{(x^2+y^2)^{3/2}} \\ &\leq \frac{\sqrt{x^2+y^2}^{\,3}}{\sqrt{x^2+y^2}^{\,3}} \\ &\leq 1. \end{aligned}$$

We can prove that $\left|\frac{\partial f}{\partial y}\right|$ is bounded in an analogous way due to the symmetry of the function definition.

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

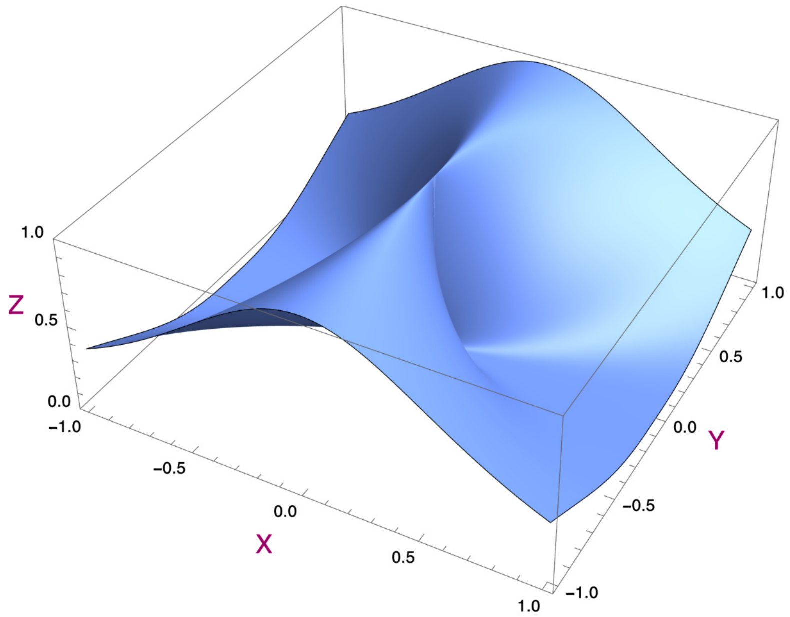

Figure 173. We see here the absolute value of the first partial derivative $\left|\dfrac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

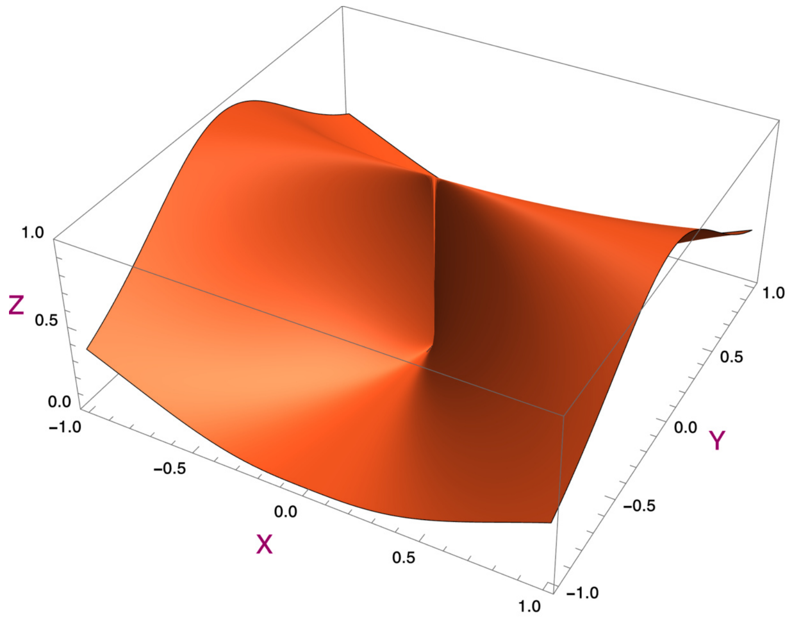

Figure 174. We see here the absolute value of the second partial derivative $\left|\dfrac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture. We can observe the high symmetry with the first partial derivative.

18.5 Differentiability

We have that some of the directional derivatives do not exist. Thus the function is not differentiable. So it is futile to continue.

18.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

18.7 Continuity of the partial derivatives

This is irrelevant. The function is not differentiable.

18.8 Overview

$$f(x,y) = \begin{cases} \dfrac{x\,y}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 19.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

19.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{x^2 y}{x^2+y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{x^2 y}{x^2+y^2}\right| &\leq \frac{x^2\,|y|}{x^2+y^2}\\ &\leq \frac{\sqrt{x^2+y^2}^{\,2}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}^{\,2}}\\ &\leq \sqrt{x^2+y^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function is continuous.

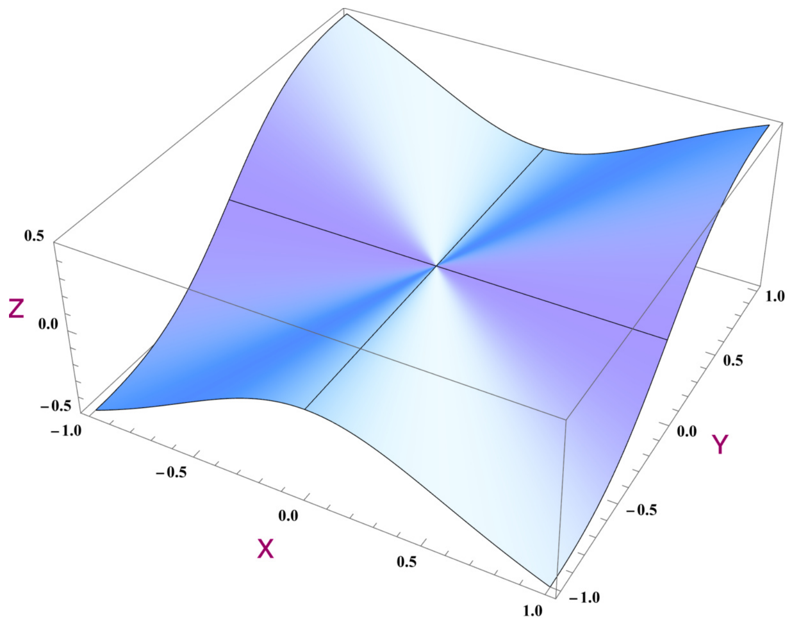

Figure 175. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

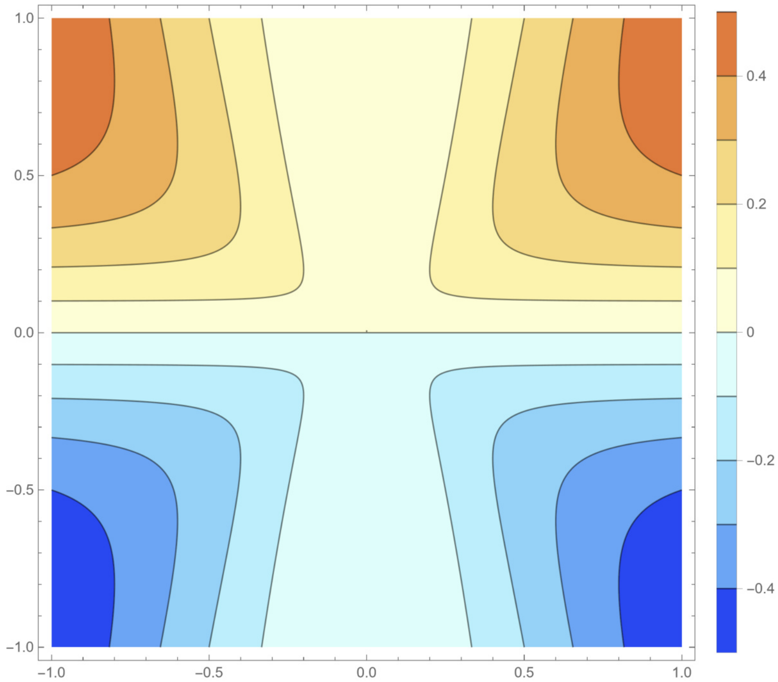

Figure 176. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

19.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

19.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{u^2\,v}{u^2+v^2} \\ &= \lim_{h\to 0} u^2\,v. \end{aligned}$$

We have in the last step made use of the normality of the vector (u,v). So the directional derivatives do always exist.

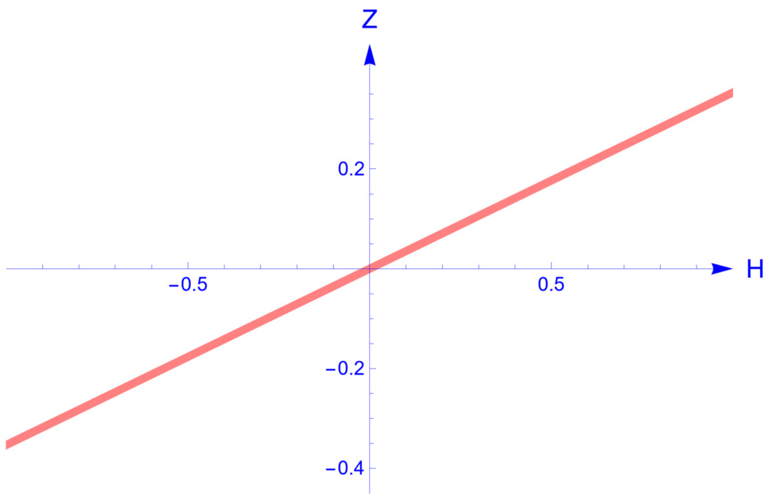

Figure 177. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

19.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{2\,x\,y^3}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{x^2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

$$\begin{aligned} \left|\frac{\partial f}{\partial x}(x,y)\right| &\leq \left|\frac{2\,x\,y^3}{(x^2+y^2)^2}\right| \\ &\leq \frac{2\,|x|\,|y|^3}{(x^2+y^2)^2} \\ &\leq \frac{2\sqrt{x^2+y^2}\,\sqrt{x^2+y^2}^{\,3}}{\sqrt{x^2+y^2}^{\,4}} \\ &\leq 2. \end{aligned}$$

Let us try to prove that $\left|\frac{\partial f}{\partial y}\right|$ is bounded.

$$\begin{aligned}
\left|\frac{\partial f}{\partial y}(x,y)\right| &\le \left|\frac{x^2\,(x^2-y^2)}{(x^2+y^2)^2}\right| \\
&\le \frac{x^2\,|\,(x^2-y^2)\,|}{(x^2+y^2)^2} \\
&\le \frac{\sqrt{x^2+y^2}^{\,2}\left(\sqrt{x^2+y^2}^{\,2}+\sqrt{x^2+y^2}^{\,2}\right)}{\sqrt{x^2+y^2}^{\,4}} \\
&\le 2.
\end{aligned}$$

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

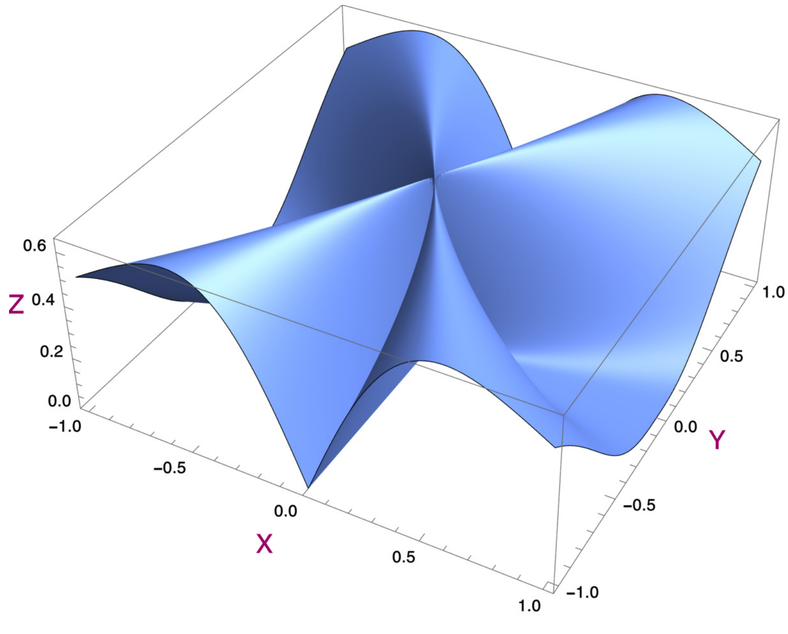

Figure 178. We see here the absolute value of the first partial derivative $\left|\frac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

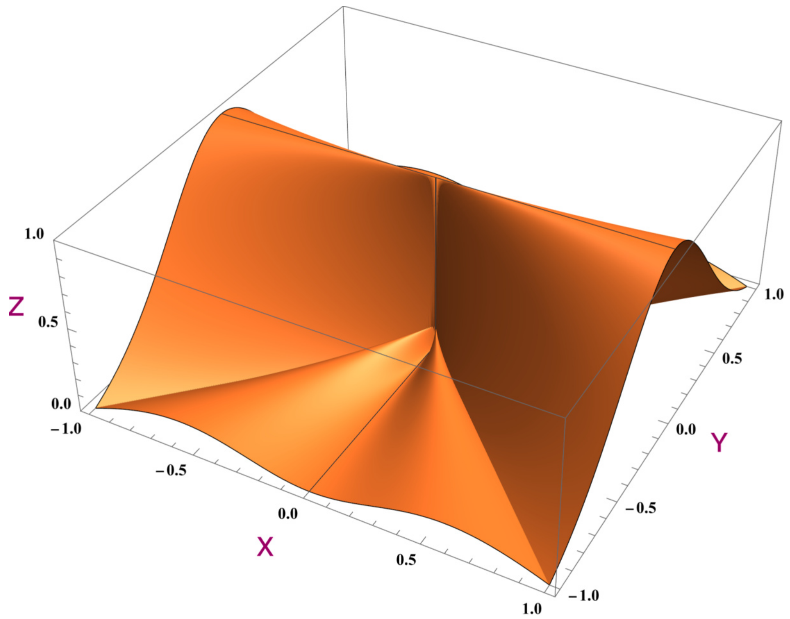

Figure 179. We see here the absolute value of the second partial derivative $\left|\dfrac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture.

19.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

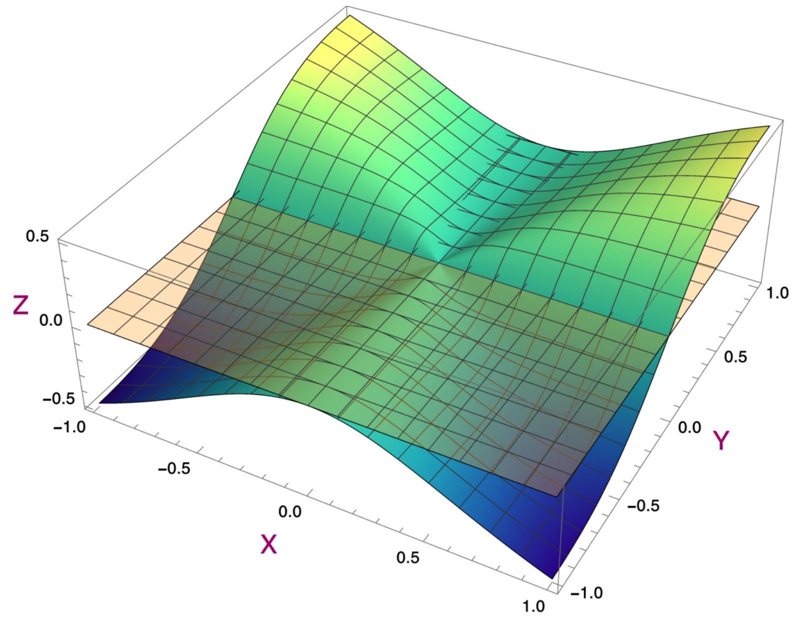

Figure 180. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function very close on the axes. But there are serious doubts about the other directions. We are certainly worried.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

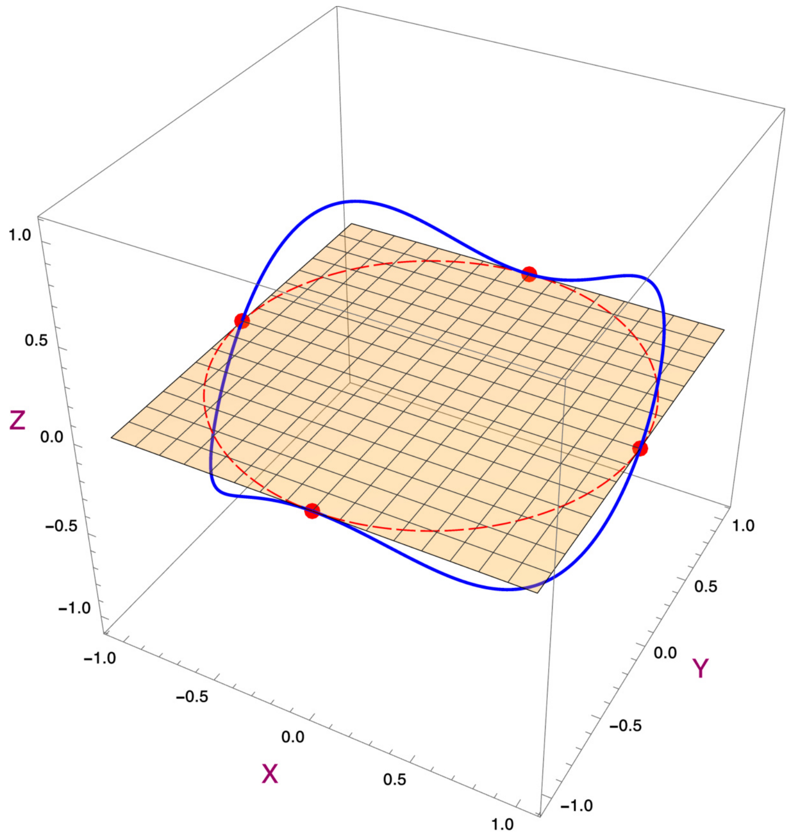

Figure 181. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0, 0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0, 0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0, 0))$ do not sweep out a nice ellipse in the candidate tangent plane. The blue curve does not lie in a plane. This is very bad news for the differentiability.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h, k)$ in $(0, 0)$ as

$$q(h, k) = \frac{f(h, k) - f(0, 0) - \left(\frac{\partial f}{\partial x}(0, 0)\, h + \frac{\partial f}{\partial y}(0, 0)\, k\right)}{\|(h, k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h^2\,k}{\left(h^2+k^2\right)^{3/2}} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We restrict the function $q(h,k)$ to the continuous curves with equations $k = \lambda\,h$. We observe then that

$$q\,|_{k=\lambda h}(h,k) = \begin{cases} q(h,\lambda\,h) = \dfrac{h^3\,\lambda}{\left(h^2\,\lambda^2+h^2\right)^{3/2}} = \dfrac{\operatorname{sgn}(h)\,\lambda}{\left(\lambda^2+1\right)^{3/2}} & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

We see that these restricted functions have no limits if $\lambda \neq 0$. But if $q(h,k)$ is continuous, all these limit values should be $q(0,0) = 0$. So this function $q(h,k)$ is not continuous in $(0,0)$. The function $f(x,y)$ is not differentiable in $(0,0)$.

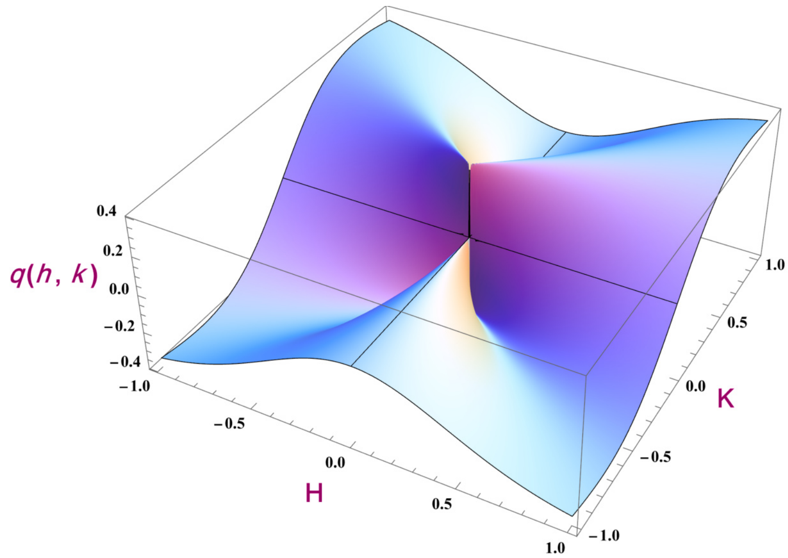

Figure 182. We see here a three dimensional figure of the graph of the function $q(h,k)$. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

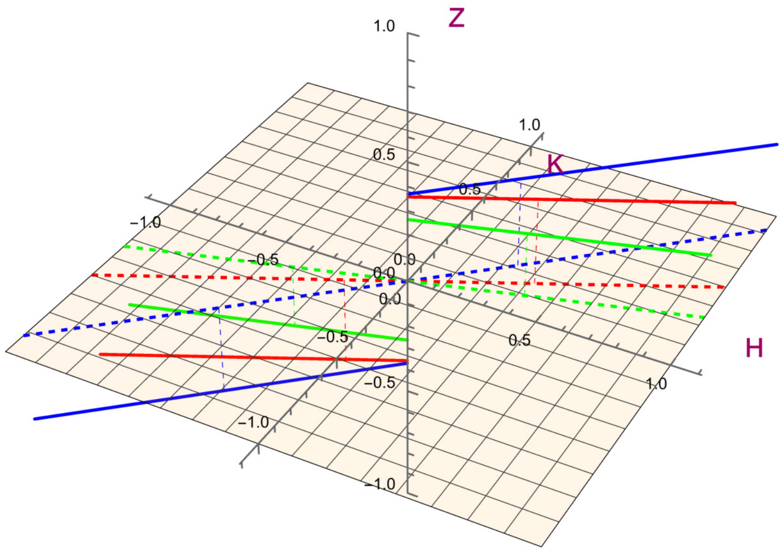

Figure 183. We have restricted the function $q(h,k)$ here to $k = 1/2\,h$ and $k = 3/10\,h$ and $k = 9/10\,h$. We see in this figure clearly that the restrictions of the function $q(h,k)$ to these lines are functions that have different limits in 0.

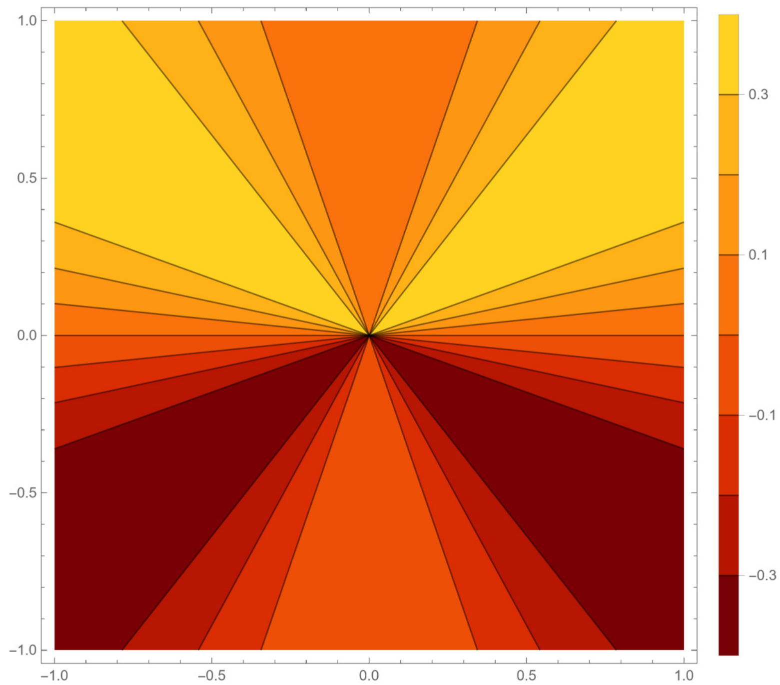

Figure 184. We see here a figure of the contour plot of the function $q(h,k)$. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

19.6 Alternative proof of differentiability (optional)

This section is **irrelevant** for this exercise, because the function is not differentiable.

19.7 Continuity of the partial derivatives

This section is **irrelevant** for this exercise, because the function is not differentiable.

19.8 Overview

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

19.9 One step further

We have met the magical curves $k = \lambda h$ when investigating the differentiability. We want to see what is really going on there. Let us take a look at the curve

$$(x(t), y(t), z(t)) = (t, \lambda t, f(t, \lambda t)) = \left(t, \lambda t, \frac{\lambda t}{\lambda^2 + 1}\right).$$

This curve projects to the curve $k = \lambda h$ on the (h, k)-plane. Let us calculate the tangent vector in $t = 0$ of this curve. This is a straight line and the tangent vector $\left(1, \lambda, \frac{\lambda}{\lambda^2+1}\right)$ is on this line. But if f is differentiable, then this tangent vector $\left(1, 1, \frac{1}{2}\right)$ lies in the tangent plane. But we see that this vector is not in the candidate tangent plane. So f cannot be differentiable.

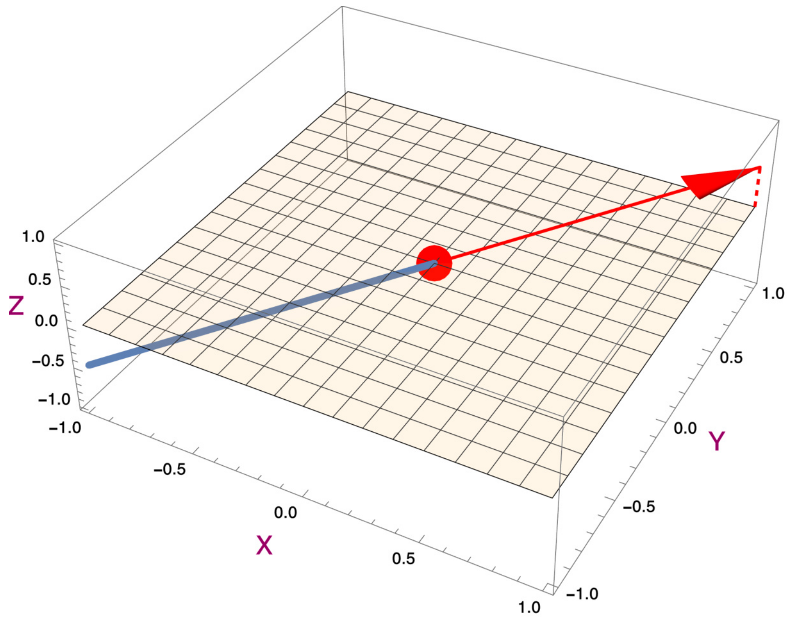

Figure 185. We see here a figure of the candidate tangent plane and the curve with equation $(x(t), y(t), z(t)) = (t, \lambda t, f(t, \lambda t))$. The tangent vector is right upon on this line. We see that this line intersects the candidate tangent plane transversally and not tangentially.

It can maybe be worthwhile to take a look at the gradient vector field. We see that this gradient vector field exists in this case. But the gradient vector field cannot be continuous because it implies that the function is differentiable. So we wonder if we can find an indication for this fact.

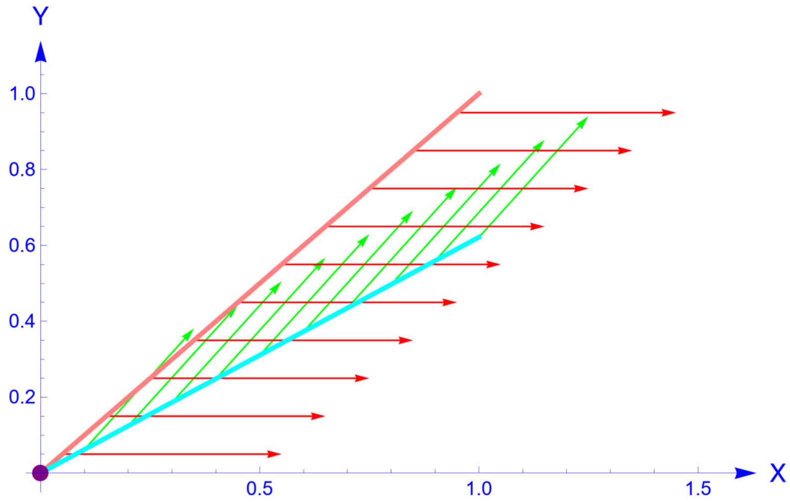

Figure 186. We made here the following sketch. We have drawn the graphics of $y = x$ in pink and $y = 0.62\,x$ in cyan. We have sketched the gradient vector field $\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$ which is not continuous as can be seen as follows. Observe the gradient vector field on the pink curve, these are the red vectors. The purple vector is the gradient vector in $(0,0)$, which is $(0,0)$. Observe the gradient vector field on the cyan curve, these are the green vectors. The red vectors converge to a vector with a non zero x-component. This component is equal to $1/2$. The green vectors converge to a vector that has an x-component that is two times smaller then the x-component of the vector to which the red vectors converge if $x \to 0$. This is clearly impossible if the vector field is continuous. Moreover, the x-component of the limit vector should in both cases be 0 if the gradient vector field is continuous. We conclude that this gradient vector field is not continuous. If it is continuous, then the function is differentiable and that is not the case. Please note however that a sketch of the gradient vector field is inherently a sketch of discrete data. So the utmost care must be taken in order to make it a little bit trustworthy.

Exercise 20.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

20.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{x^3+y^3}{x^2+y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$
\begin{aligned}
\left| \frac{x^3+y^3}{x^2+y^2} - 0 \right| &\leq \frac{|x|^3+|y|^3}{x^2+y^2} \\
&\leq \frac{\sqrt{x^2+y^2}^3 + \sqrt{x^2+y^2}^3}{\sqrt{x^2+y^2}^2} \\
&\leq \frac{2\sqrt{x^2+y^2}^3}{\sqrt{x^2+y^2}^2} \\
&\leq 2\sqrt{x^2+y^2}.
\end{aligned}
$$

It is sufficient to take $\delta = \epsilon/2$. We can find a δ, so we conclude that the function is continuous.

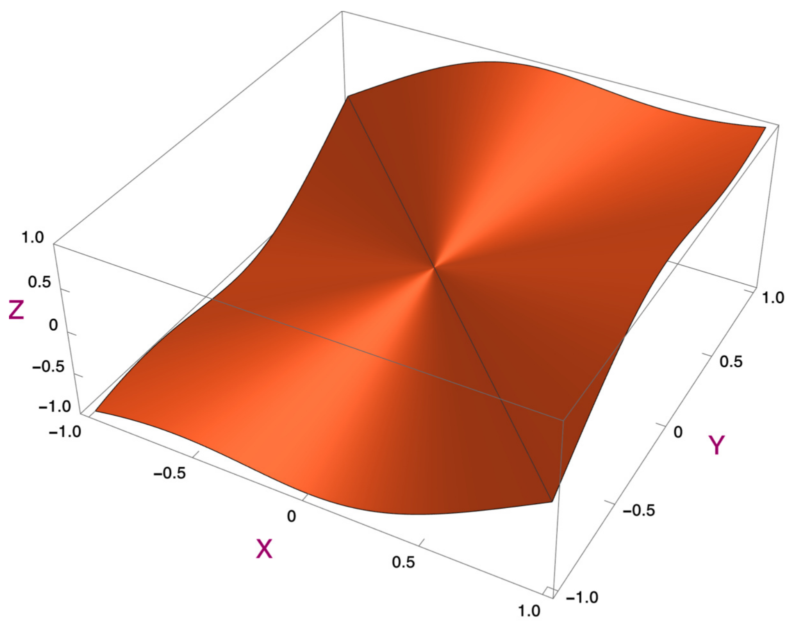

Figure 187. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

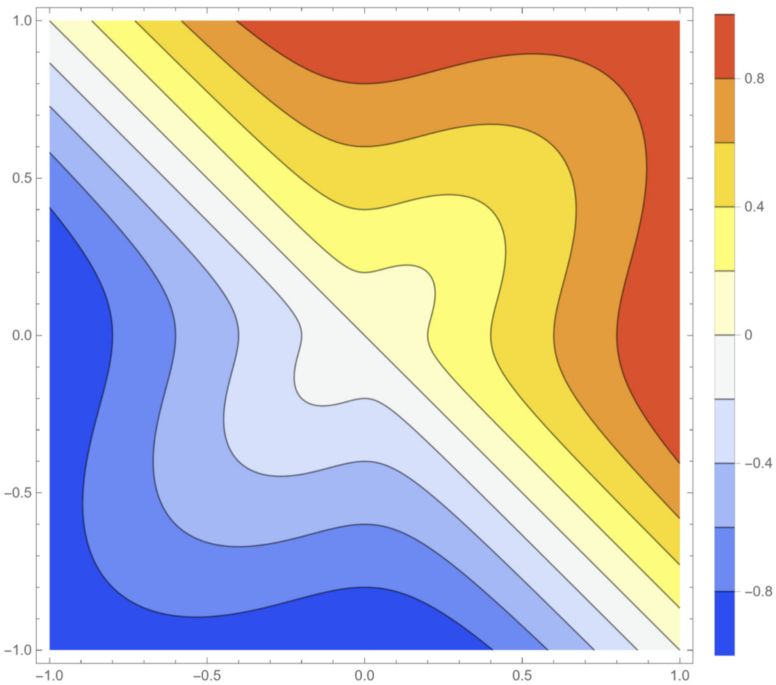

Figure 188. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0, 0)$.

20.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).
We observe then that

$$f\big|_{y=0}(x, y) = \begin{cases} f(x, 0) = x & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial x}(0, 0) &= \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} \\ &= \lim_{h \to 0} 1 \\ &= 1. \end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = y & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} \\ &= \lim_{h \to 0} 1 \\ &= 1. \end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 1 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 1.$$

20.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h \to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$
\begin{aligned}
D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{u^3+v^3}{u^2+v^2} \\
&= u^3+v^3.
\end{aligned}
$$

We remember for the last step that (u, v) is a normalised vector.

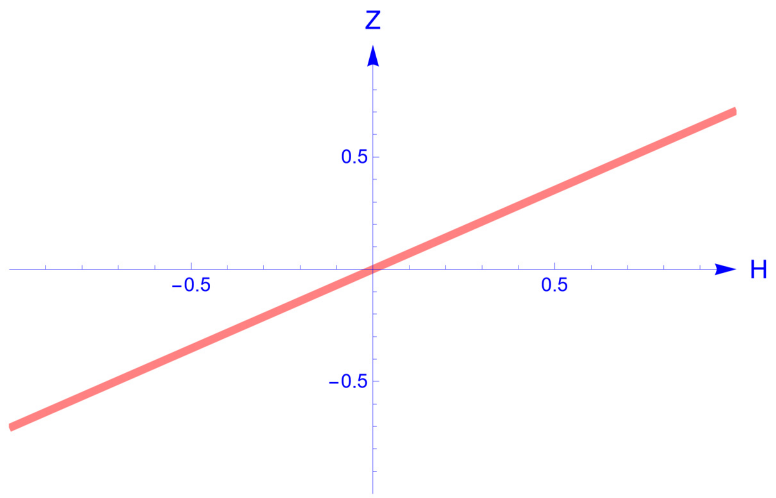

Figure 189. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$.

20.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{x\left(x^3 + 3\,x\,y^2 - 2\,y^3\right)}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{y\left(-2\,x^3 + 3\,x^2\,y + y^3\right)}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

$$\begin{aligned} \left|\frac{\partial f}{\partial x}(x,y)\right| &\leq \left|\frac{x\,(x^3 + 3\,x\,y^2 - 2\,y^3)}{(x^2+y^2)^2}\right| \\ &\leq \frac{|x|\,(|x|^3 + 3\,|x|\,y^2 + 2\,|y|^3)}{(x^2+y^2)^2} \\ &\leq \frac{\sqrt{x^2+y^2}\,6\,\sqrt{x^2+y^2}^3}{(x^2+y^2)^2} \\ &\leq 6\,\sqrt{x^2+y^2}^2 \\ &\leq 6. \end{aligned}$$

We have chosen here the restriction to the neighbourhood defined by $\sqrt{x^2+y^2} < 1$.

We are not going to prove that $\left|\frac{\partial f}{\partial y}\right|$ is bounded. The proof can be given in a similar way because of symmetry considerations.

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

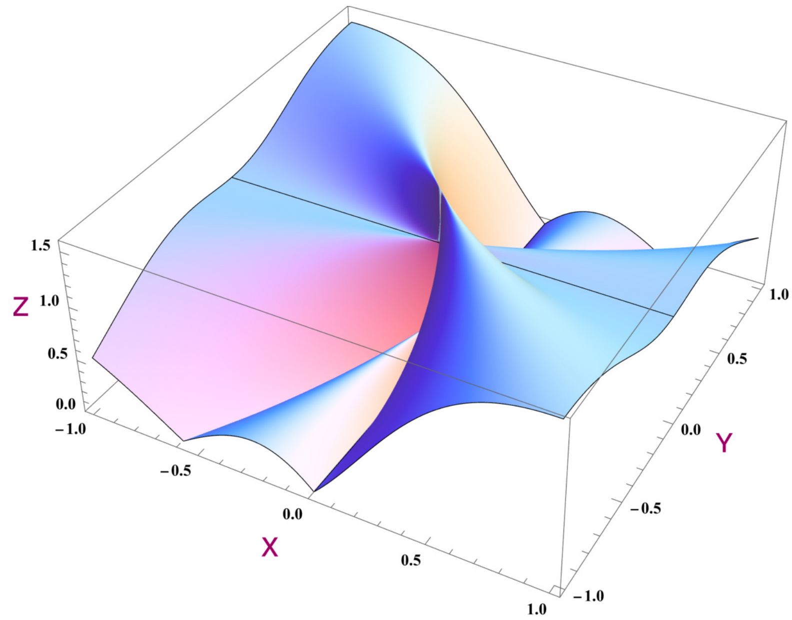

Figure 190. We see here the absolute value of the first partial derivative $\left|\frac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

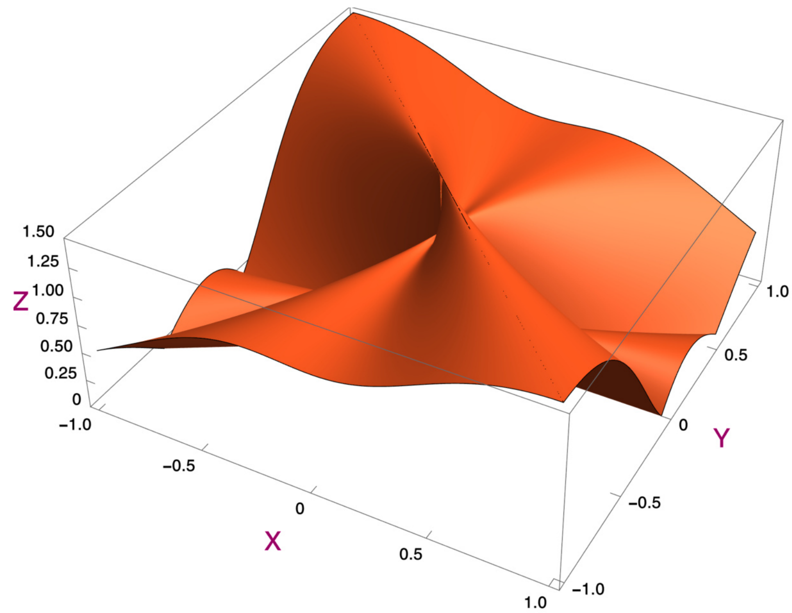

Figure 191. We see here the absolute value of the second partial derivative $\left|\frac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture.

20.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

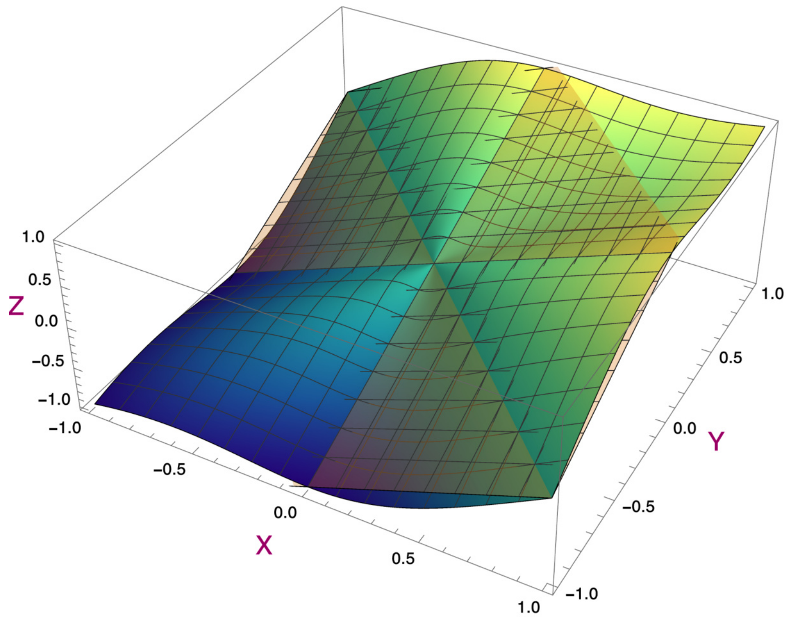

Figure 192. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x, y)$. We see here that the candidate tangent plane fits the function quite nicely at first sight.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we

know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

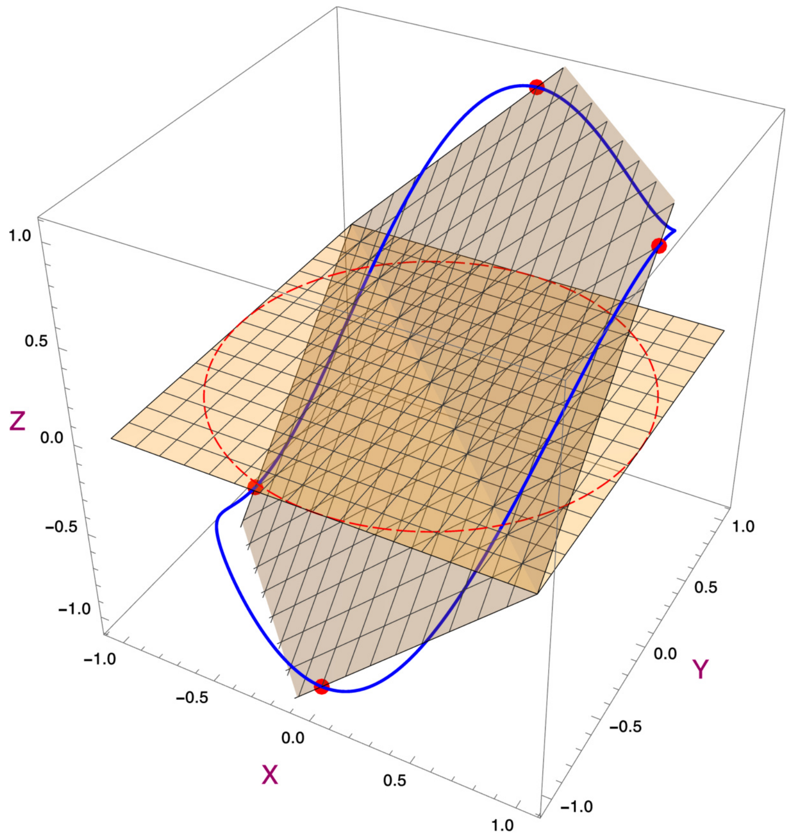

Figure 193. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ do not sweep out a nice ellipse lying in one plane! The blue curve should be a nice ellipse lying in the candidate tangent plane in the case of differentiability. This is very bad news for differentiability.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} -\dfrac{h\,k\,(h+k)}{\left(h^2+k^2\right)^{3/2}} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0). \end{cases}$$

is continuous in $(0,0)$.

We restrict the function $q(h,k)$ to the continuous curves with equations $k = \lambda\,h$. We observe then that

$$q\,|_{k=\lambda h}(h,k) = \begin{cases} q(h,\lambda\,h) & \\ \quad = -\dfrac{h^3\,\lambda\,(\lambda+1)}{\left(h^2\,\left(\lambda^2+1\right)\right)^{3/2}} & \text{if } h \neq 0; \\ \quad = -\dfrac{\operatorname{sgn}(h)\,\lambda\,(\lambda+1)}{\left(\left(\lambda^2+1\right)\right)^{3/2}} & \\ 0 & \text{if } h = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $q(h,k)$ is continuous, all these limit values should be $q(0,0) = 0$. So this function $q(h,k)$ is not continuous in $(0,0)$. The function $f(x,y)$ is not differentiable in $(0,0)$.

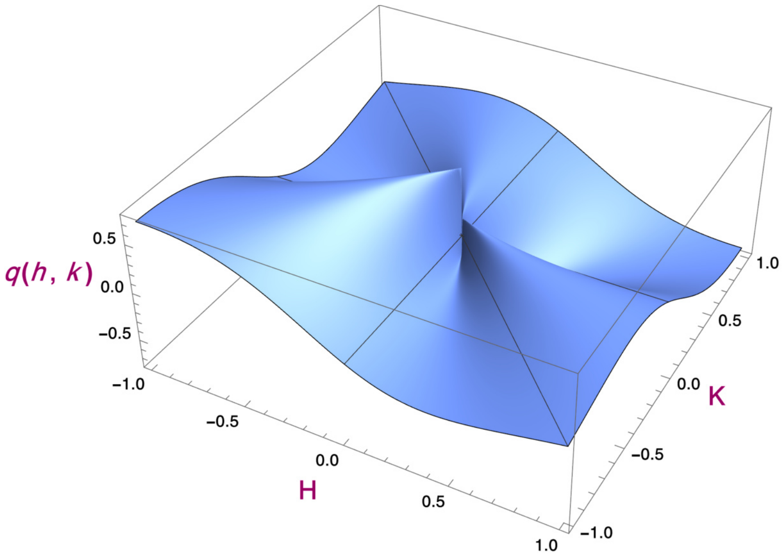

Figure 194. We see here a three dimensional figure of the graph of the function $q(h,k)$. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

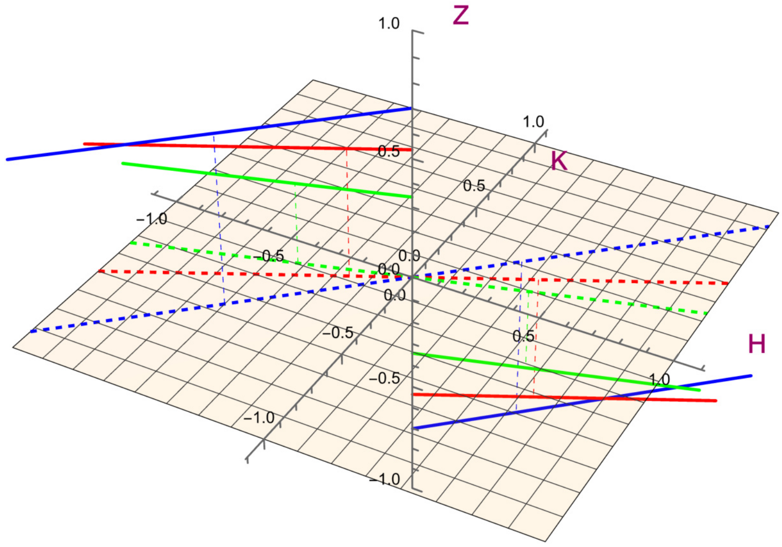

Figure 195. We have restricted the function $q(h, k)$ here to $k = 1/2\,h$ and $k = 3/10\,h$ and $k = 9/10\,h$. We see in this figure clearly that the restrictions of the function to these lines are functions that have no limits in 0.

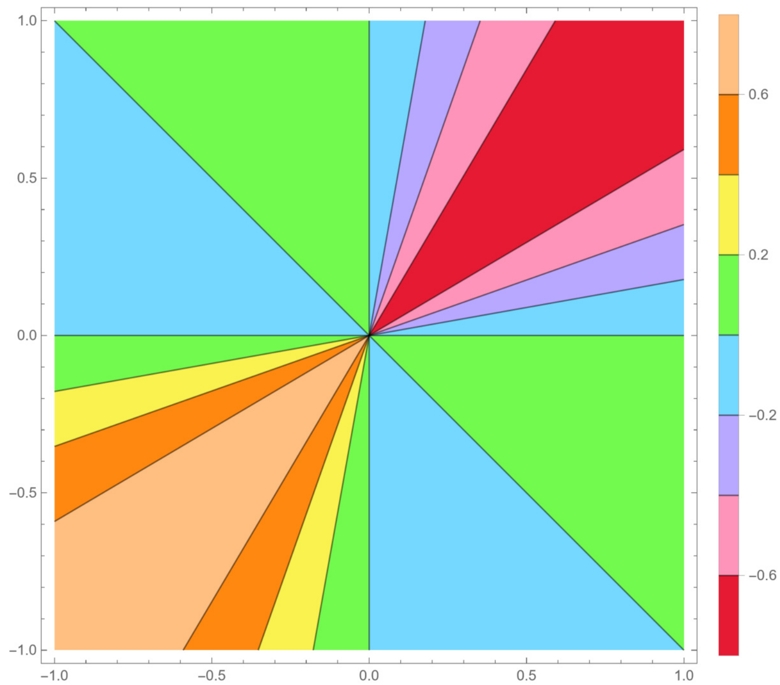

Figure 196. We see here a figure of the contour plot of the function $q(h,k)$. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Alternative proof for the non differentiability

Suppose that we already met in the course the differentiation rule of the composition of two differentiable functions. This is also called the chain rule. Then we have proven the following. If the function is differentiable in (a,b), then the directional derivative can be calculated is follows.

$$D_{(u,v)}f(a,b) = \frac{\partial f}{\partial x}(a,b)\,u + \frac{\partial f}{\partial y}(a,b)\,v\,.$$

Important remark. This formula is only valid if the function is differentiable. One of the most common mistakes is that one uses this formula in the case of non differentiability. It seems to be easy to calculate quickly the partial derivatives if they exist and then use this formula.

We have calculated the directional derivatives and we saw that

$$D_{(u,v)}f(a,b) = u^3 + v^3$$

and this is certainly not the linear function in u and v which we should have in the case of differentiability. So we conclude that the function is not differentiable.

20.6 Alternative proof of differentiability (optional)

This section is **irrelevant** for this exercise, because the function is not differentiable.

20.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

20.8 Overview

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

20.9 One step further

We are interested in the magical curves $k = \lambda h$ which we used in the proof of non differentiability.

Let us define a plane curve $\beta(t) = (t, \lambda t)$. This curve goes through $(0,0)$ and is differentiable. Now we take the space curve that projects to $\beta(t)$: $\alpha(t) = (t, \lambda t, f(t, \lambda t)) = \left(t, \lambda t, \frac{(\lambda^3+1)\, t}{\lambda^2+1}\right)$. This curve lies on the surface defined by the function f. We see that this curve is differentiable, but remark that this is not guaranteed because f is not differentiable!

We calculate the derivative in $t = 0$.

$$\alpha'(0) = \left(1, \lambda, \frac{\lambda^3 + 1}{\lambda^2 + 1}\right).$$

We draw the curve, the tangent line and the candidate tangent plane.

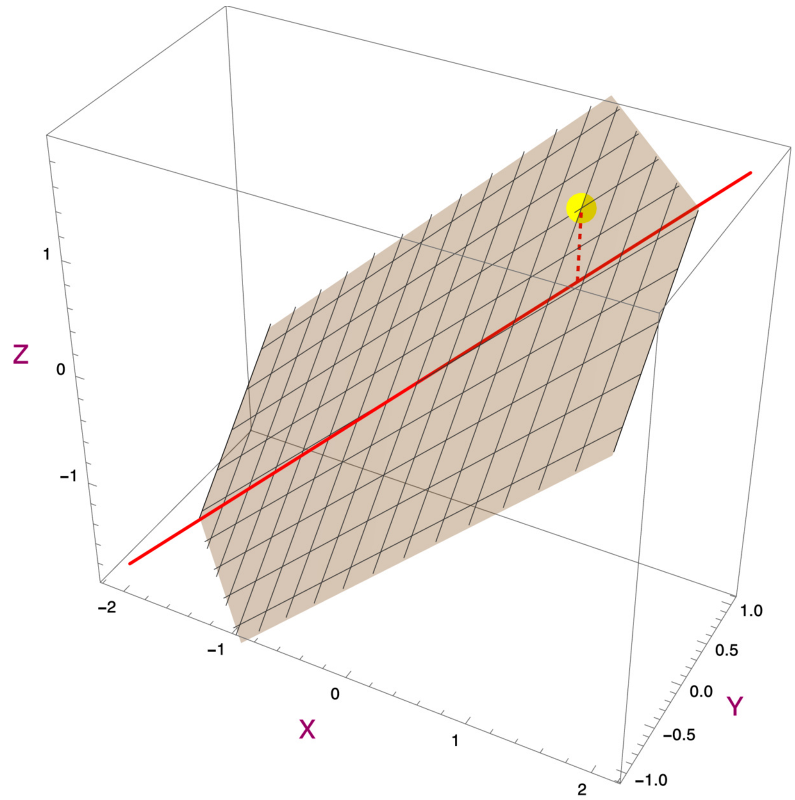

Figure 197. We see here a figure of the candidate tangent plane. The tangent line is drawn in red and completely covers the curve which is also a line in this case. We see that the tangent line is not in the candidate tangent plane. It intersects the tangent plane transversally and not tangentially. So the candidate tangent plane is not a tangent plane. The yellow point lies on the candidate tangent plane. We have drawn that yellow point in order to see more clearly that the tangent line is not on the candidate tangent plane. We conclude that the candidate tangent plane is not a tangent plane. The function is not differentiable. The figure is made with $\lambda = 1/2$.

It can maybe be worthwhile to take a look at the gradient vector field. We see that this gradient vector field exists in this case. But the gradient vector field cannot be continuous because it implies that the function is differentiable. So we wonder if we can find an indication for this fact.

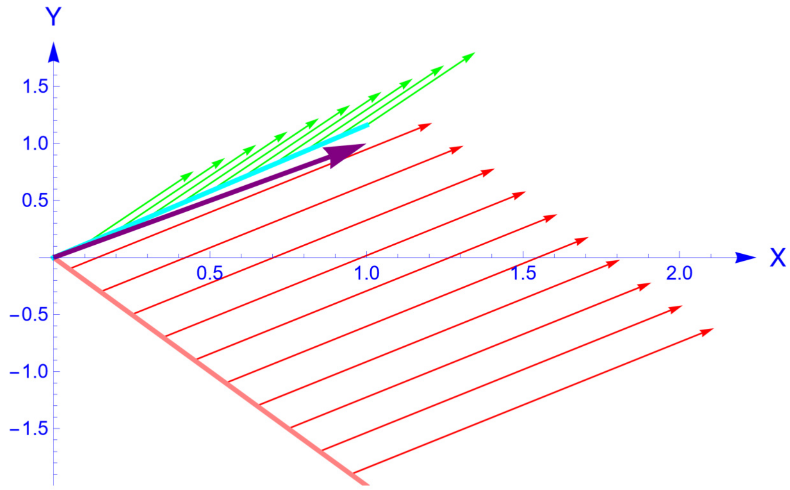

Figure 198. We made here the following sketch. We have drawn the graphics of $y = -2x$ in pink and $y = 1.16x$ in cyan. We have sketched the gradient vector field $\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$ which is not continuous as can be seen as follows. Observe the gradient vector field on the pink curve, these are the red vectors. The purple vector is the gradient vector in $(0,0)$. Observe the gradient vector field on the cyan curve, these are the green vectors. The red vectors converge to a vector with a non zero x-component. This component is $32/25$. The green vectors converge to a vector that has an x-component that is two times smaller then the x-component of the vector to which the red vectors converge if $x \to 0$. This is clearly impossible if the vector field is continuous. Moreover, the x-component of the limit vector should in both cases be 1 if the gradient vector field is continuous. We conclude that this gradient vector field is not continuous. The function is differentiable in that case, which it is not. Please note however that a sketch of the gradient vector field is inherently a sketch of discrete data. So the utmost care must be taken in order to make it a little bit trustworthy.

Exercise 21.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^6 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

21.1 Continuity

We restrict the function to the continuous curves with equations $y = \lambda x^3$. We observe then that

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{\lambda x^6}{\lambda^2 x^6 + x^6} = \dfrac{\lambda}{\lambda^2 + 1} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

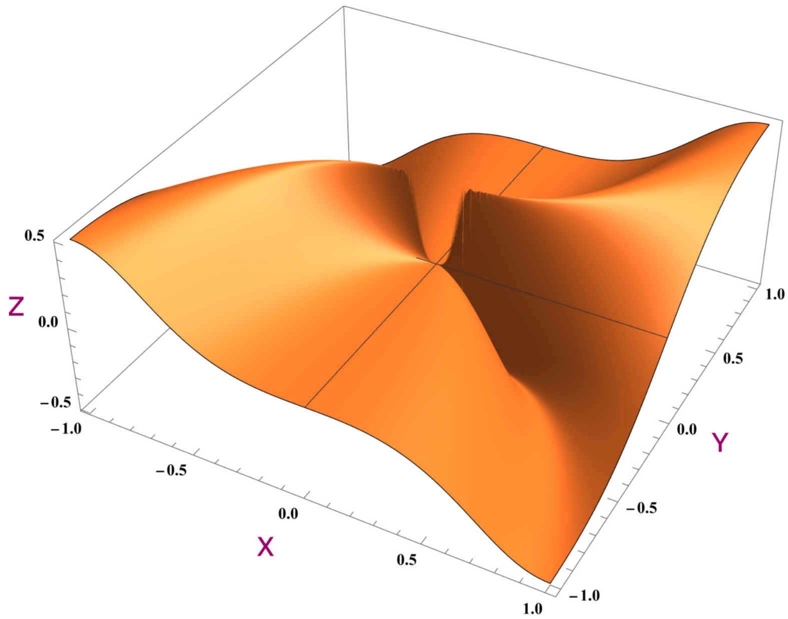

Figure 199. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. Please note that the graphical computing machine was not able to close the gap sufficiently. This does not seem to be a graph of a continuous function.

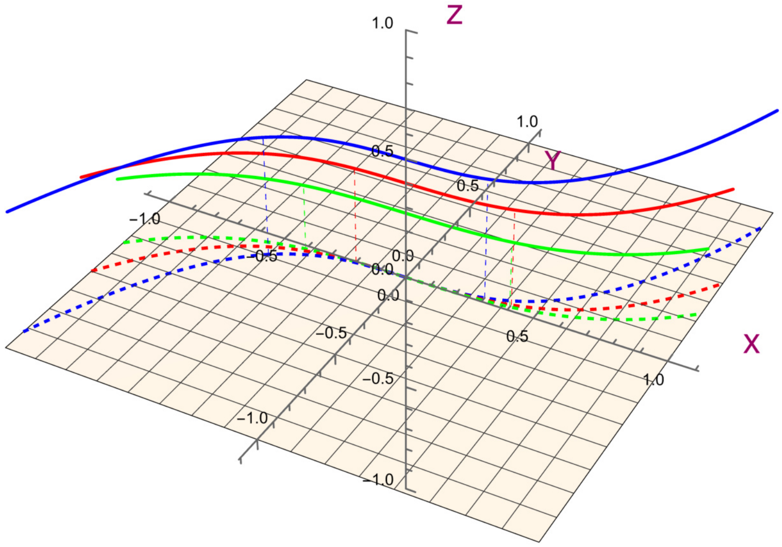

Figure 200. We have restricted the function here to $y = 1/2\,x^3$ and $y = 3/10\,x^3$ and $y = 9/10\,x^3$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

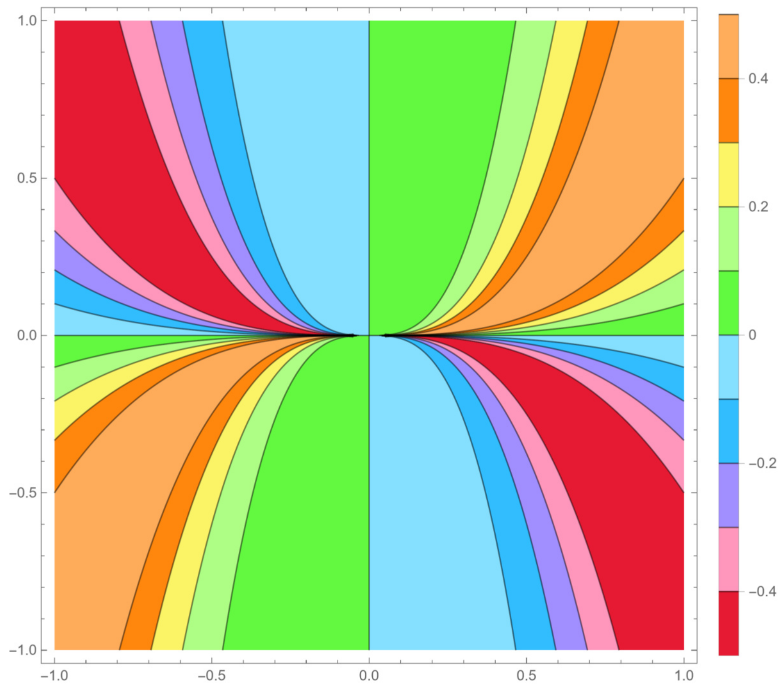

Figure 201. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

21.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y)=0 & \text{if } y\neq 0;\\ 0 & \text{if } y=0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0)=0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0)=0.$$

21.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu,0+hv)-f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu,0+hv)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{hu^3 v}{h^4 u^6 + v^2} \\ &= 0. \end{aligned}$$

This limit calculation is valid if $v \neq 0$ but we have covered that case before.

So the directional derivatives do always exist.

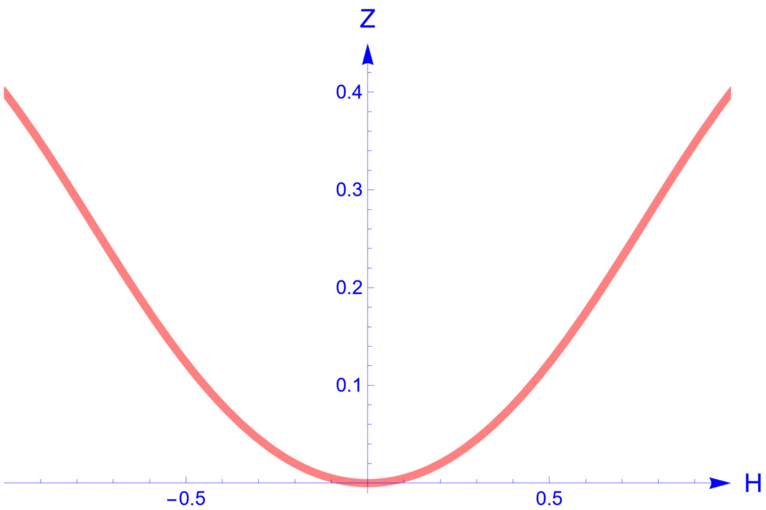

Figure 202. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu,hv)$. The slope of the tangent in 0 is indeed 0.

21.4 Alternative proof of continuity (optional)

Irrelevant. The function is not continuous.

21.5 Differentiability

Irrelevant. The function is not continuous.

21.6 Alternative proof of differentiability (optional)

No. The function is not continuous and not differentiable.

21.7 Continuity of the partial derivatives

Irrelevant. The function is not continuous and not differentiable.

21.8 Overview

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^6 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	no
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

Exercise 22.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y)=\begin{cases}\dfrac{x^5 y}{x^8+y^4} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

22.1 Continuity

We restrict the function to the continuous curves with equations $y = \lambda x^3$. We observe then that

$$f|_{y=\lambda x}(x,y)=\begin{cases} f(x,\lambda x^3)=\dfrac{\lambda x^8}{\lambda^4 x^{12}+x^8}=\dfrac{\lambda}{\lambda^4 x^4+1} & \text{if } x\neq 0;\\ 0 & \text{if } x=0.\end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

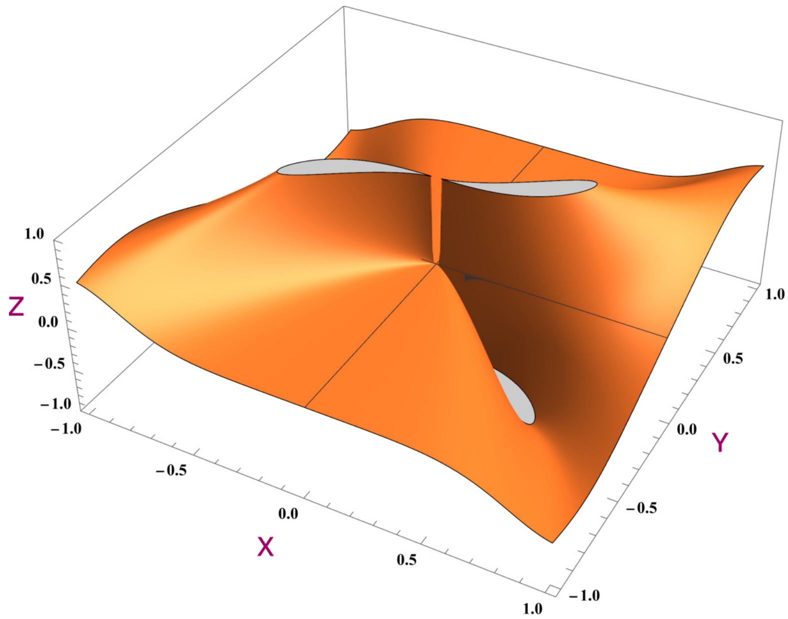

Figure 203. We see here a three dimensional figure of the graph of the function. The vertical line above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

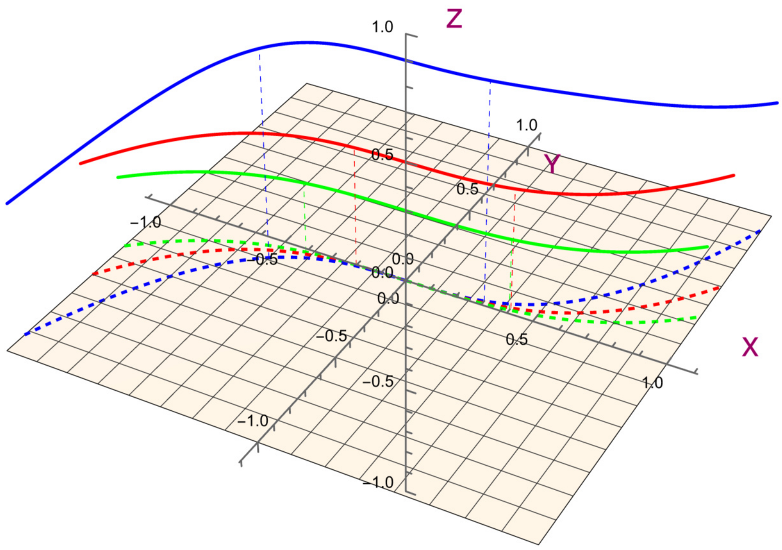

Figure 204. We have restricted the function here to $y = 1/2\,x^3$ and $y = 3/10\,x^3$ and $y = 9/10\,x^3$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

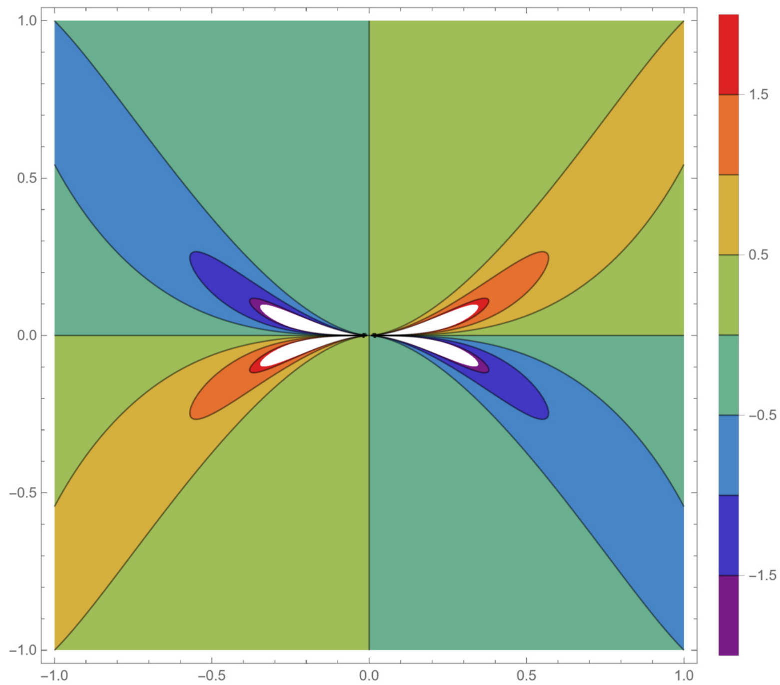

Figure 205. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

22.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

22.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{hu^5 v}{h^4 u^8 + v^4} \\ &= 0. \end{aligned}$$

This calculation is only valid if $v \neq 0$. But we covered that case before. So the directional derivatives do always exist.

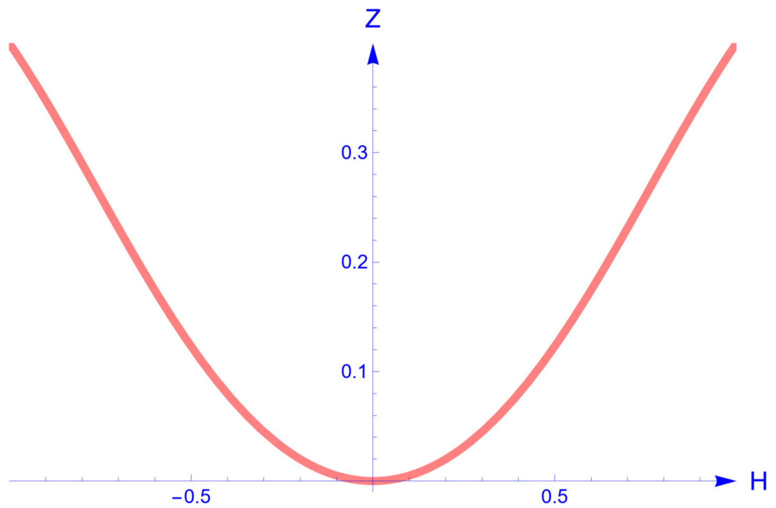

Figure 206. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$. The slope of the tangent line in 0 is horizontal.

22.4 Alternative proof of continuity (optional)

Irrelevant. The function is not continuous.

22.5 Differentiability

The function is not differentiable because it is not continuous.

22.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

22.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

22.8 Overview

$$f(x,y) = \begin{cases} \dfrac{x^5 y}{x^8 + y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	no
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

Exercise 23.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{x^5+y^4}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

23.1 Continuity

We restrict the function to the continuous curves with equations $y = \lambda x$. We observe then that

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{\lambda^4 x^4 + x^5}{\left(\lambda^2 x^2 + x^2\right)^2} = \dfrac{\lambda^4 + x}{\left(\lambda^2+1\right)^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

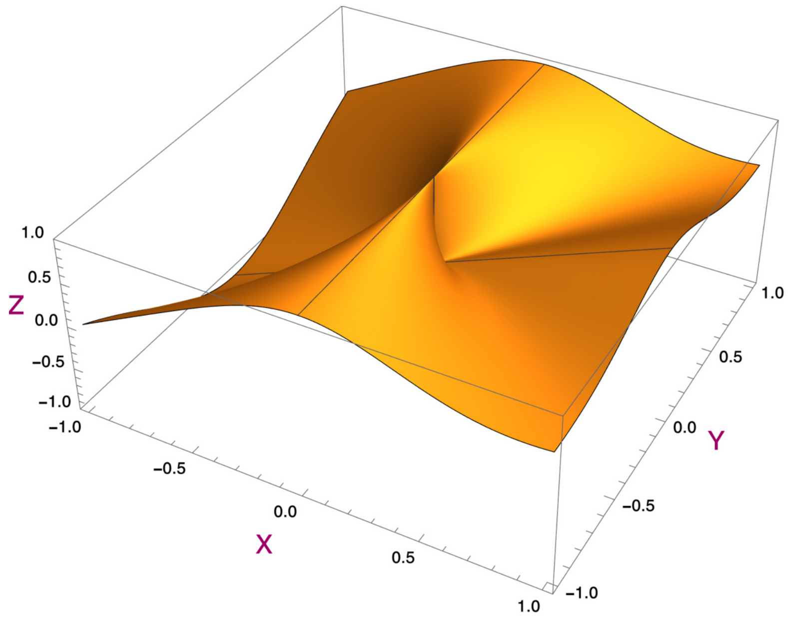

Figure 207. We see here a three dimensional figure of the graph of the function. The vertical line above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

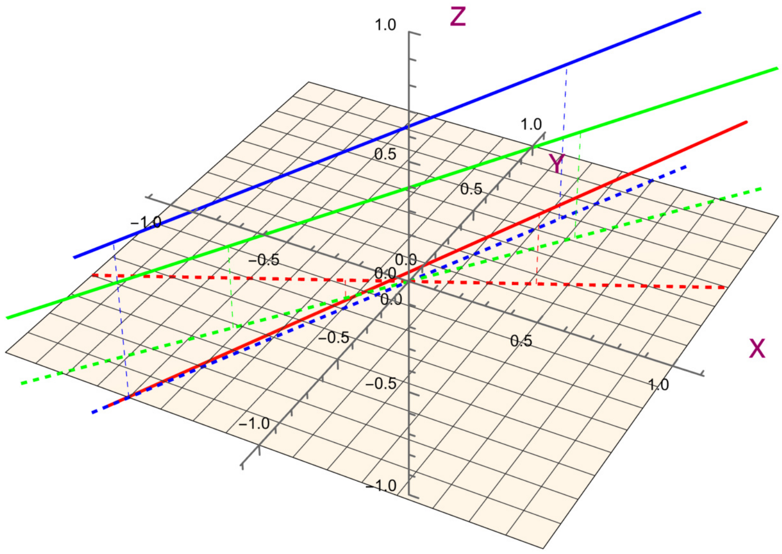

Figure 208. We have restricted the function here to $y = 1/2\,x$ and $y = 13/10\,x$ and $y = 2\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

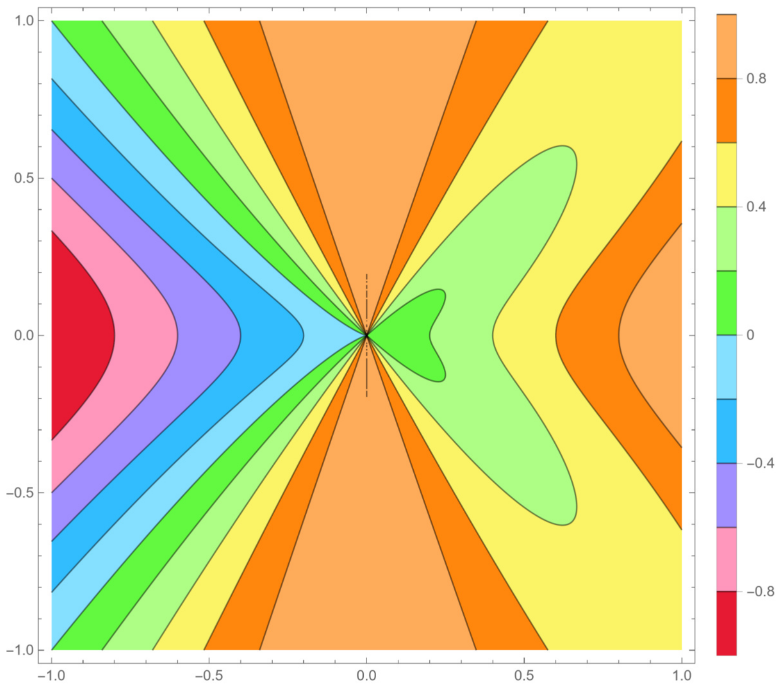

Figure 209. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0, 0)$. This looks discontinuous indeed.

23.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x, y) = \begin{cases} f(x, 0) = x & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 1\\ &= 1.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f\big|_{x=0}(x,y) = \begin{cases} f(0,y)=1 & \text{if } y\neq 0;\\ 0 & \text{if } y=0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0}\frac{1}{h}\end{aligned}$$

So the partial derivative to y does not exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 1$$

and the partial derivative to y does not exist.

23.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

Let us take a look at $f(hu, hv)$.

$$f(hu,hv) = \begin{cases} \dfrac{hu^5 + v^4}{\left(u^2+v^2\right)^2} & \text{if } h \neq 0, \\ 0 & \text{if } h = 0. \end{cases}$$

We see that this function is not continuous if $v \neq 0$. But we covered that case before.

So the directional derivatives not in the X-direction do not exist.

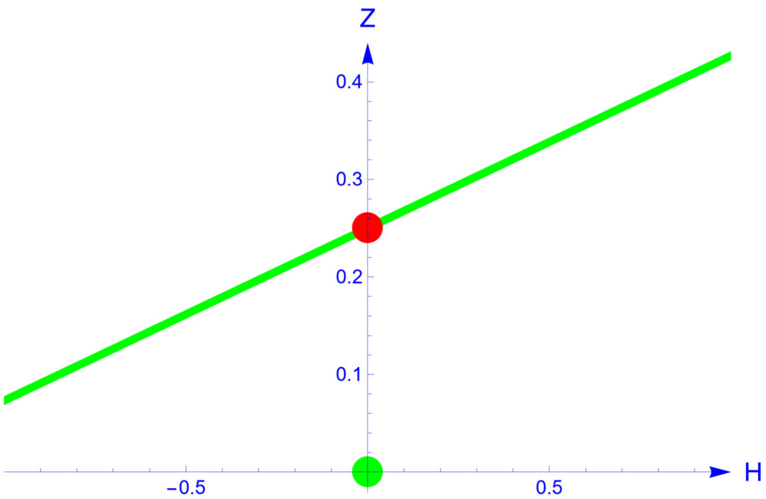

Figure 210. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu,hv)$. This function is not continuous in 0, thus it is not differentiable.

23.4 Alternative proof of continuity (optional)

Irrelevant. The function is not continuous.

23.5 Differentiability

The function is not differentiable because it is not continuous.

23.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

23.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

23.8 Overview

$$f(x,y) = \begin{cases} \dfrac{x^5 + y^4}{\left(x^2 + y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	no
partial derivatives exist	no
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 24.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{x\,y^2}{\sqrt{x^2+y^2}\left(x^2+y^4\right)} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

24.1 Continuity

We restrict the function to the continuous curves with equations $x = \lambda\,y^3$. We observe then that

$$f\,|_{x=\lambda\,y^3}(x,y) = \begin{cases} f(\lambda\,y^3,y) = \dfrac{\lambda\ \mathrm{sgn}(y)}{\left(\lambda^2\,y^2+1\right)\sqrt{\lambda^2\,y^4+1}}. & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

We see that these restricted functions have no limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

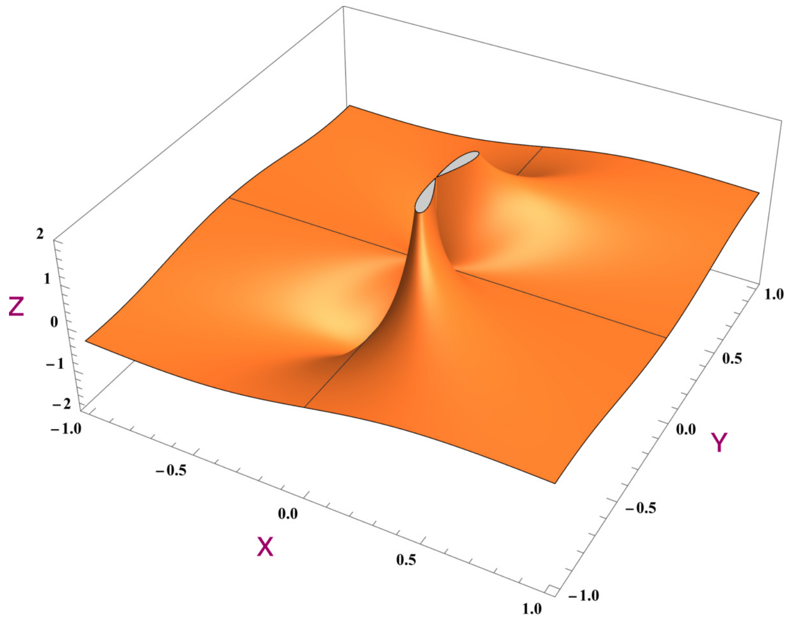

Figure 211. We see here a three dimensional figure of the graph of the function. The vertical line above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

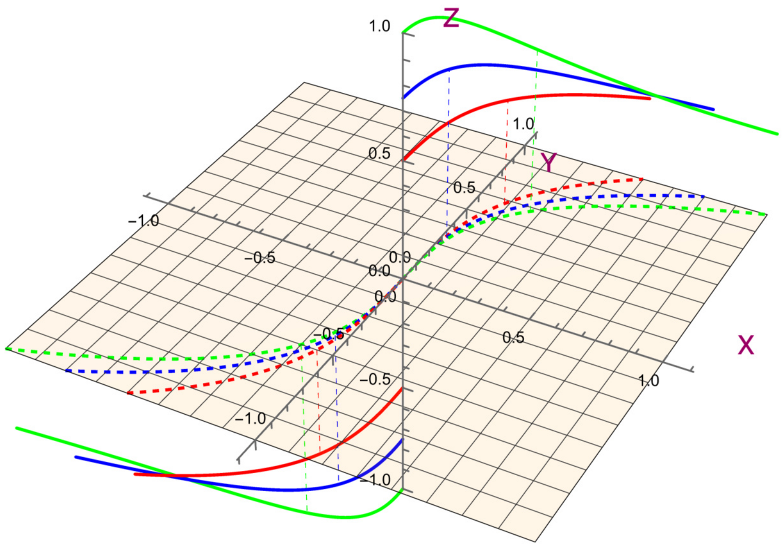

Figure 212. We have restricted the function here to $x = 1/2\,y^3$ and $x = y^3$ and $x = 3/4\,y^3$. We see in this figure clearly that the restrictions of the function to these lines are functions that have no limits in 0.

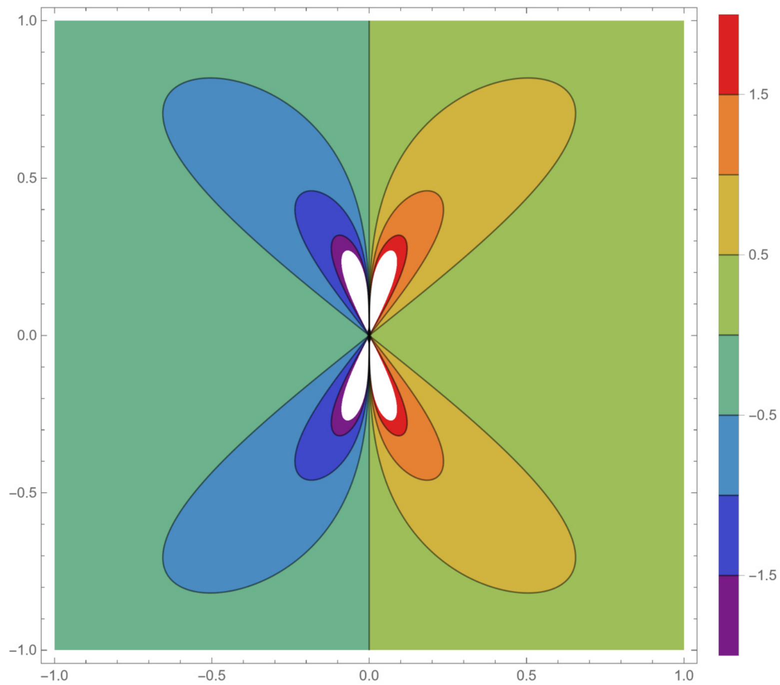

Figure 213. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

24.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y)=0 & \text{if } y\neq 0;\\ 0 & \text{if } y=0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0)=0 \quad\text{and}\quad \frac{\partial f}{\partial y}(0,0)=0.$$

24.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h}.$$

We observe however that the function $f(0+hu, 0+hv)$ is not continuous. Indeed

$$f(0+hu, 0+hv) = \frac{huv^2}{\sqrt{h^2\,(u^2+v^2)}\,(h^2v^4+u^2)} = \frac{\operatorname{sgn}(h)uv^2}{\sqrt{(u^2+v^2)}\,(h^2\,v^4+u^2)}.$$

This function is not continuous if $u \neq 0$ or $v \neq 0$. But we covered these cases before.

The consequence is that the function $f(0+h\,u, 0+h\,v)$ is not differentiable.

So the directional derivatives do not always exist.

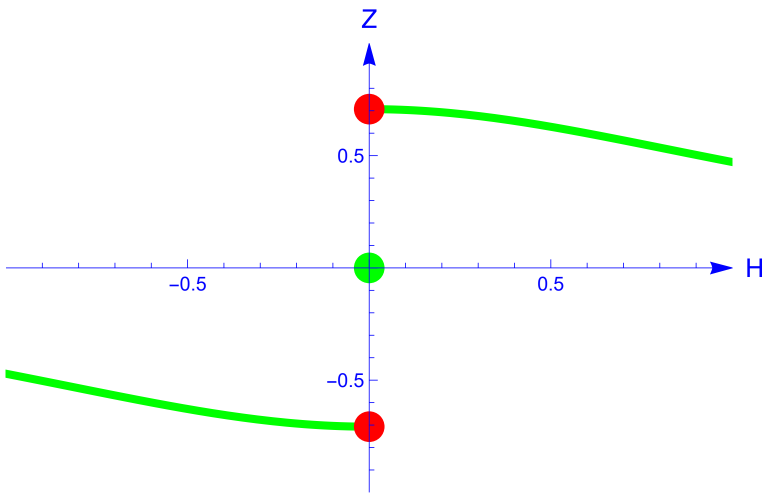

Figure 214. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$. This function is not continuous and thus not differentiable.

24.4 Alternative proof of continuity (optional)

The function is not continuous. So this is irrelevant.

24.5 Differentiability

The function is not differentiable because it is not continuous.

24.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

24.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

24.8 Overview

continuous	no
partial derivatives exist	yes
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 25.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \frac{x\,y\,\left(x^2-y^2\right)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

25.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{x\,y\,(x^2-y^2)}{x^2+y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left|\frac{x\,y\,(x^2-y^2)}{x^2+y^2}\right| &\le \frac{|x|\,|y|\,(|x|^2+|y|^2)}{x^2+y^2} \\
&\le \frac{\sqrt{x^2+y^2}\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2}^{\,2}+\sqrt{x^2+y^2}^{\,2}\right)}{\sqrt{x^2+y^2}^{\,2}} \\
&\le \frac{2\sqrt{x^2+y^2}^{\,4}}{\sqrt{x^2+y^2}^{\,2}} \\
&\le 2\sqrt{x^2+y^2}^{\,2}.
\end{aligned}$$

It is sufficient to take $\delta = (\epsilon/2)^{1/2}$. We can find a δ, so we conclude that the function is continuous.

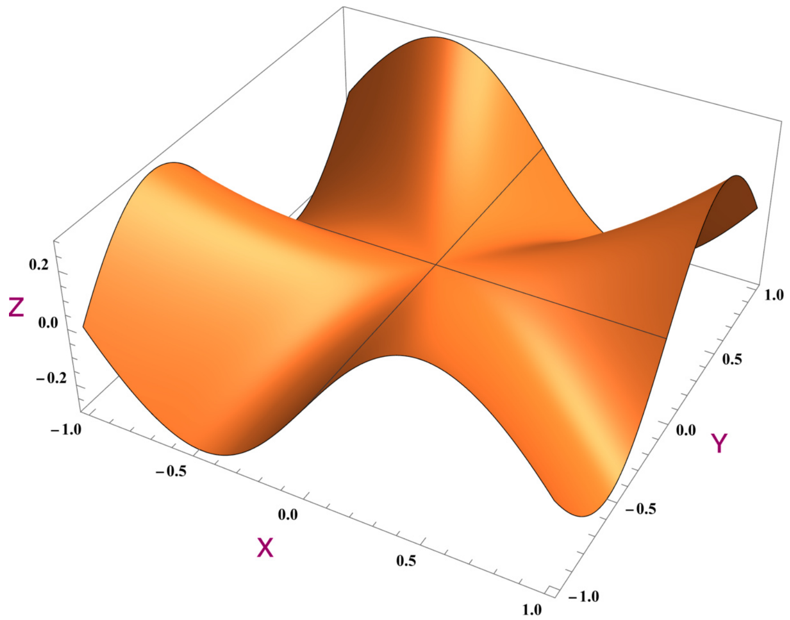

Figure 215. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

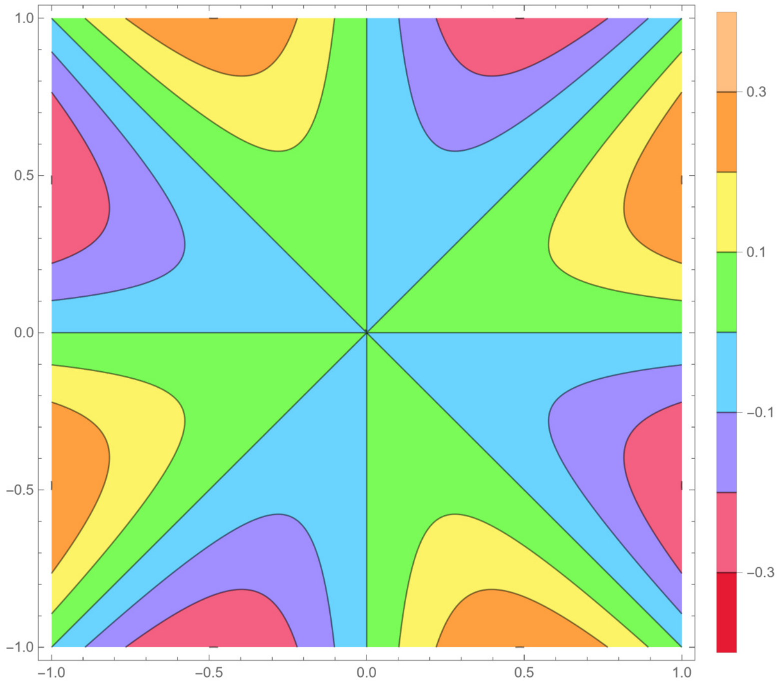

Figure 216. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

25.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y)=0 & \text{if } y\neq 0;\\ 0 & \text{if } y=0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0)=0 \quad\text{and}\quad \frac{\partial f}{\partial y}(0,0)=0.$$

25.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{h\,u\,v\,(h^2 u^2 - h^2 v^2)}{h^2 u^2 + h^2 v^2} \\ &= \lim_{h\to 0} \frac{h\,u\,v\,(u^2 - v^2)}{u^2 + v^2} \\ &= 0. \end{aligned}$$

So the directional derivatives do always exist.

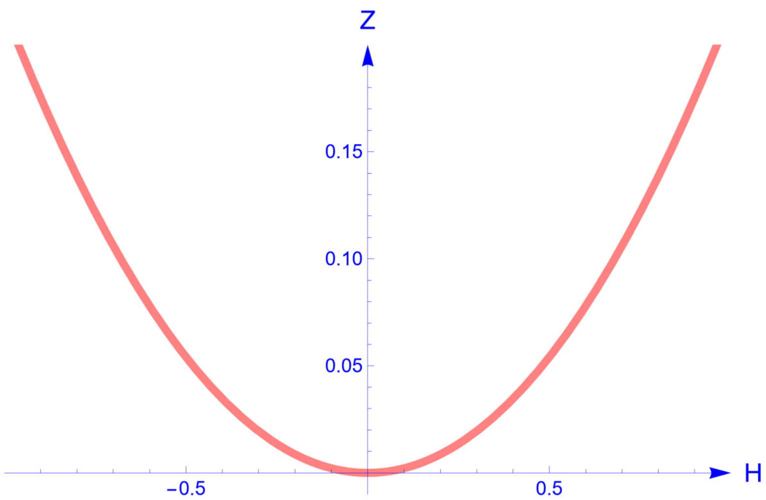

Figure 217. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = (\sqrt{3}/2, 1/2)$. The slope is 0 in 0. We have plotted here the function $f(h\,u, h\,v)$.

25.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{y\left(x^4+4x^2y^2-y^4\right)}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{x^5-4x^3y^2-xy^4}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

$$
\begin{aligned}
&\left|\frac{\partial f}{\partial x}(x,y)\right| \\
&\quad\leq \left|\frac{y\ (x^4+4\,x^2\,y^2-y^4)}{(x^2+y^2)^2}\right| \\
&\quad\leq \frac{\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2}^4+4\sqrt{x^2+y^2}^2\sqrt{x^2+y^2}^2+\sqrt{x^2+y^2}^4\right)}{\sqrt{x^2+y^2}^4} \\
&\quad\leq \frac{6\sqrt{x^2+y^2}^5}{\sqrt{x^2+y^2}^4} \\
&\quad\leq 6\sqrt{x^2+y^2} \\
&\quad\leq 6.
\end{aligned}
$$

We have chosen here the restriction to the neighbourhood defined by $\sqrt{x^2+y^2} < 1$.

Let us try to prove that $\left|\frac{\partial f}{\partial y}(x,y)\right|$ is bounded.

$$
\begin{aligned}
&\left|\frac{\partial f}{\partial y}(x,y)\right| \\
&\quad \le \left|\frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}\right| \\
&\quad \le \frac{|x|^5 + 4|x|^3y^2 + |x|y^4}{(x^2+y^2)^2} \\
&\quad \le \frac{\sqrt{x^2+y^2}^5 + 4\sqrt{x^2+y^2}^3\sqrt{x^2+y^2}^2 + \sqrt{x^2+y^2}\sqrt{x^2+y^2}^4}{\sqrt{x^2+y^2}^4} \\
&\quad \le \frac{6\sqrt{x^2+y^2}^5}{\sqrt{x^2+y^2}^4} \\
&\quad \le 6\sqrt{x^2+y^2} \\
&\quad \le 6.
\end{aligned}
$$

For the last step in the equalities we have chosen here the restriction to the neighbourhood defined by $\sqrt{x^2+y^2} < 1$.

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

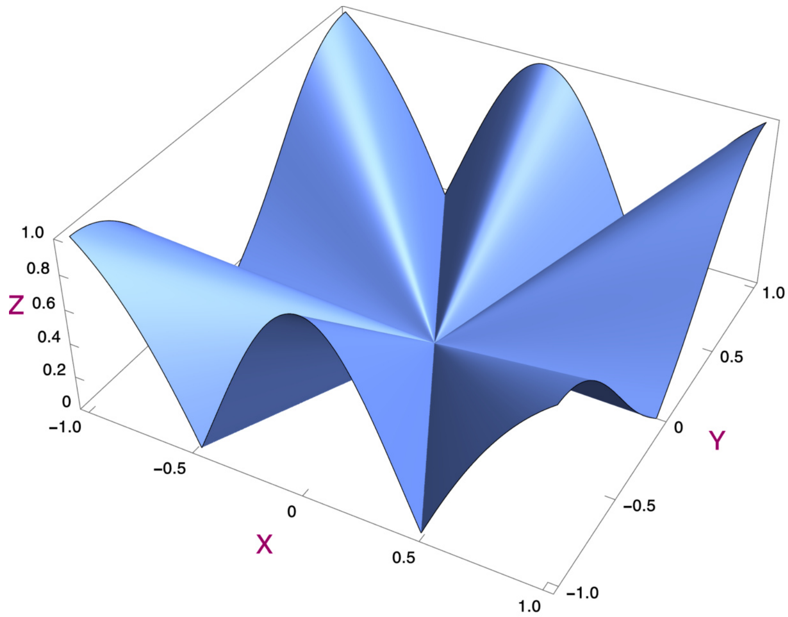

Figure 218. We see here the absolute value of the first partial derivative $\left|\dfrac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

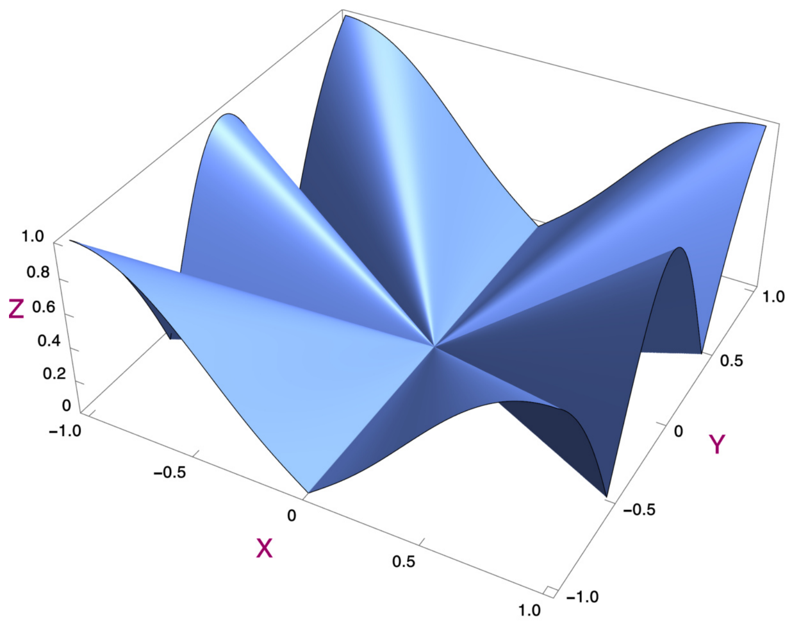

Figure 219. We see here the absolute value of the second partial derivative $\left|\dfrac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture.

25.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

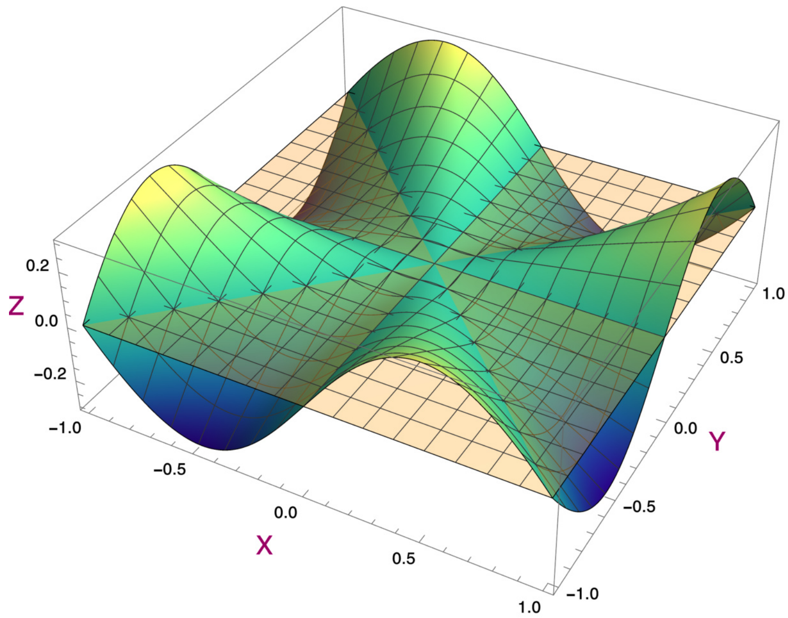

Figure 220. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We have doubts about the differentiability. We will have to follow our calculations.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we

know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

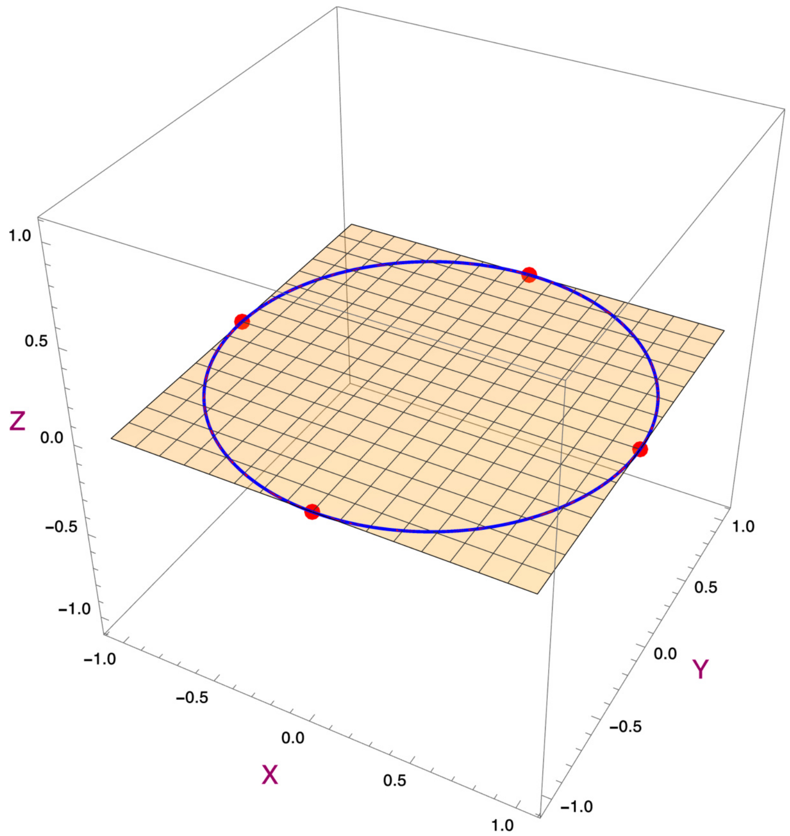

Figure 221. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane. This is good news in favour of differentiability.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h\,k\left(h^2 - k^2\right)}{\left(h^2 + k^2\right)^{3/2}} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2 + k^2} < \delta$, we have that

$$\left|\frac{h\,k\,(h^2 - k^2)}{(h^2 + k^2)^{3/2}} - 0\right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{h\,k\,(h^2-k^2)}{(h^2+k^2)^{3/2}}\right| &\leq \frac{\sqrt{h^2+k^2}\,\sqrt{h^2+k^2}\left(\sqrt{h^2+k^2}^{\,2}+\sqrt{h^2+k^2}^{\,2}\right)}{\sqrt{h^2+k^2}^{\,3}}\\ &\leq \frac{2\sqrt{h^2+k^2}^{\,4}}{\sqrt{h^2+k^2}^{\,3}}\\ &\leq 2\sqrt{h^2+k^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon/2$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous. The function f is differentiable.

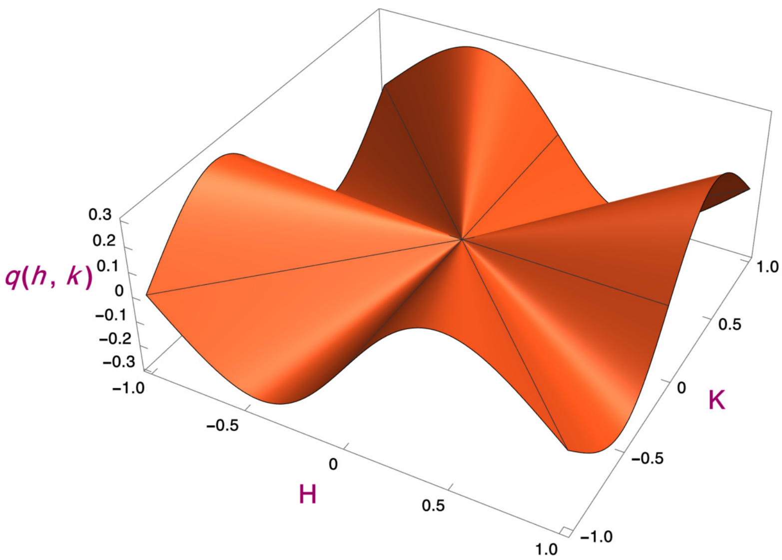

Figure 222. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

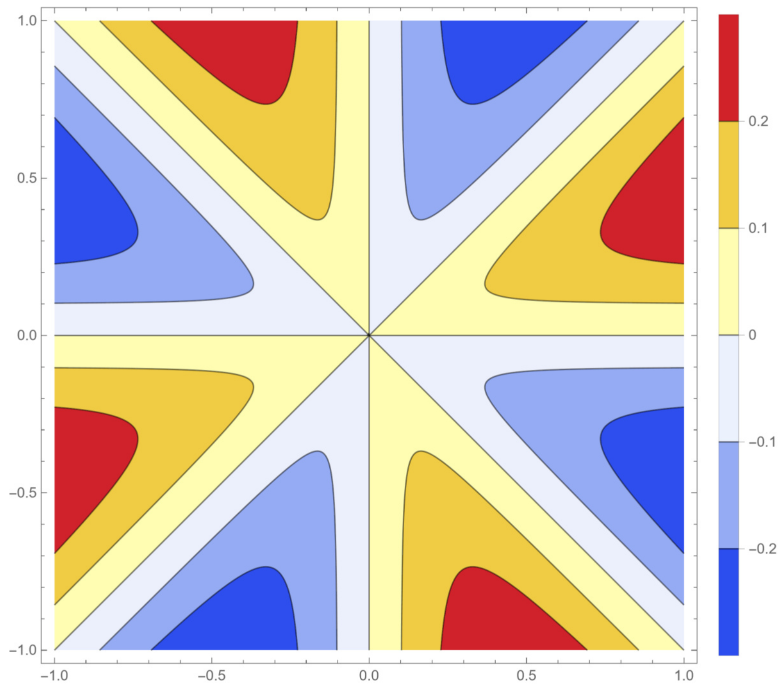

Figure 223. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

25.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\, u + \frac{\partial f}{\partial y}(0,0)\, v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the func-

tion $f(x,y)$ is locally Lipschitz continuous in $(0,0)$. We cite here the criterion that we will use.

A function is differentiable in (a,b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.

2. The directional derivatives have the following form: $D_{(u,v)}(a,b) = \nabla f(a,b)\cdot(u,v) = \frac{\partial f}{\partial x}(a,b)\,u + \frac{\partial f}{\partial y}(a,b)\,v$.

3. The function is locally Lipschitz continuous in (a,b). This means that there exists at least one neighbourhood of (a,b) and a number $K > 0$ such that for all (x_1,y_1) and (x_2,y_2) in the neighbourhood, we have $|f(x_1,y_1) - f(x_2,y_2)| < K\,||(x_1,y_1) - (x_2,y_2)||$. Remark that the same K must be valid for all (x_1,y_1) and (x_2,y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be “locally flat" and thus “locally linear" which is partly expressed by this Lipschitz continuity condition.

Let us try to prove the Lipschitz condition. We are going to use a well known method.

$$\begin{aligned}
&|f(x_1,y_1) - f(x_2,y_2)| \\
&\qquad = |f(x_1,y_1) - f(x_1,y_2) + f(x_1,y_2) - f(x_2,y_2)| \\
&\qquad \le |f(x_1,y_1) - f(x_1,y_2)| + |f(x_1,y_2) - f(x_2,y_2)|.
\end{aligned}$$

We focus now on the first term $|f(x_1,y_1) - f(x_1,y_2)|$. We fix now x_1 and look upon $f(x_1,y)$ as a function in one variable y. So it springs to mind that we can estimate this by using Lagrange’s intermediate value theorem. So we have

$$|f(x_1,y_1) - f(x_1,y_2)| = \left|\frac{\partial f}{\partial y}(x_1,\xi)\right|\;|y_1 - y_2|$$

where ξ is a number lying in the open interval (y_1, y_2) if e.g. $y_1 \leq y_2$. Remark that we already have proven that the partial derivatives are bounded in a neighbourhood, we can take that bound found for $\frac{\partial f}{\partial y}$, say M_2 and that neighbourhood in this case also. So $\left|\frac{\partial f}{\partial y}(x_1, \xi)\right| \leq M_2$. We work in a completely analogous way for the second term $|f(x_1, y_2) - f(x_2, y_2)|$. We have in that case $\left|\frac{\partial f}{\partial x}(\xi, y_2)\right| \leq M_1$ where ξ is now a number in the open interval (x_1, x_2) if e.g. $x_1 \leq x_2$.

We take up our inequality again

$$
\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)| \\
&\qquad \leq |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\qquad \leq |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)| \\
&\qquad \leq M_2\,|(y_1 - y_2)| + M_1\,|(x_1 - x_2)| \\
&\qquad \leq M_2\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + M_1\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&\qquad \leq (M_1 + M_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.
\end{aligned}
$$

So this function is locally Lipschitz continuous. We have an alternative proof for the differentiability.

25.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

We know that the partial derivative to x exists and is equal to

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{y\left(x^4+4x^2y^2-y^4\right)}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We want to see if it is continuous or not.

Discussion of the continuity of the first partial derivative in $(0,0)$. We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that the inequality $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$ holds under certain conditions. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned} &\left|\frac{y\left(x^4+4x^2y^2-y^4\right)}{\left(x^2+y^2\right)^2}\right| \\ &\leq \frac{|y|\left(x^4+4x^2y^2+y^4\right)}{(x^2+y^2)^2} \\ &\leq \frac{\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2}^4+4\sqrt{x^2+y^2}^2\sqrt{x^2+y^2}^2+\sqrt{x^2+y^2}^4\right)}{\sqrt{x^2+y^2}^4} \\ &\leq \frac{6\sqrt{x^2+y^2}^5}{\sqrt{x^2+y^2}^4} \\ &\leq 6\sqrt{x^2+y^2}. \end{aligned}$$

It is sufficient to take $\delta = \epsilon/6$. We can find a δ, so we conclude that the function is continuous.

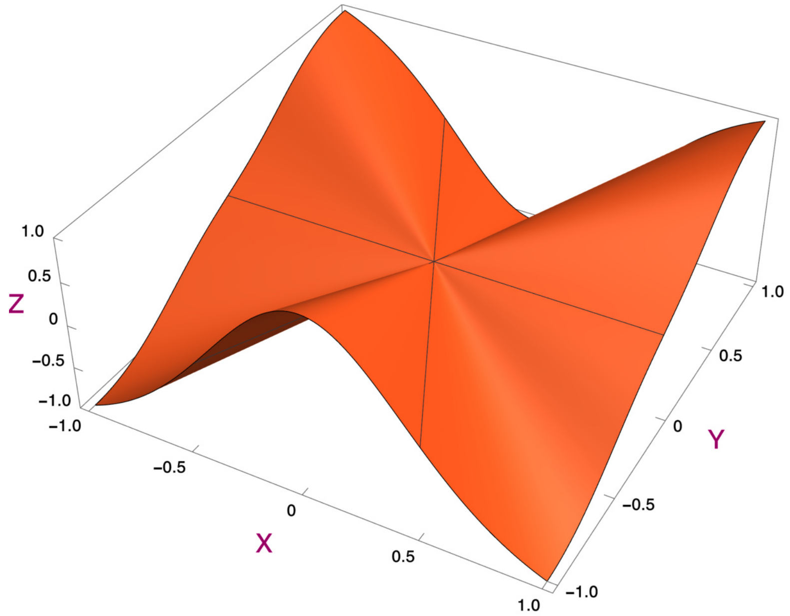

Figure 224. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial x}(x, y)$. This looks like a continuous function.

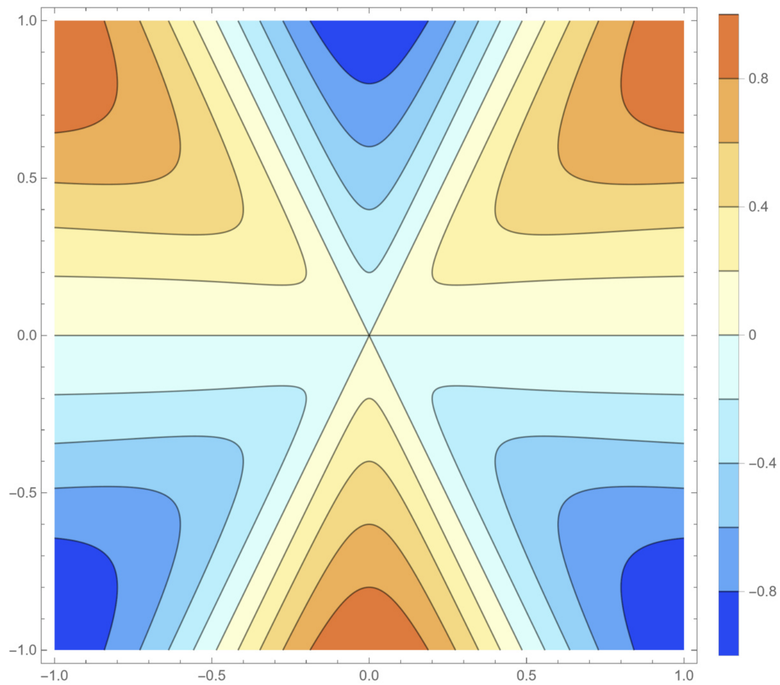

Figure 225. We see here a figure of the contour plot of the $\frac{\partial f}{\partial x}(x, y)$. Only level curves of level around 0 come close to $(0, 0)$.

Discussion of the continuity of the second partial derivative in $(0, 0)$.

We know that the partial derivative to y exists and is equal to

$$\frac{\partial f}{\partial y}(x, y) = \begin{cases} \dfrac{x^5 - 4x^3y^2 - xy^4}{\left(x^2+y^2\right)^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

We want to see if it is continuous or not.

Discussion of the continuity in $(0, 0)$.
We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove the inequality $|\frac{\partial f}{\partial y}(x, y) - \frac{\partial f}{\partial y}(0, 0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if the inequality $||(x, y) - (0, 0)|| < \delta$ holds, then it follows that $|\frac{\partial f}{\partial y}(x, y) - \frac{\partial f}{\partial y}(0, 0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0, 0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{\partial f}{\partial y}(x, y) - \frac{\partial f}{\partial y}(0, 0) \right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{\partial f}{\partial y} \right| &\leq \left| \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2} \right| \\
&\leq \frac{|x|^5 + 4|x|^3y^2 + |x|y^4}{(x^2 + y^2)^2} \\
&\leq \frac{\sqrt{x^2 + y^2}^5 + 4\sqrt{x^2 + y^2}^3\sqrt{x^2 + y^2}^2 + \sqrt{x^2 + y^2}\sqrt{x^2 + y^2}^4}{\sqrt{x^2 + y^2}^4} \\
&\leq \frac{6\sqrt{x^2 + y^2}^5}{\sqrt{x^2 + y^2}^4} \\
&\leq 6\sqrt{x^2 + y^2}.
\end{aligned}$$

It is sufficient to take $\delta = \epsilon/6$. We can find a δ, so we conclude that the function is continuous.

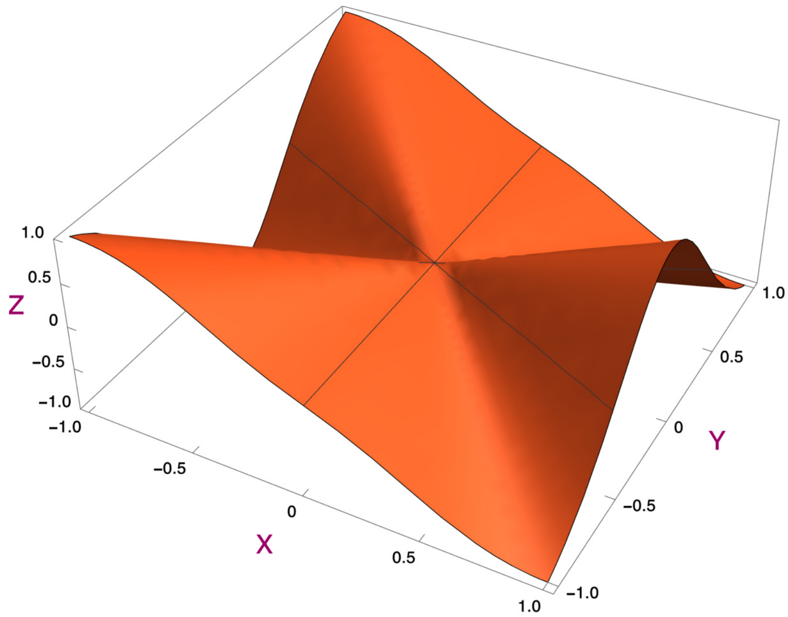

Figure 226. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial y}(x, y)$. This looks like a continuous function.

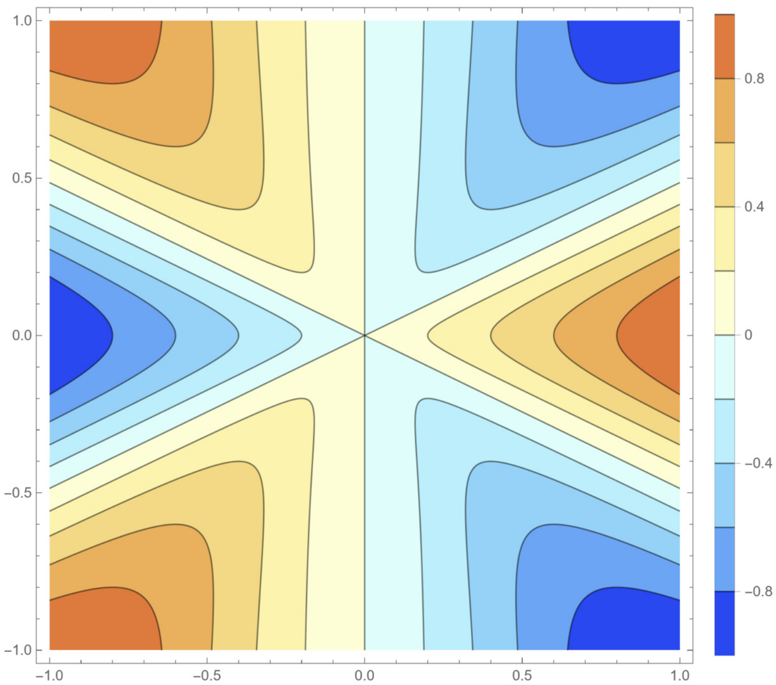

Figure 227. We see here a figure of the contour plot of the $\frac{\partial f}{\partial y}(x, y)$. Only level curves of level around 0 come close to $(0, 0)$.

25.8 Overview

$$f(x, y) = \begin{cases} \dfrac{x\,y\,\left(x^2 - y^2\right)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	yes

25.9 One step further

We want to know if this function is further uneventful from the point of view of differentiability. Let us take a look at the second partial derivative

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -\frac{4\,x\,y^3\,(x^2 - 3\,y^2)}{(x^2 + y^2)^3}.$$

Let us take a look of a three dimensional plot of this partial derivative to y of the function.

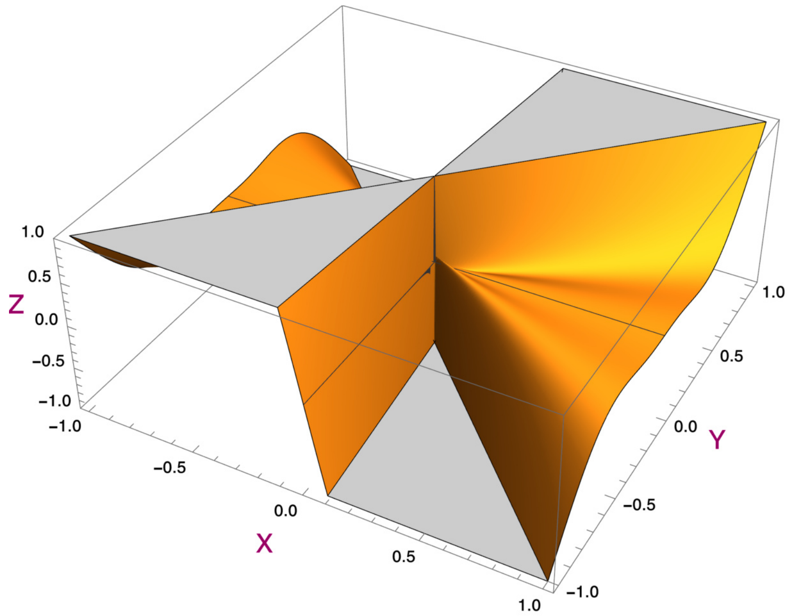

Figure 228. We see here a figure of the second order partial derivative $\frac{\partial^2 f}{\partial x^2}(x,y)$. It seems quite improbable that this second order partial derivative is continuous. We stop however our investigations here and leave this to the initiative of the interested reader.

Exercise 26.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{y\sin(xy)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

26.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{y\sin(xy)}{x^2+y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left|\frac{y\,\sin(x\,y)}{x^2+y^2}\right| &\le \frac{|y|\,|\sin(x\,y)|}{x^2+y^2}\\
&\le \frac{|y|\,|x\,y|}{x^2+y^2}\\
&\le \frac{|y|\,|x|\,|y|}{x^2+y^2}\\
&\le \frac{\sqrt{x^2+y^2}\,\sqrt{x^2+y^2}\,\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}^{\,2}}\\
&\le \sqrt{x^2+y^2}.
\end{aligned}$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function is continuous.

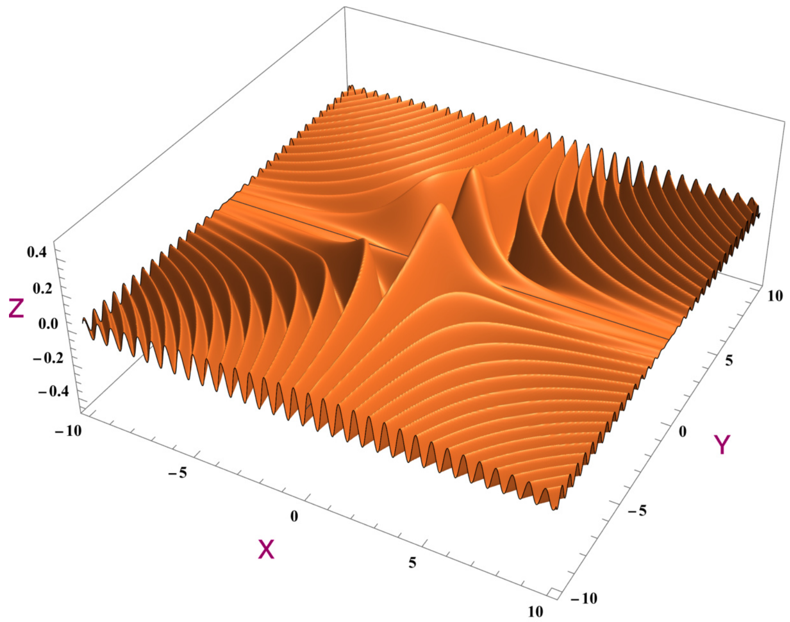

Figure 229. We see here a three dimensional figure of the graph of the function on a larger scale. This looks like a continuous function.

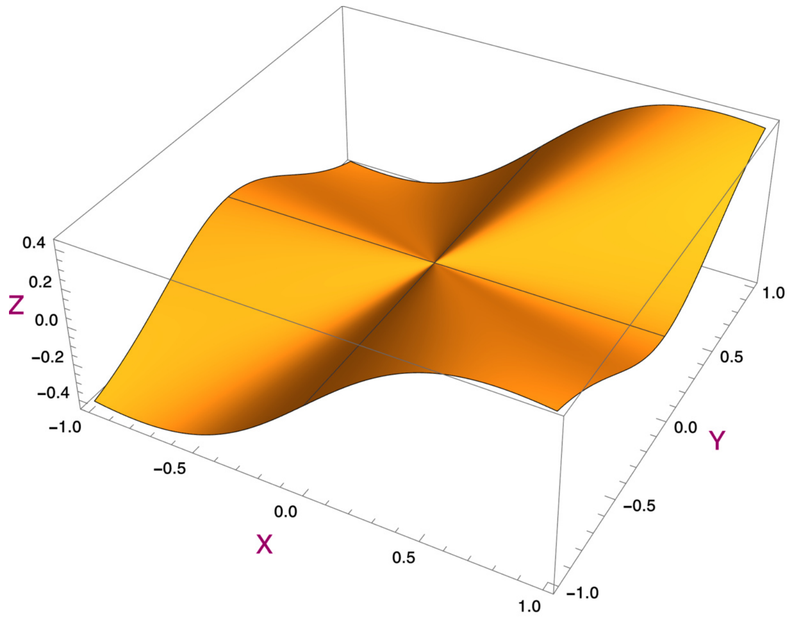

Figure 230. We see here a three dimensional figure of the graph of the function on a more local scale. This looks like a continuous function.

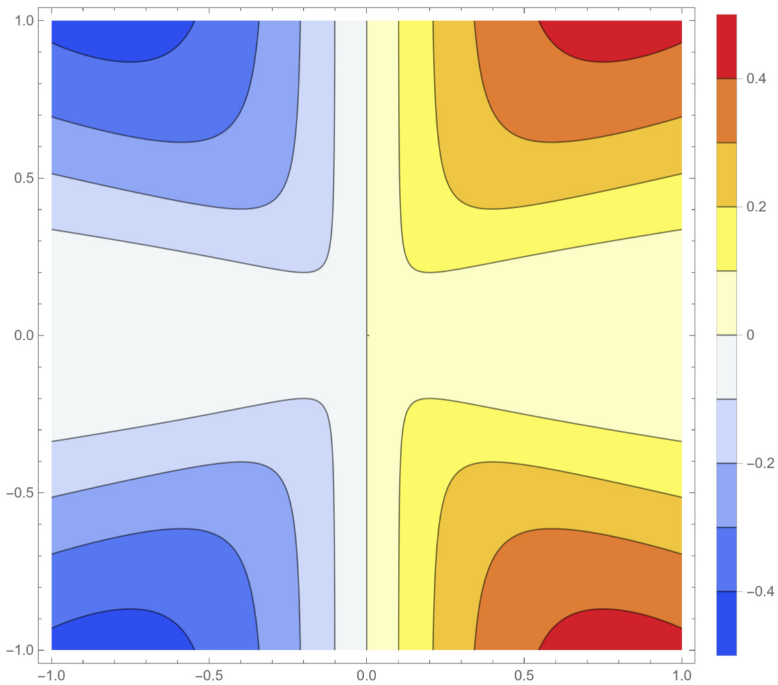

Figure 231. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

26.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y)=0 & \text{if } y\neq 0;\\ 0 & \text{if } y=0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0)=0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0)=0.$$

26.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned}
D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{v \sin(h^2\,u\,v)}{h^2\,u^2 + h^2\,v^2} \\
&= \lim_{h\to 0} \frac{v \sin(h^2\,u\,v)}{h^2\,u\,v} \frac{h^2\,u\,v}{h^2\,u^2 + h^2\,v^2} \\
&= \lim_{h\to 0} \frac{u\,v^2}{u^2 + v^2} \\
&= \frac{u\,v^2}{u^2 + v^2} \\
&= u\,v^2.
\end{aligned}$$

So the directional derivatives do always exist.

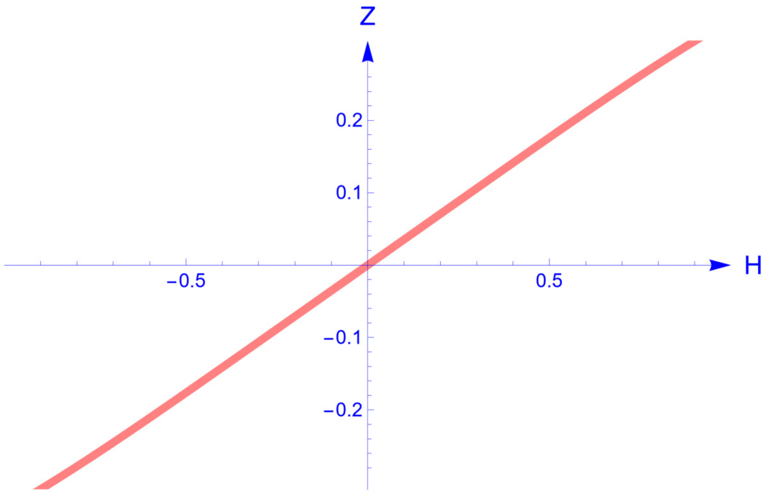

Figure 232. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

26.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y)$$

$$= \begin{cases} \dfrac{x^2\,y^2\,\cos(x\,y) + y^4\cos(x\,y) - 2\,x\,y\,\sin(x\,y)}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{x^3\, y\cos(x\,y) + x^2\,\sin(x\,y) + x\,y^3\cos(x\,y) - y^2\sin(x\,y)}{\left(x^2+y^2\right)^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

$$\begin{aligned}
\left|\frac{\partial f}{\partial x}(x,y)\right| &\leq \left|\frac{x^2\,y^2\,\cos(x\,y) + y^4\cos(x\,y) - 2\,xy\,\sin(xy)}{(x^2+y^2)^2}\right| \\
&\leq \frac{x^2\,y^2\,|\cos(x\,y)| + y^4|\cos(x\,y)| + 2\,|x||y||\sin(xy)|}{(x^2+y^2)^2} \\
&\leq \frac{\sqrt{x^2+y^2}^2\sqrt{x^2+y^2}^2 + \sqrt{x^2+y^2}^4 + 2\sqrt{x^2+y^2}\sqrt{x^2+y^2}\sqrt{x^2+y^2}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}^4} \\
&\leq \frac{4\sqrt{x^2+y^2}^4}{\sqrt{x^2+y^2}^4} \\
&\leq 4.
\end{aligned}$$

Let us try to prove that $\left|\frac{\partial f}{\partial y}\right|$ is bounded.

$$
\begin{aligned}
&\left|\frac{\partial f}{\partial y}(x,y)\right| \\
&\quad \le \left|\frac{x^3\,y\cos(x\,y)+x^2\,\sin(x\,y)+x\,y^3\cos(x\,y)-y^2\,\sin(x\,y)}{(x^2+y^2)^2}\right| \\
&\quad \le \frac{|x|^3\,|y|\,|\cos(x\,y)|+x^2\,|\sin(x\,y)|+|x|\,|y|^3|\cos(x\,y)| + y^2\,|\sin(x\,y)|}{(x^2+y^2)^2} \\
&\quad \le \frac{\sqrt{x^2+y^2}^3\sqrt{x^2+y^2}+\sqrt{x^2+y^2}^2\sqrt{x^2+y^2}\sqrt{x^2+y^2} + \sqrt{x^2+y^2}\sqrt{x^2+y^2}^3+\sqrt{x^2+y^2}^2\,|\sqrt{x^2+y^2}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}^4} \\
&\quad \le \frac{4\sqrt{x^2+y^2}^4}{\sqrt{x^2+y^2}^4} \\
&\quad \le 4.
\end{aligned}
$$

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

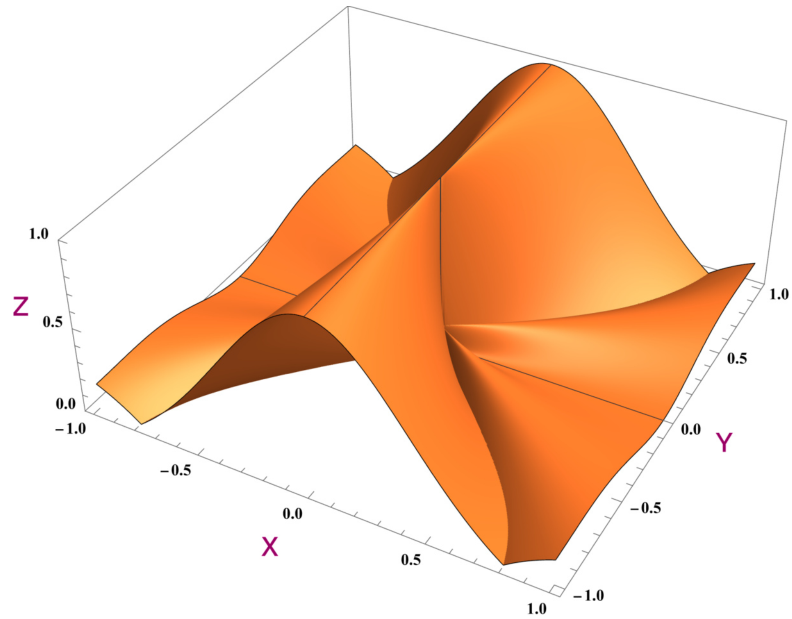

Figure 233. We see here the absolute value of the first partial derivative $\left|\frac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

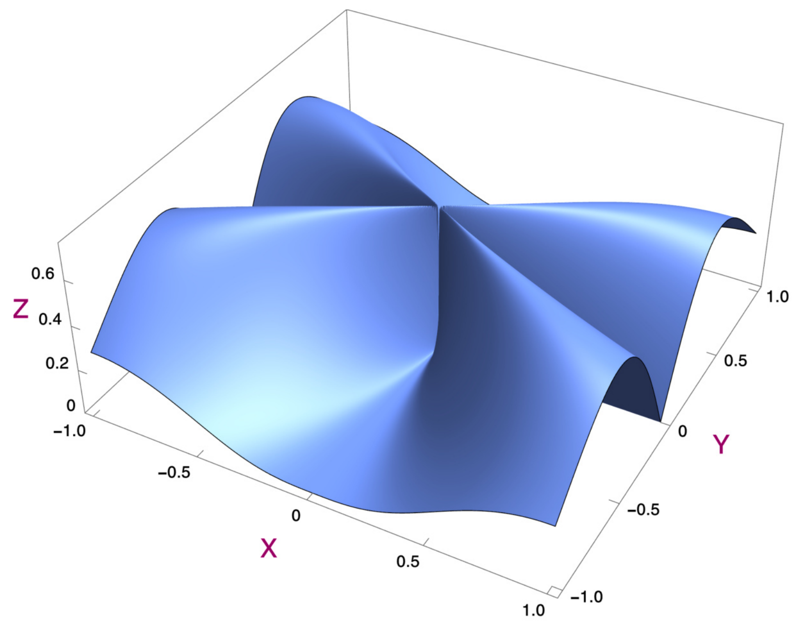

Figure 234. We see here the absolute value of the second partial derivative $\left|\frac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture.

26.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

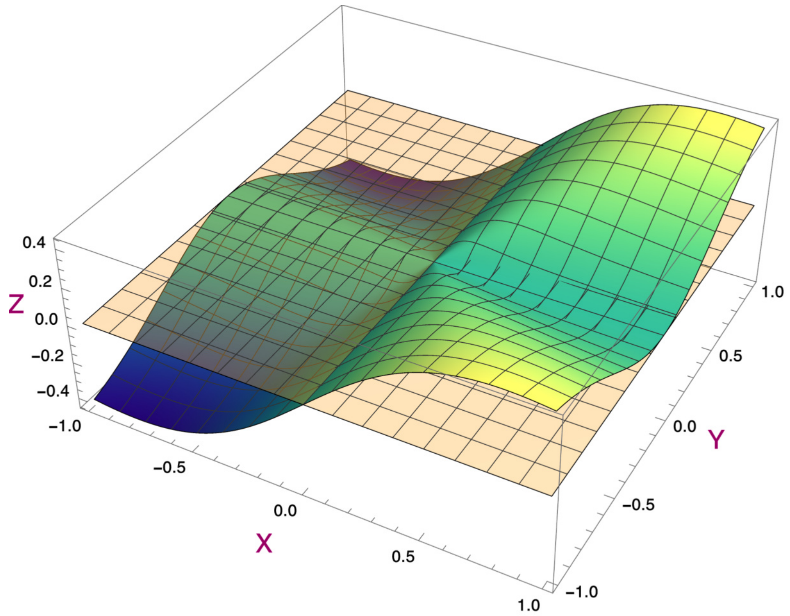

Figure 235. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x, y)$. We see here that the candidate tangent plane does not fit the function very nicely. There is not enough flatness in a neighbourhood of $(0, 0)$. It is indeed no tangent plane following our calculations.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

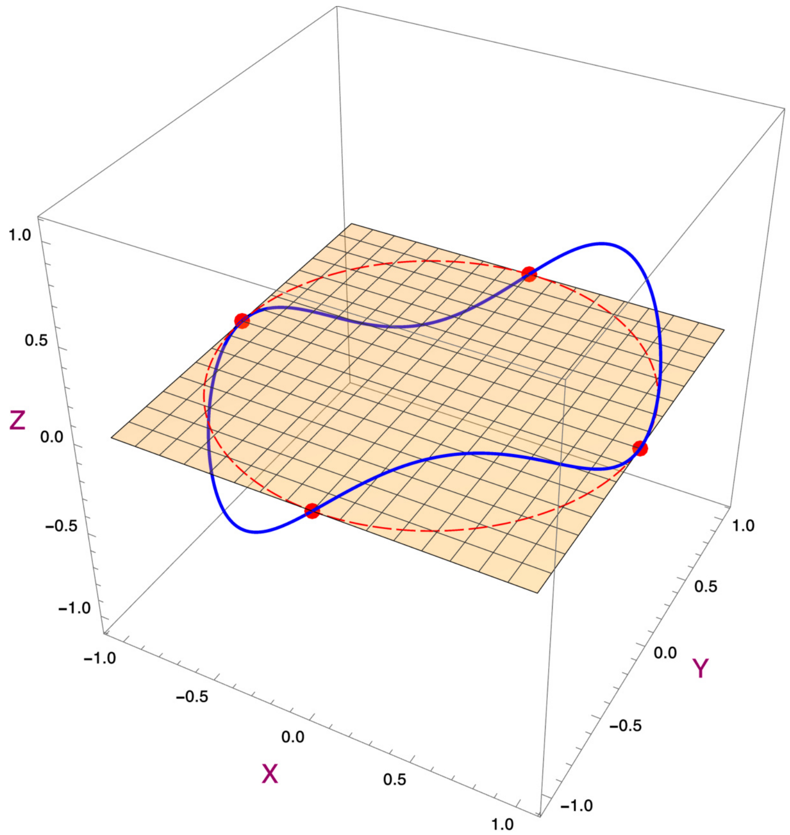

Figure 236. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also the blue curve containing the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ do not sweep out a nice ellipse lying in one plane! This is very bad news for differentiability.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$.

We restrict the function $q(h,k)$ to the continuous curves with equations $k = \lambda\,h$. We observe then that

$$q\,|_{k=\lambda\,h}(h,k) = \begin{cases} q(h,\lambda\,h) = \dfrac{h\lambda \sin\left(h^2\lambda\right)}{|h|^3\left(\lambda^2+1\right)^{3/2}} & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

Let us calculate the limit

$$\begin{aligned} \lim_{h\to 0} q(h,\,\lambda\,h) &= \lim_{h\to 0} \frac{h\,\lambda \sin(h^2\,\lambda)}{|h|^3\;(h^2\lambda)}\,\frac{(h^2\,\lambda)}{(\lambda^2+1)^{3/2}} \\ &= \lim_{h\to 0} \frac{h\,\lambda}{|h|^3}\,\frac{(h^2\,\lambda)}{(\lambda^2+1)^{3/2}} \\ &= \lim_{h\to 0} \frac{\operatorname{sgn}(h)\,\lambda^2}{(\lambda^2+1)^{3/2}}. \end{aligned}$$

This limit does not exist if $\lambda \neq 0$. But if $q(h,k)$ is continuous, all these limit values should be $q(0,0) = 0$. So this function $q(h,k)$ is not continuous in $(0,0)$. The function $f(x,y)$ is not differentiable in $(0,0)$.

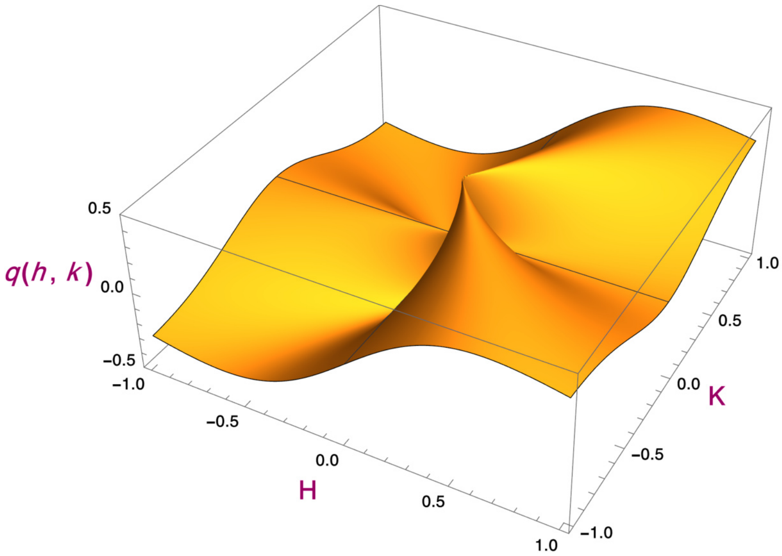

Figure 237. We see here a three dimensional figure of the graph of the function $q(h,k)$. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

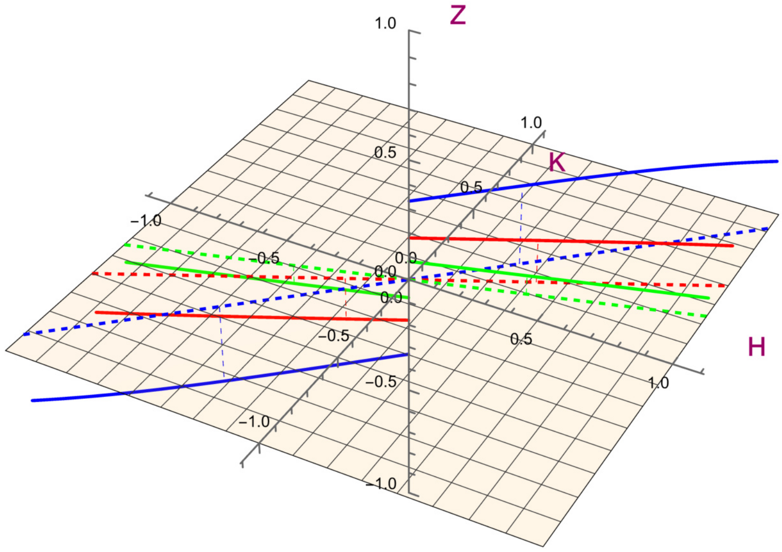

Figure 238. We have restricted the function $q(h, k)$ here to $k = 1/2\,h$ and $k = 3/10\,h$ and $k = 9/10\,h$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

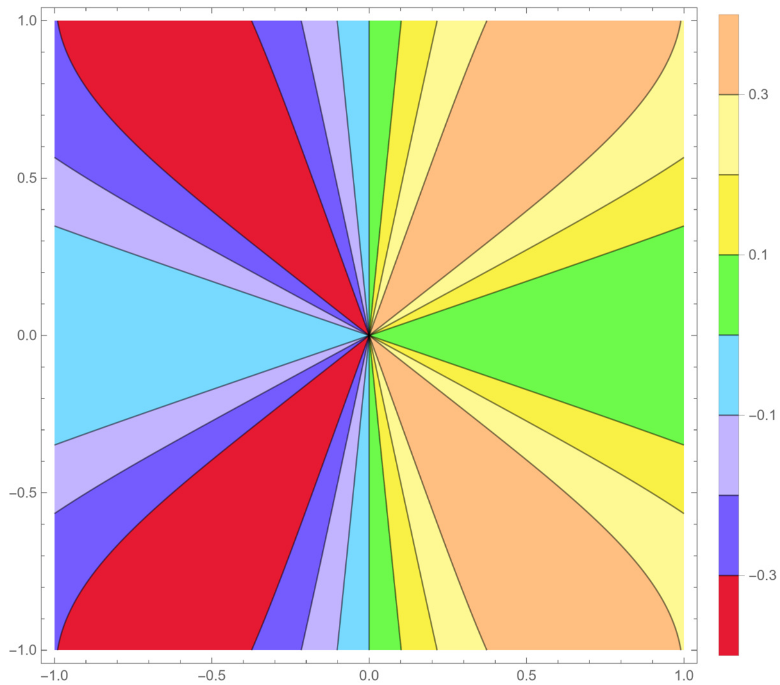

Figure 239. We see here a figure of the contour plot of the function $q(h,k)$. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Alternative proof for the non differentiability

Suppose that we already met in the course the differentiation rule of the composition of two differentiable functions. This is also called the chain rule. Then we have proven the following. If the function is differentiable in (a,b), then the directional derivative can be calculated is follows.

$$D_{(u,v)}f(a,b) = \frac{\partial f}{\partial x}(a,b)\,u + \frac{\partial f}{\partial y}(a,b)\,v.$$

Important remark. This formula is only valid if the function is differentiable. One of the most common mistakes is that one uses this formula in the case of non differentiability. It seems to be easy to calculate quickly the partial derivatives if they exist and then use this formula.

We have calculated the directional derivatives and we saw that

$$D_{(u,v)}f(a,b) = u\,v^2$$

and this is certainly not the linear function in u and v which we should have in the case of differentiability. So we conclude that the function is not differentiable.

26.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

26.8 Overview

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

26.9 One step further

We have met the magical curves $k = \lambda h$ when investigating the differentiability. We want to see what is really going on there. Let us take a look at the curve

$$(x(t), y(t), z(t)) = \begin{cases} (t, \lambda t, f(t, \lambda t)) = \left(t, \lambda t, \frac{\lambda \sin(\lambda t^2)}{\lambda^2 t + t}\right) & \text{if } t \neq 0, \\ (0,0,0) & \text{if } t = 0. \end{cases}$$

This curve is certainly continuous because f is continuous. But it is not unconditionally guaranteed that this curve is differentiable because f is not differentiable. So we will check differentiability first. There can only be a problem in the third component of the definition of the curve. We check the derivative.

$$\begin{aligned} z'(0) &= \lim_{t\to 0} \frac{z(t)-z(0)}{t-0} \\ &= \lim_{t\to 0} \frac{\lambda \sin(\lambda t^2)}{t\ (\lambda^2 t + t)} \\ &= \lim_{t\to 0} \frac{\lambda \sin(\lambda t^2)}{\lambda t^2} \frac{\lambda t^2}{t\ (\lambda^2 t + t)} \\ &= \frac{\lambda^2}{\lambda^2+1}. \end{aligned}$$

We see that the curve is differentiable and we have

$$(x'(t), y'(t), z'(t)) = \begin{cases} \left(1, \lambda, \dfrac{\lambda\left(2\lambda t^2 \cos\left(\lambda t^2\right) - \sin\left(\lambda t^2\right)\right)}{\left(\lambda^2+1\right) t^2}\right) & \text{if } t \neq 0, \\ \left(1, \lambda, \dfrac{\lambda^2}{\lambda^2+1}\right) & \text{if } t = 0. \end{cases}$$

We have for $t = 0$

$$(x'(0), y'(0), z'(0)) = \left(1, \lambda, \frac{\lambda^2}{\lambda^2+1}\right).$$

The tangent vector is in the case $\lambda = 1$ equal to $\left(1, 1, \frac{1}{2}\right)$. But we see that this vector is not in the candidate tangent plane. So f cannot be differentiable.

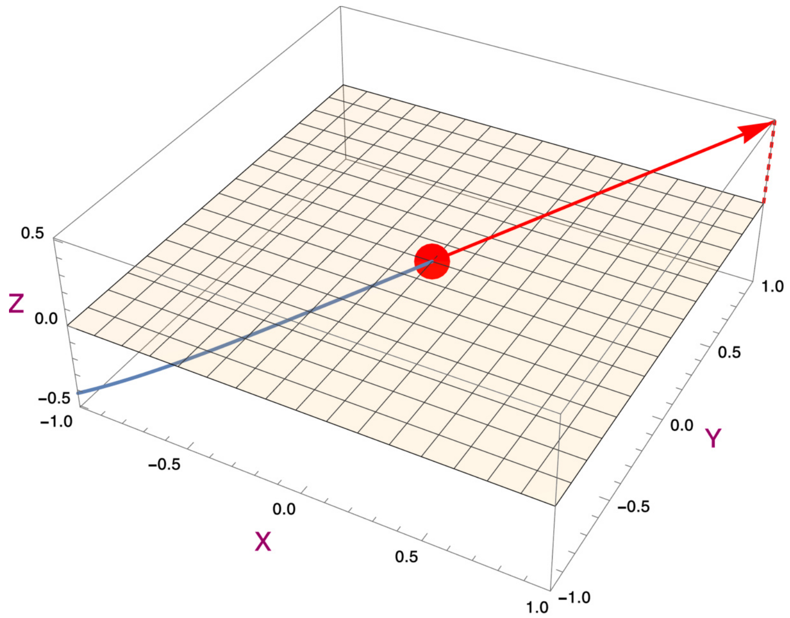

Figure 240. We see here a figure of the candidate tangent plane and the curve with equation $(x(t), y(t), z(t)) = (t, \lambda t, f(t, \lambda t))$. It is not a line, but it almost looks line a line. We see that this curve intersects the candidate tangent plane transversally and not tangentially. This is not possible if the function f is differentiable.

It can maybe be worthwhile to take a look at the gradient vector field. We see that this gradient vector field exists in this case. But the gradient vector field cannot be continuous because it implies that the function is differentiable. So we wonder if we can find an indication for this fact.

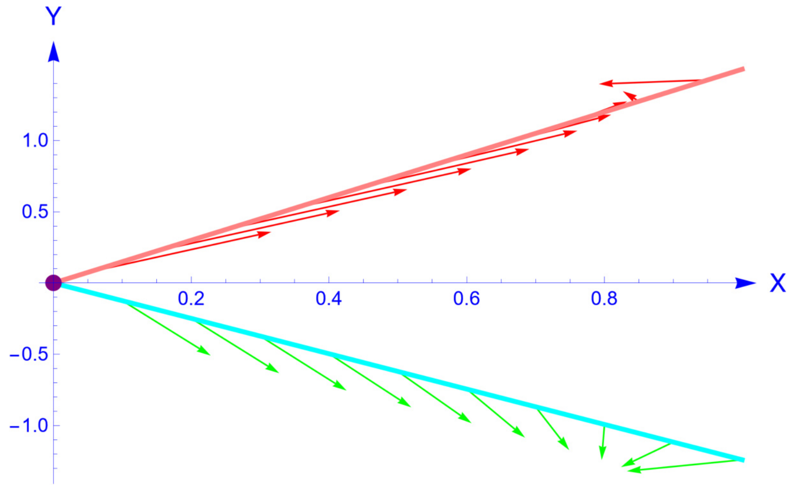

Figure 241. We made here the following sketch. We have drawn the graphics of $y = 1.5\,x$ in pink and $y = -1.24\,x$ in cyan. We have sketched the gradient vector field $\left(\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)\right)$ which is not continuous as can be seen as follows. Observe the gradient vector field on the pink curve, these are the red vectors. Observe the gradient vector field on the cyan curve, these are the green vectors. The purple vector is the gradient vector in $(0, 0)$, which is $(0, 0)$. The red vectors converge to a vector with a non zero x-component. This component is approximately equal to 0.27. The green vectors converge to a vector that have an x-component that is two times smaller then the x-component of the vector to which the red vectors converge if $x \to 0$. This is clearly impossible if the vector field is continuous. Moreover, the x-component of the limit vector should in both cases be zero if the gradient vector field is continuous. We conclude that this gradient vector field is not continuous. The function is differentiable in that case, which it is not. Please note however that a sketch of the gradient vector field is inherently a sketch of discrete data. So the utmost care must be taken in order to make it a little bit trustworthy.

Exercise 27.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

27.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{y^3}{x^2+y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{y^3}{x^2+y^2}\right| &\leq \frac{|y|^3}{x^2+y^2}\\ &\leq \frac{\sqrt{x^2+y^2}^3}{\sqrt{x^2+y^2}^2}\\ &\leq \sqrt{x^2+y^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function is continuous.

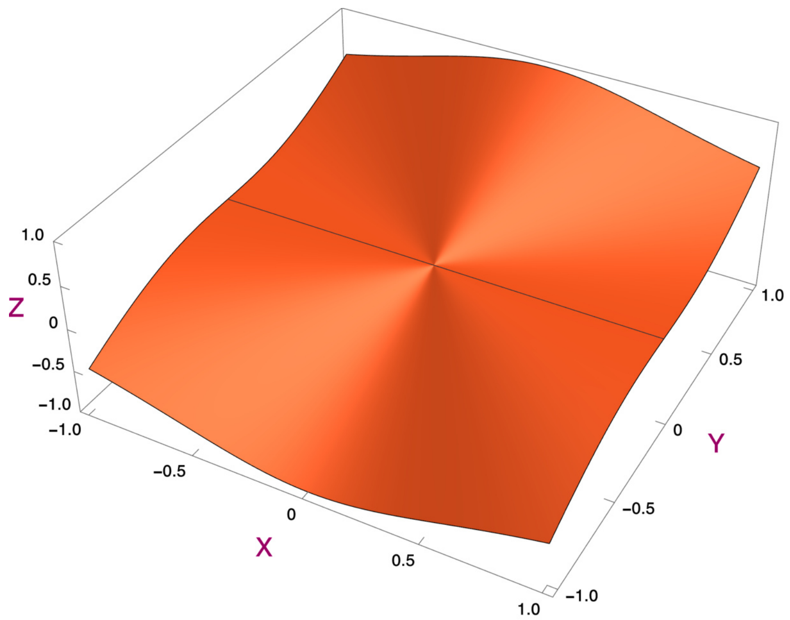

Figure 242. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

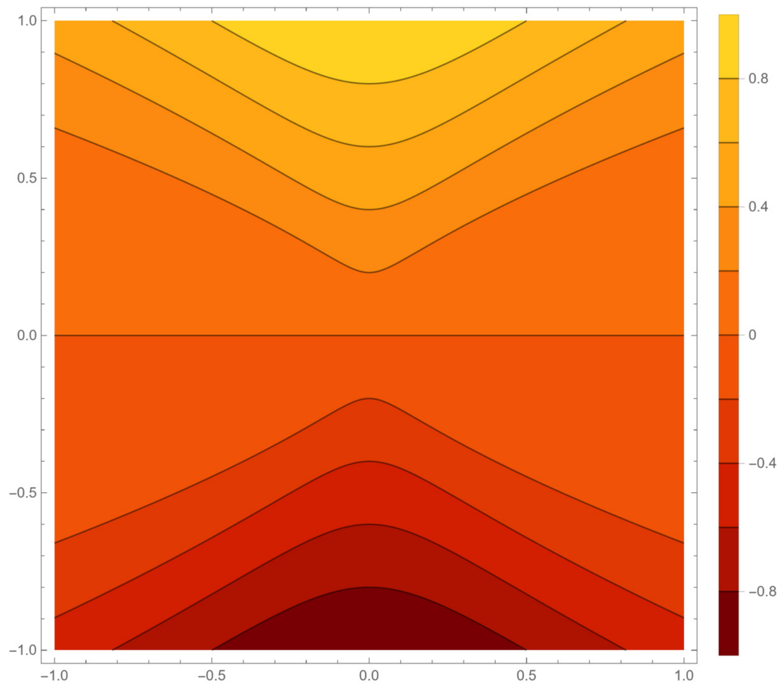

Figure 243. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

27.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = y & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 1\\ &= 1.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 1.$$

27.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{v^3}{u^2+v^2} \\ &= v^3. \end{aligned}$$

We remember that $u^2 + v^2 = 1$ because (u,v) is a unit vector.

So the directional derivatives do always exist.

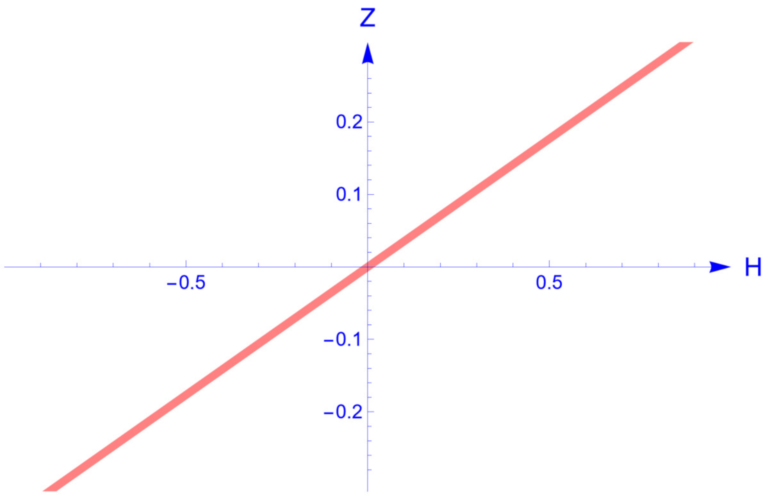

Figure 244. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$.

27.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} -\dfrac{2\,x\,y^3}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{3\,x^2\,y^2+y^4}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

$$\begin{aligned} \left|\frac{\partial f}{\partial x}(x,y)\right| &\leq \left|-\frac{2\,x\,y^3}{(x^2+y^2)^2}\right| \\ &\leq \frac{2\,|x|\,|y|^3}{(x^2+y^2)^2} \\ &\leq \frac{2\sqrt{x^2+y^2}\,\sqrt{x^2+y^2}^{\,3}}{\sqrt{x^2+y^2}^{\,4}} \\ &\leq 2. \end{aligned}$$

Let us try to prove that $\left|\frac{\partial f}{\partial y}\right|$ is bounded.

$$\begin{aligned}
\left|\frac{\partial f}{\partial y}(x,y)\right| &\le \frac{3x^2y^2+y^4}{(x^2+y^2)^2} \\
&\le \frac{3\sqrt{x^2+y^2}^2\sqrt{x^2+y^2}^2+\sqrt{x^2+y^2}^4}{\sqrt{x^2+y^2}^4} \\
&\le \frac{4\sqrt{x^2+y^2}^4}{\sqrt{x^2+y^2}^4} \\
&\le 4.
\end{aligned}$$

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

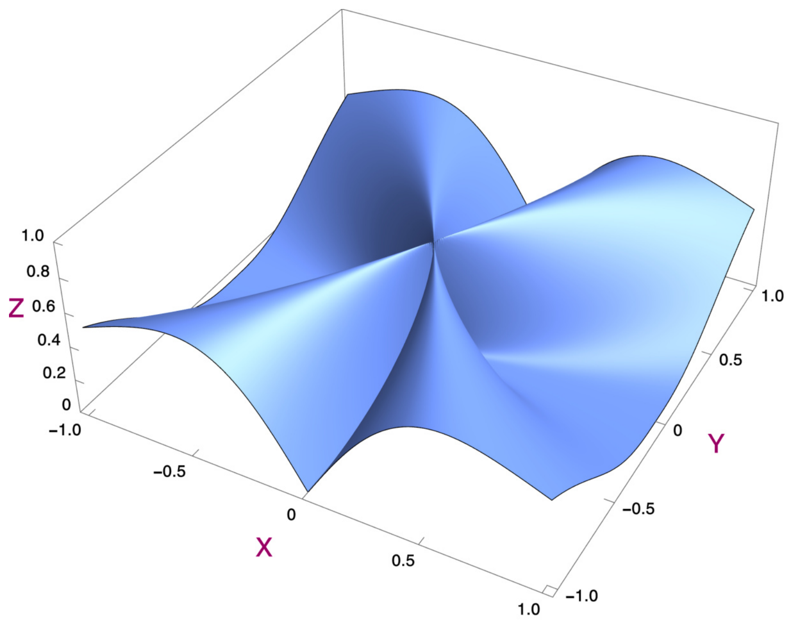

Figure 245. We see here the absolute value of the first partial derivative $\left|\frac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

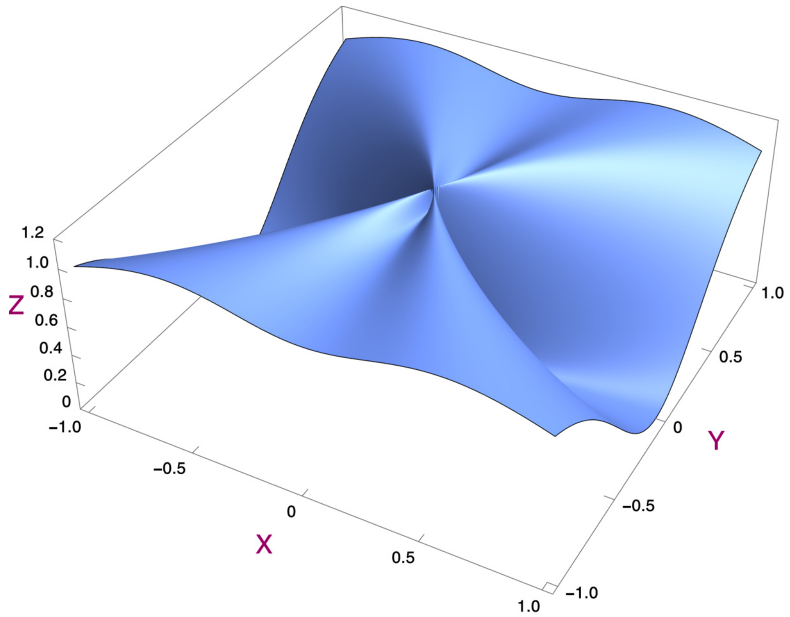

Figure 246. We see here the absolute value of the second partial derivative $\left|\dfrac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture.

27.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

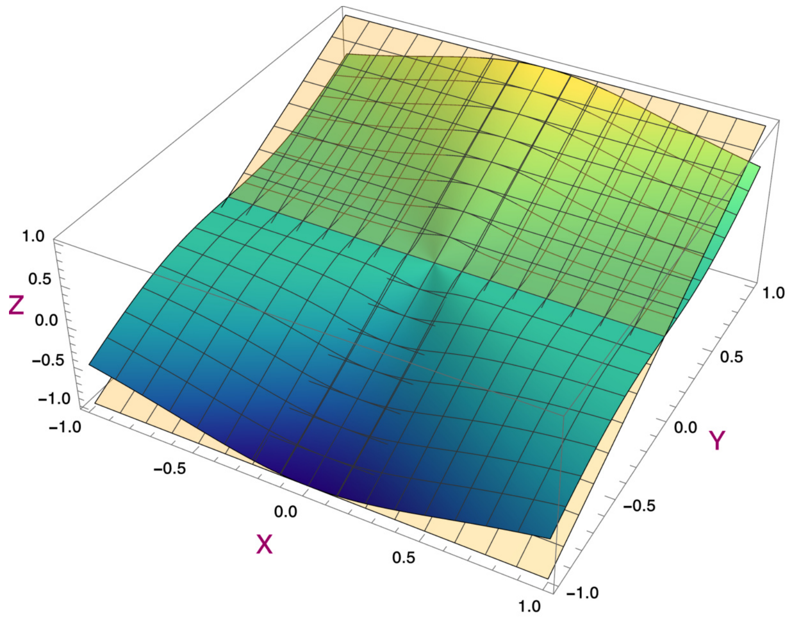

Figure 247. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function quite nicely. But there are directions that cause a little bit of doubt.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

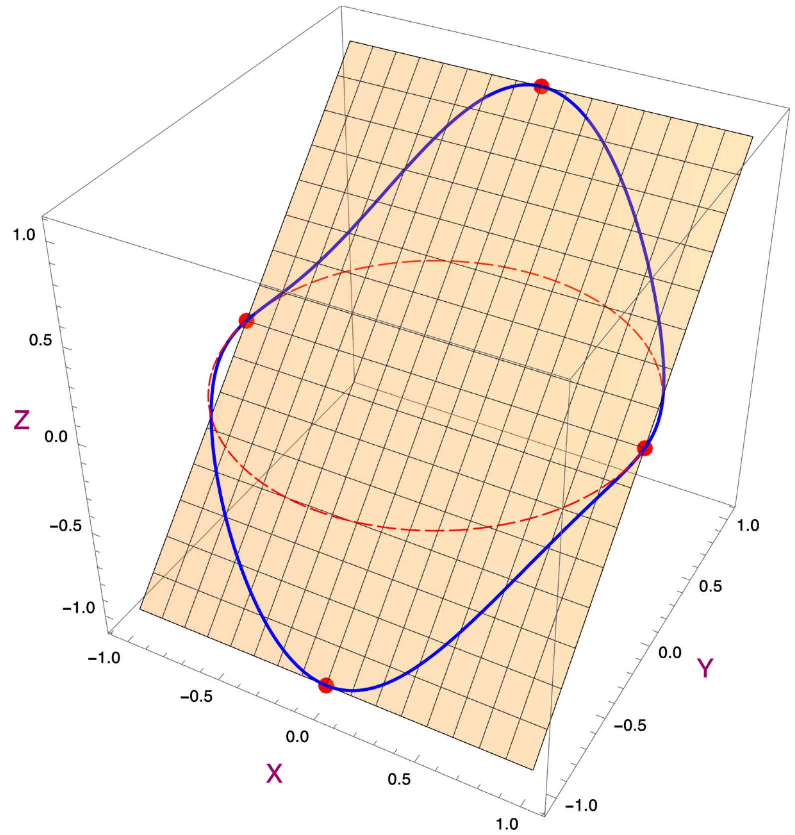

Figure 248. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ do not sweep out a nice ellipse lying in one plane! The blue curve is not in the candidate tangent plane. This is almost a sure sign of non differentiability.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} -\dfrac{h^2\,k}{\left(h^2+k^2\right)^{3/2}} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We restrict the function $q(h,k)$ to the continuous curves with equations $k = \lambda\,h$. We observe then that

$$q\,|_{k=\lambda\,h}(h,k)$$

$$= \begin{cases} q(h,\lambda\,h) = -\dfrac{h^3\,\lambda}{\left(h^2\,\left(\lambda^2+1\right)\right)^{3/2}} = -\dfrac{\operatorname{sgn}(h)\,\lambda}{\left(\left(\lambda^2+1\right)\right)^{3/2}} & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

We see that these restricted functions have no limits. But if $q(h,k)$ is continuous, all these limit values should be $q(0,0) = 0$. So this function $q(h,k)$ is not continuous in $(0,0)$. The function $f(x,y)$ is not differentiable in $(0,0)$.

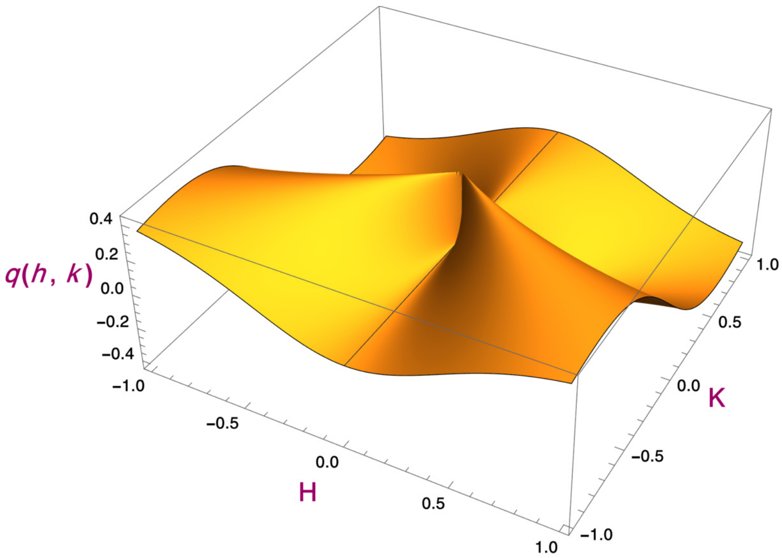

Figure 249. We see here a three dimensional figure of the graph of the function $q(h,k)$. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

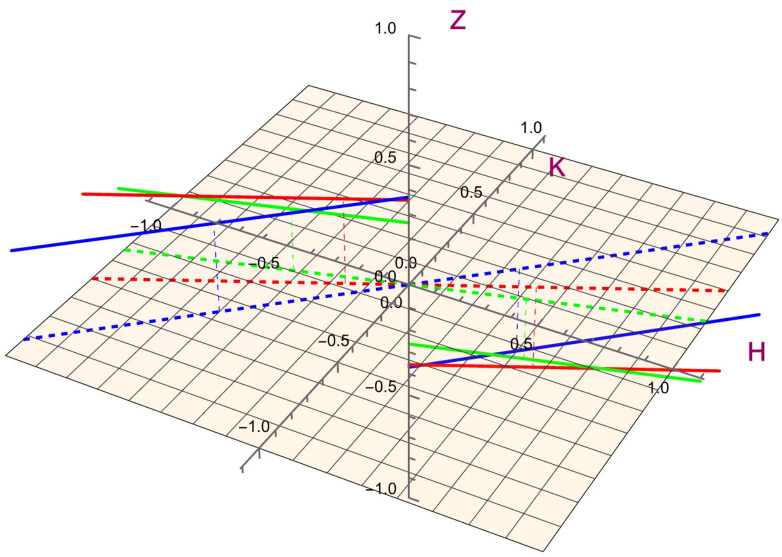

Figure 250. We have restricted the function $q(h,k)$ here to $k = 1/2\,h$ and $k = 3/10\,h$ and $k = 9/10\,h$. We see in this figure clearly that the restrictions of the function to these lines are functions that have no limits in 0.

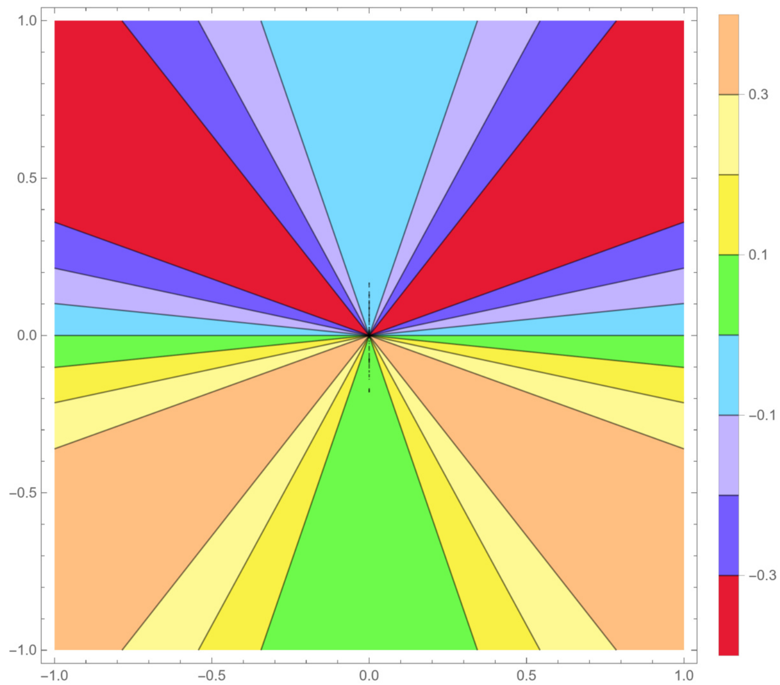

Figure 251. We see here a figure of the contour plot of the function $q(h,k)$. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

Alternative proof for the non differentiability

Suppose that we already met in the course the differentiation rule of the composition of two differentiable functions. This is also called the chain rule. Then we have proven the following. If the function is differentiable in (a,b), then the directional derivative can be calculated is follows.

$$D_{(u,v)}f(a,b) = \frac{\partial f}{\partial x}(a,b)\,u + \frac{\partial f}{\partial y}(a,b)\,v.$$

Important remark. This formula is only valid if the function is differentiable. One of the most common mistakes is that one uses this formula in the case of non differentiability. It seems to be easy to calculate quickly the partial derivatives if they exist and then use this formula. *But this is only a correct procedure if one is certain that the function is differentiable.*

We have calculated the directional derivatives and we saw that

$$D_{(u,v)}f(a,b) = v^3.$$

and this is certainly not the linear function in u and v which we should have in the case of differentiability. So we conclude that the function is not differentiable.

27.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

27.8 Overview

$$f(x,y) = \begin{cases} \dfrac{y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

27.9 One step further

It can maybe be worthwhile to take a look at the gradient vector field. We see that this gradient vector field exists in this case. But the gradient vector field cannot be continuous because it implies that the function is differentiable. So we wonder if we can find an indication for this fact.

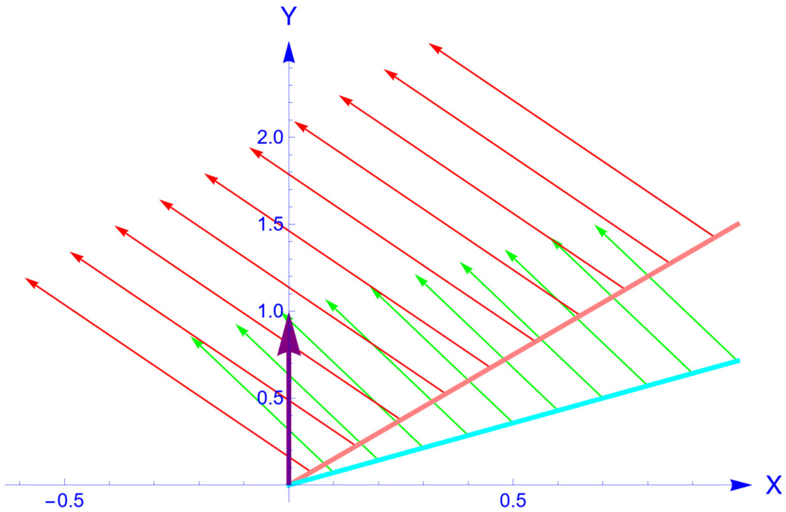

Figure 252. We made here the following sketch. We have drawn the graphics of $y = 1.5\,x$ in pink and $y = 0.74\,x$ in cyan. We have sketched the gradient vector field $\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$ which is not continuous as can be seen as follows. Observe the gradient vector field on the pink curve, these are the red vectors. Observe the gradient vector field on the cyan curve, these are the green vectors. The purple vector is the gradient vector in $(0,0)$. The red vectors converge to a vector with a non zero x-component. This component is equal to -0.64. The green vectors converge to a vector that has a x-component that is two times smaller then the x-component of the vector to which the red vectors converge if $x \to 0$. This is clearly impossible if the vector field is continuous. Moreover, the x-component of the limit vector should in both cases be zero if the gradient vector field is continuous. We conclude that this gradient vector field is not continuous. The function is differentiable in that case, which it is not. Please note however that a sketch of the gradient vector field is inherently a sketch of discrete data. So the utmost care must be taken in order to make it a little bit trustworthy.

Exercise 28.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} (x^2+y^2)\log(x^2+y^2) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

28.1 Continuity

We know from one variable calculus that $\lim_{\alpha\to 0} \alpha \log(\alpha) = 0$. We can use for this the theorem of de l'Hospital. The composition of continuous functions is also continuous and this proves the continuity.

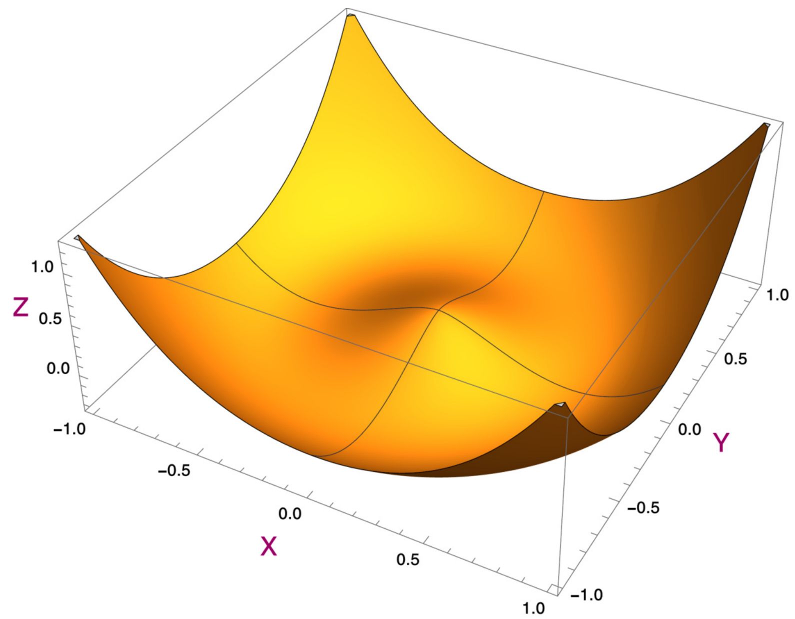

Figure 253. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

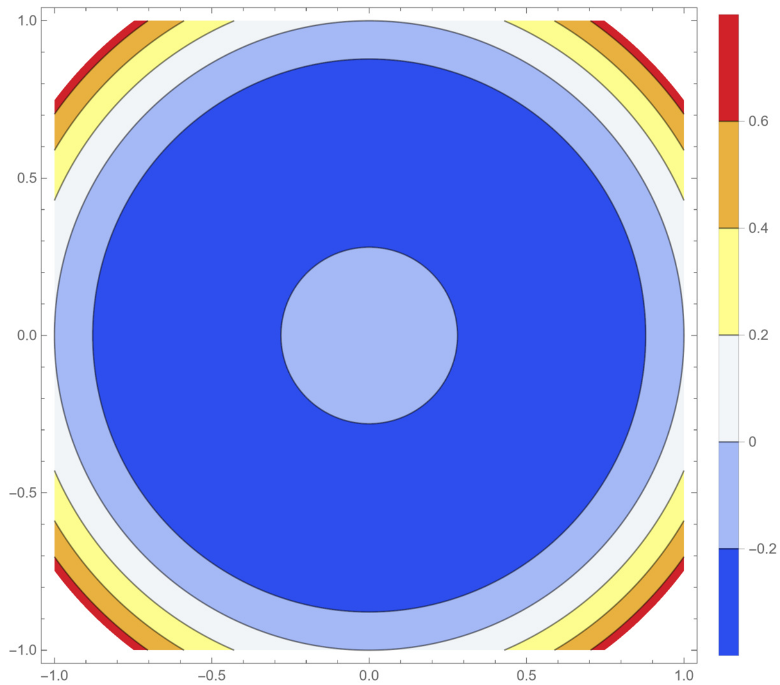

Figure 254. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

28.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = x^2 \log(x^2) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} h\log\left(h^2\right)\\ &= 0.\end{aligned}$$

We can use the theorem of de l'Hospital for this. So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We see that by rotational symmetry the partial derivatives are the same.

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

28.3 Directional derivatives

By rotational symmetry, the directional derivatives are the same as in the X-direction.

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0}\frac{f(0+hu,0+hv)-f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\lim_{h\to 0}\frac{f(0+hu,0+hv)-f(0,0)}{h} = 0.$$

So the directional derivatives do always exist.

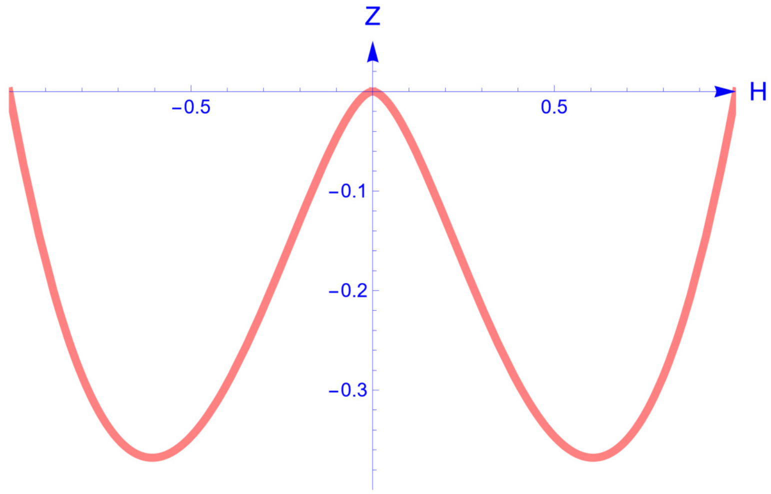

Figure 255. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$. We see that the slope in 0 is indeed zero.

28.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.
Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

The partial derivative to x is

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} 2x\left(\log\left(x^2+y^2\right)+1\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We have

$$2x\left(\log\left(x^2+y^2\right)+1\right) = 2x\log\left(x^2+y^2\right) + 2x.$$

Now $2x$ is certainly continuous and thus bounded. So we are left with showing that $2x\log(x^2+y^2)$ is bounded. Now $\log(x^2) \le \log(x^2+y^2)$ and if $x^2+y^2 < 1$ and $x > 0$, we have $x\log(x^2) \le x\log(x^2+y^2)$ and consequently we have

$$|x\log(x^2)| \ge |x\log(x^2+y^2)|.$$

Now by the theorem of de l'Hospital, we have that $|x\log(x^2)|$ has a finite limit in $x = 0$, so $|x\log(x^2+y^2)|$ is bounded. By symmetry considerations, this bound is valid for all directions. So the partial derivative to x is bounded and again by symmetry considerations, all directional derivatives are bounded by the same bound.

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

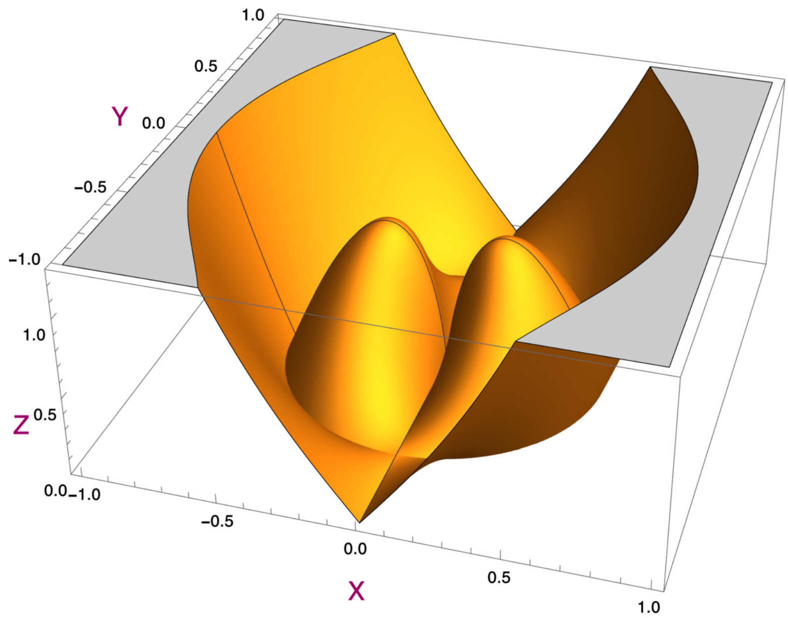

Figure 256. We see here the absolute value of the first partial derivative $\left|\frac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

We have by symmetry the same figure for the partial derivative to y.

28.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

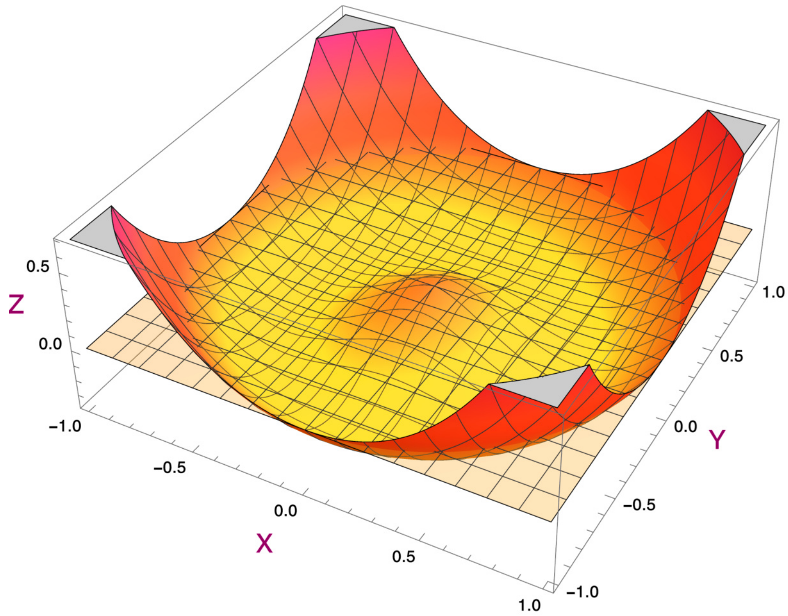

Figure 257. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. It is impossible to say if the candidate tangent plane is a good fit. It is indeed no tangent plane following our calculations.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

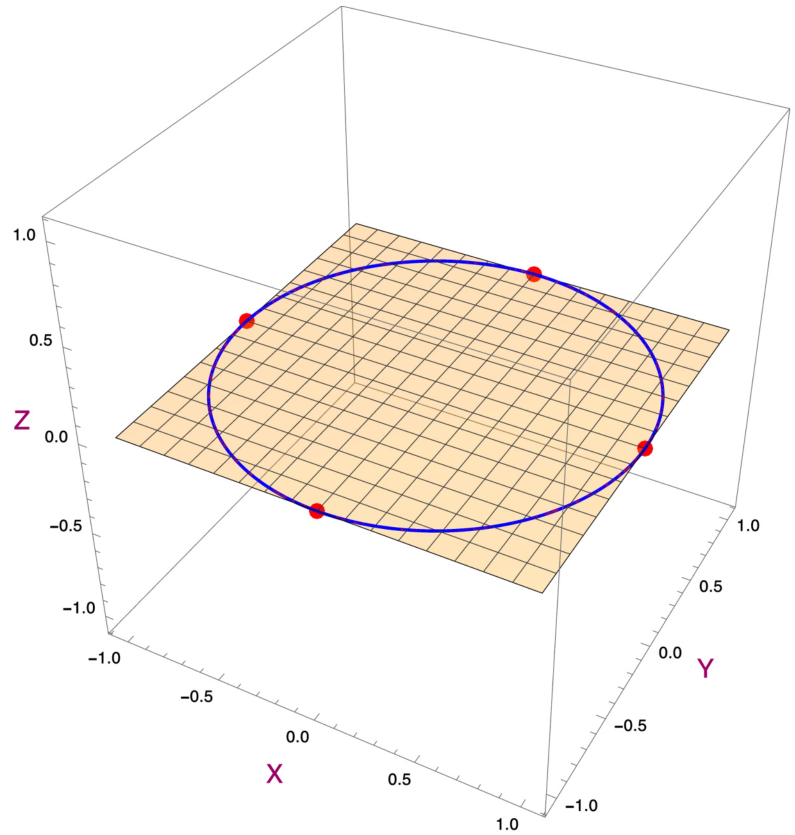

Figure 258. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h, k)$ in $(0, 0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \sqrt{h^2+k^2}\,\log\left(h^2+k^2\right) & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate the continuity. We see that we have essentially the function in one variable $\sqrt{\alpha}\,\log(\alpha)$ composed with the two variable function $\alpha = h^2+k^2$. Now the function with function definition $\sqrt{\alpha}\,\log(\alpha)$ is by de l'Hospital continuous with value 0, we have by symmetry the continuity of the quotient $q(h,k)$. We give a plot of this situation.

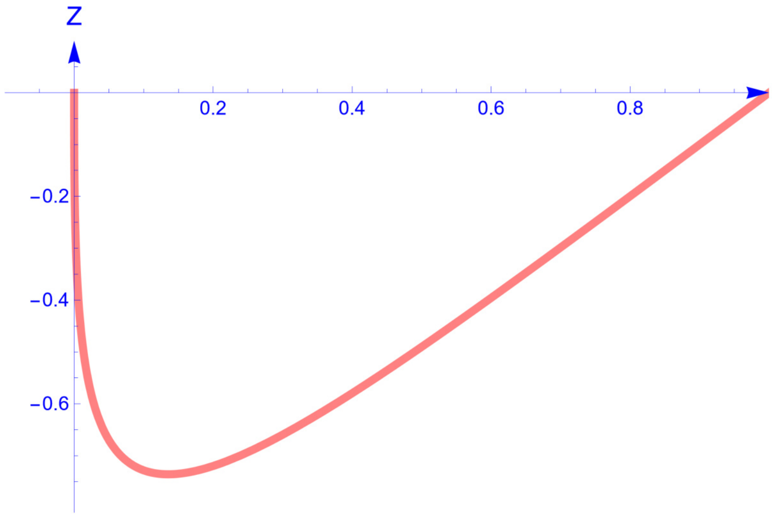

Figure 259. We see here a figure of the graph of the function $\sqrt{\alpha}\,\log(\alpha)$.

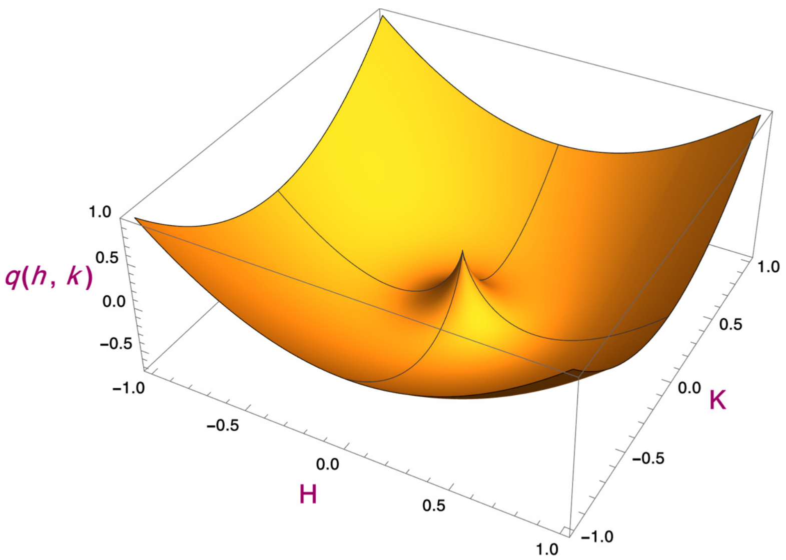

Figure 260. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

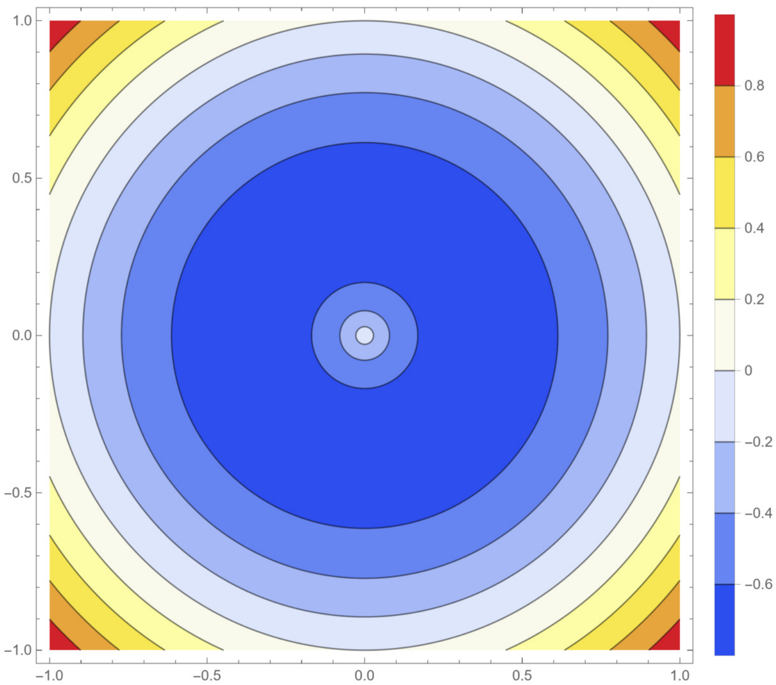

Figure 261. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

28.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\,u + \frac{\partial f}{\partial y}(0,0)\,v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the func-

tion $f(x, y)$ is locally Lipschitz continuous in $(0, 0)$. We cite here the criterion that we will use.

A function is differentiable in (a, b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.
2. The directional derivatives have the following form: $D_{(u,v)}(a, b) = \nabla f(a, b) \cdot (u, v) = \frac{\partial f}{\partial x}(a, b)\, u + \frac{\partial f}{\partial y}(a, b)\, v$.
3. The function is locally Lipschitz continuous in (a, b). This means that there exists at least one neighbourhood of (a, b) and a number $K > 0$ such that for all (x_1, y_1) and (x_2, y_2) in the neighbourhood, we have $|f(x_1, y_1) - f(x_2, y_2)| < K \|(x_1, y_1) - (x_2, y_2)\|$. Remark that the same K must be valid for all (x_1, y_1) and (x_2, y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be "locally flat" and thus "locally linear" which is partly expressed by this Lipschitz continuity condition.

Let us try to prove the Lipschitz condition. We are going to use a well known method.

$$
\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)| \\
&\qquad = |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\qquad \le |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)|.
\end{aligned}
$$

We focus now on the first term $|f(x_1, y_1) - f(x_1, y_2)|$. We fix now x_1 and look upon $f(x_1, y)$ as a function in one variable y. So it springs to mind that we can estimate this by using Lagrange's intermediate value theorem. So we have

$$
|f(x_1, y_1) - f(x_1, y_2)| = \left| \frac{\partial f}{\partial y}(x_1, \xi) \right| \, |y_1 - y_2|
$$

where ξ is a number lying in the open interval (y_1, y_2) if e.g. $y_1 \leq y_2$. Remark that we already have proven that the partial derivatives are bounded in a neighbourhood, we can take that bound found for $\frac{\partial f}{\partial y}$, say M_2 and that neighbourhood in this case also. So $\left|\frac{\partial f}{\partial y}(x_1, \xi)\right| \leq M_2$. We work in a completely analogous way for the second term $|f(x_1, y_2) - f(x_2, y_2)|$. We have in that case $\left|\frac{\partial f}{\partial x}(\xi, y_2)\right| \leq M_1$ where ξ is now a number in the open interval (x_1, x_2) if e.g. $x_1 \leq x_2$.

We take up our inequality again

$$
\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)| \\
&\quad \leq |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\quad \leq |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)| \\
&\quad \leq M_2\,|(y_1 - y_2)| + M_1\,|(x_1 - x_2)| \\
&\quad \leq M_2\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + M_1\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&\quad \leq (M_1 + M_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.
\end{aligned}
$$

So this function is locally Lipschitz continuous. We have an alternative proof for the differentiability.

28.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

$$
\frac{\partial f}{\partial x}(x, y) = \begin{cases} 2x\,\left(\log\left(x^2 + y^2\right) + 1\right) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}
$$

We want to see if it is continuous or not.
Discussion of the continuity of the first partial derivative in $(0,0)$.

Because

$$2\,x\,\left(\log\left(x^2+y^2\right)+1\right)=2\,x\,\log\left(x^2+y^2\right)+2\,x$$

it is enough to prove the continuity of $2\,x\,\log\left(x^2+y^2\right)$. The function $2\,x$ is indeed continuous.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon>0$ and we have to prove the inequality $|\frac{\partial f}{\partial x}(x,y)-\frac{\partial f}{\partial x}(0,0)|<\epsilon$. The problem is now to find a $\delta>0$ such that if the inequality $||(x,y)-(0,0)||<\delta$ holds, it follows that $|\frac{\partial f}{\partial x}(x,y)-\frac{\partial f}{\partial x}(0,0)|<\epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)||=\sqrt{x^2+y^2}<\delta$, we have that

$$\left|\frac{\partial f}{\partial x}(x,y)-\frac{\partial f}{\partial x}(0,0)\right|<\epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|2\,x\,\log\left(x^2+y^2\right)\right| &\leq 2\,|x|\,\left|\log\left(x^2+y^2\right)\right|\\ &\leq 2\,(x^2+y^2)\,\left|\log\left(x^2+y^2\right)\right|.\end{aligned}$$

The last function in our inequalities is the composition of $2\alpha\log(\alpha)$ with $\alpha=(x^2+y^2)$ and these are continuous functions in 0 with value 0. We can for example use de l'Hospital to prove it.

We can find a δ, so we conclude that the partial derivative to x is continuous.

We can similarly prove that the partial derivative to y is continuous. So we have an alternative proof for the differentiability.

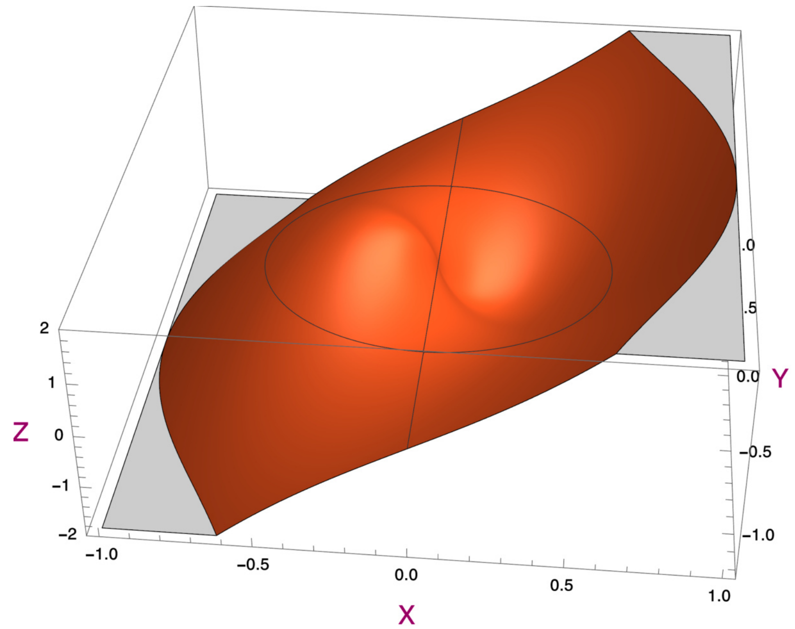

Figure 262. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial x}(x, y)$. This looks like a continuous function.

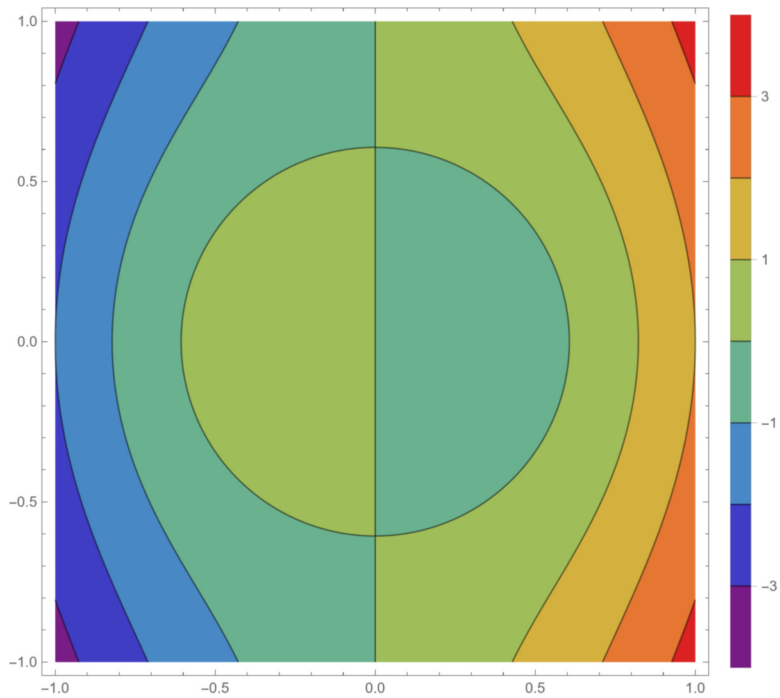

Figure 263. We see here a figure of the contour plot of $\frac{\partial f}{\partial x}(x, y)$. Only level curves of level around 0 come close to $(0, 0)$.

28.8 Overview

$$f(x, y) = \begin{cases} (x^2 + y^2) \log (x^2 + y^2) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	yes

28.9 One step further

We want to know if this function is further uneventful from the point of view of differentiability. Let us take a look at the second partial derivative

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{2\left(\left(x^2+y^2\right)\log\left(x^2+y^2\right)+3x^2+y^2\right)}{x^2+y^2}.$$

Let us take a look of a three dimensional plot of this partial derivative to y of the function.

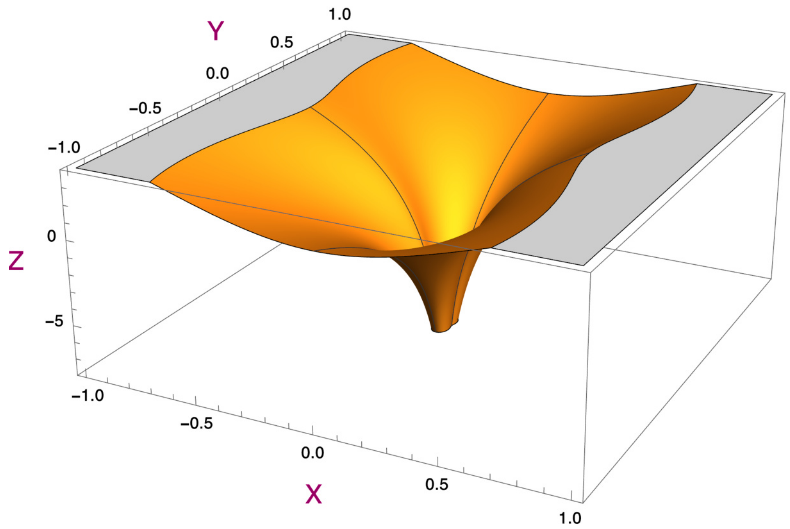

Figure 264. We see here a figure of the second order partial derivative $\frac{\partial^2 f}{\partial x^2}(x,y)$. It seems quite improbable that this partial derivative to y is continuous. It seems to be unbounded. We stop however our investigations here and leave this to the initiative of the interested reader.

Exercise 29.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{x^2+y^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

29.1 Continuity

We restrict the function to the continuous curves with equations $y = \lambda x$. We observe then that

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{\lambda^2+1}{\lambda^4 x^2+1} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

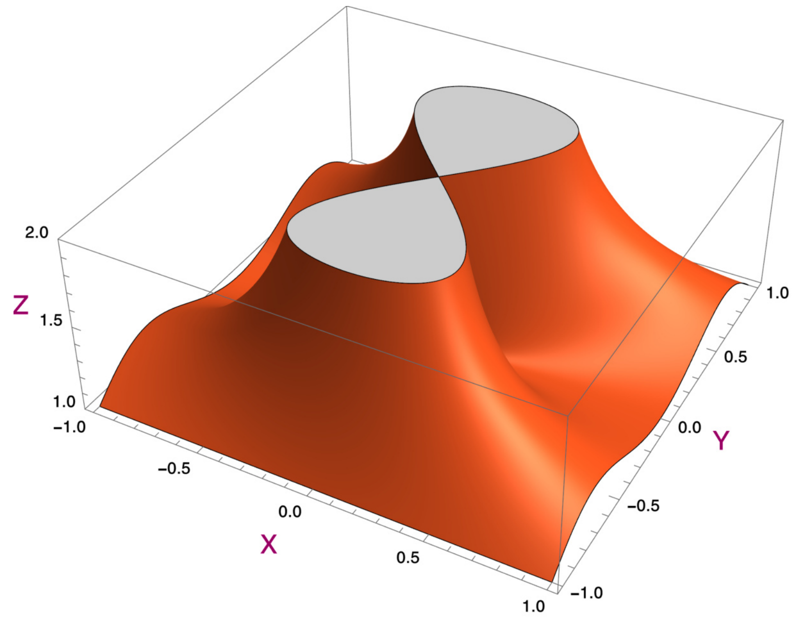

Figure 265. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

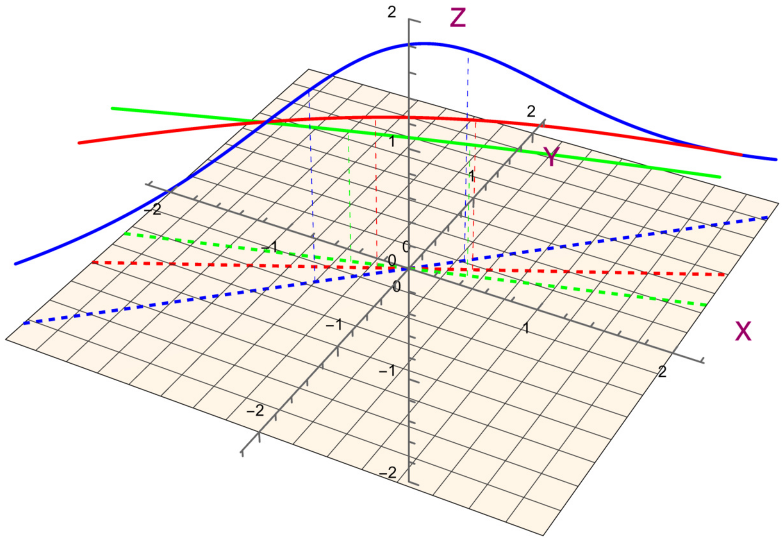

Figure 266. We have restricted the function here to $y = 1/2\,x$ and $y = 3/10\,x$ and $y = 9/10\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

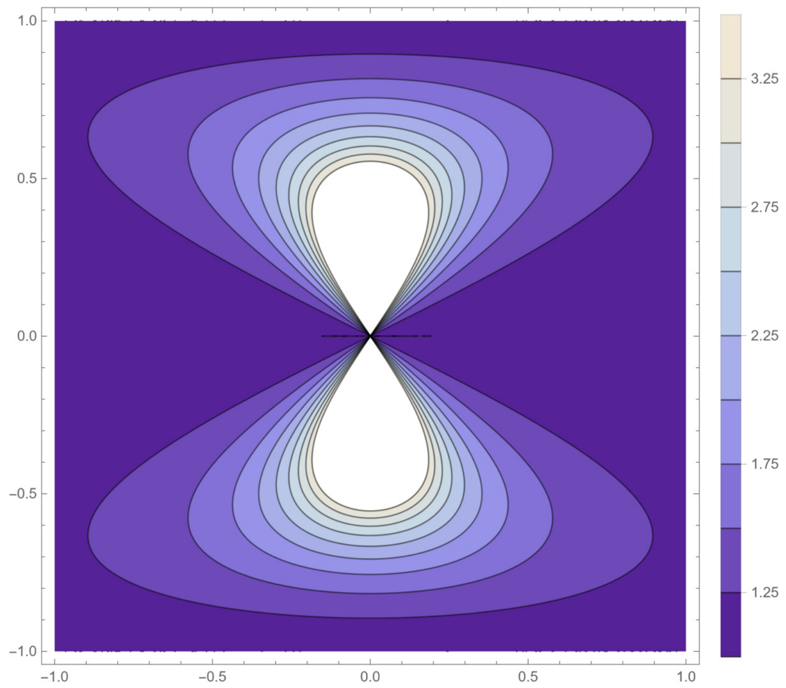

Figure 267. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0, 0)$. This looks discontinuous indeed.

29.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x, y) = \begin{cases} f(x, 0) = 1 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We take a look at this function and see that it is not continuous.

So the partial derivative to x does not exists

Discussion of the partial derivative to y in (0,0).

We observe that

$$f|_{x=0}(x, y) = \begin{cases} f(0, y) = \dfrac{1}{y^2} & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

This function is not bounded in any neighbourhood of $y = 0$ and is not continuous.

So the partial derivative to y does not exist.

We conclude that the partial derivative to x does not exist. and that the partial derivative to y does not exist.

29.3 Directional derivatives

Let (u, v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2 + v^2} = 1$. Then the directional derivative in $(0, 0)$ in the direction (u, v), notation $D_{(u,v)}(0, 0)$ is by definition

$$D_{(u,v)}(0, 0) = \lim_{h \to 0} \frac{f(0 + hu, 0 + hv) - f(0, 0)}{h}.$$

We observe that

$$f(0 + hu, 0 + hv) = \begin{cases} \dfrac{u^2 + v^2}{h^2 v^4 + u^2} & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

The function is not continuous if $u \neq 0$. We have covered the exceptional case $u = 0$ before.

So the directional derivatives do not always exist.

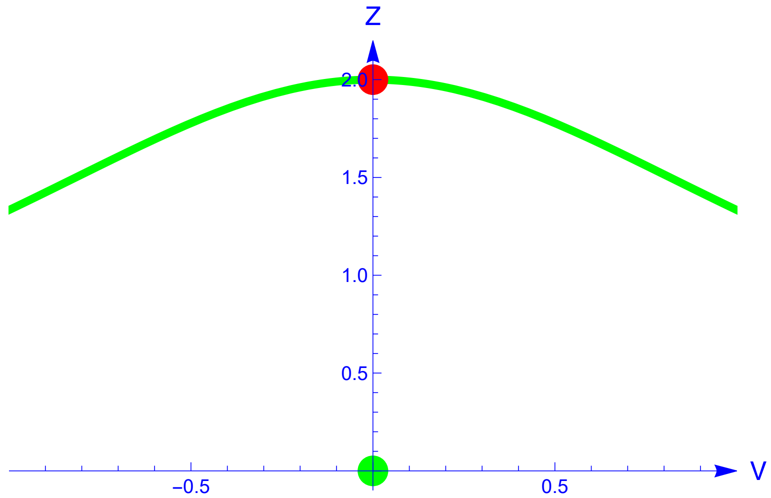

Figure 268. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$.

29.4 Alternative proof of continuity (optional)

Irrelevant. The function is not continuous.

29.5 Differentiability

The function is not continuous, so it cannot be differentiable.

29.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

29.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability. Irrelevant. The function is not differentiable.

29.8 Overview

$$f(x,y) = \begin{cases} \dfrac{x^2+y^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	no
partial derivatives exist	no
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 30.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2y^2+(x-y)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

30.1 Continuity

We restrict the function to the continuous curves with equations $y = x + \lambda x^2$. We observe then that

$$f|_{y=x+\lambda x^2}(x,y) = \begin{cases} \dfrac{x^2\left(\lambda x^2+x\right)^2}{\lambda^2 x^4 + x^2\left(\lambda x^2+x\right)^2} = \dfrac{(\lambda x+1)^2}{\lambda^2\left(x^2+1\right)+2\lambda x+1} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

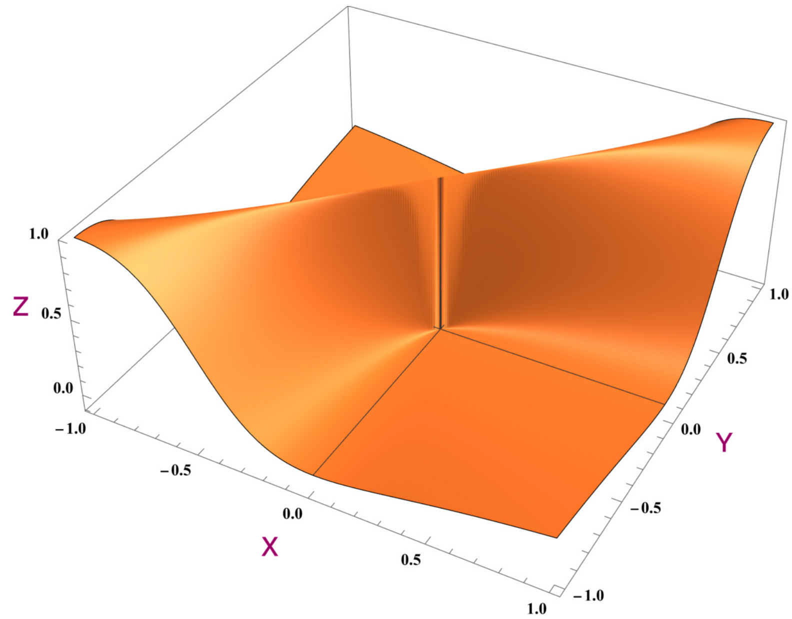

Figure 269. We see here a three dimensional figure of the graph of the function. The vertical line above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

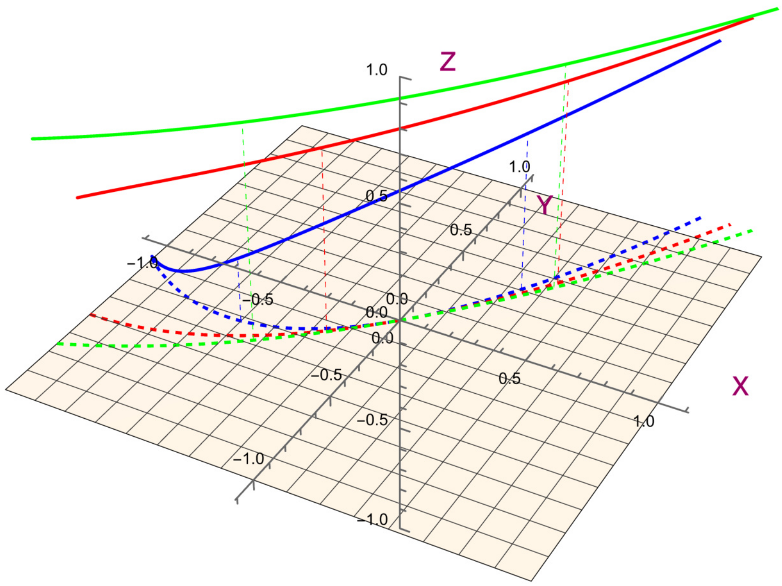

Figure 270. We have restricted the function here to $y = x + 1/2\,x^2$ and $y = x + 3/10\,x^2$ and $y = x + 9/10\,x^2$. We see in this figure clearly that the restrictions of the function to these continuous curves are functions that have different limits in 0.

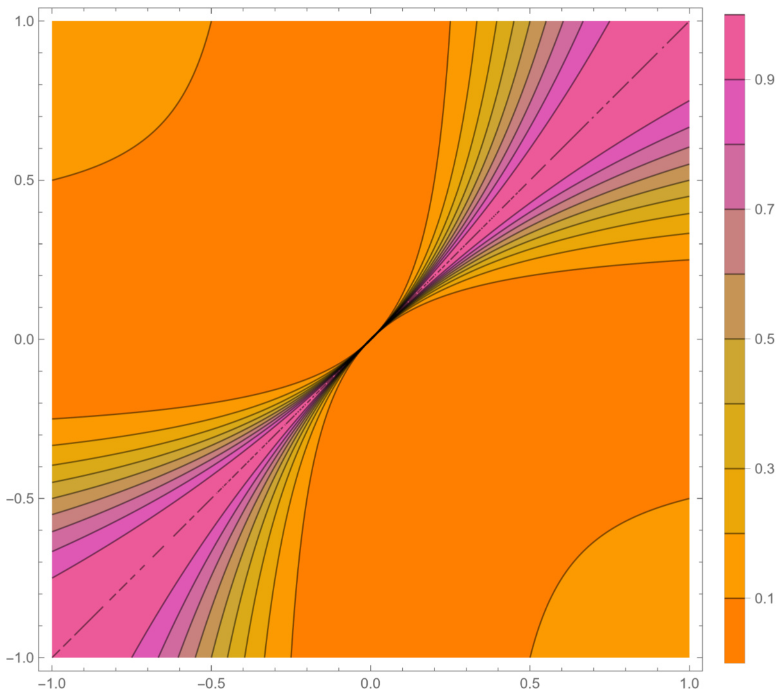

Figure 271. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

30.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

30.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{h u^2 v^2}{u^2 (h^2 v^2 + 1) - 2uv + v^2} \\ &= 0. \end{aligned}$$

This calculation is only valid if $u \neq v$.

If $u = v$, then we have

$$f(0+hu, 0+hu) = \begin{cases} 1 & \text{if } u \neq 0, \\ 0 & \text{if } u = 0. \end{cases}$$

This function is not continuous and the directional derivative does not exist.

So the directional derivatives do not always exist.

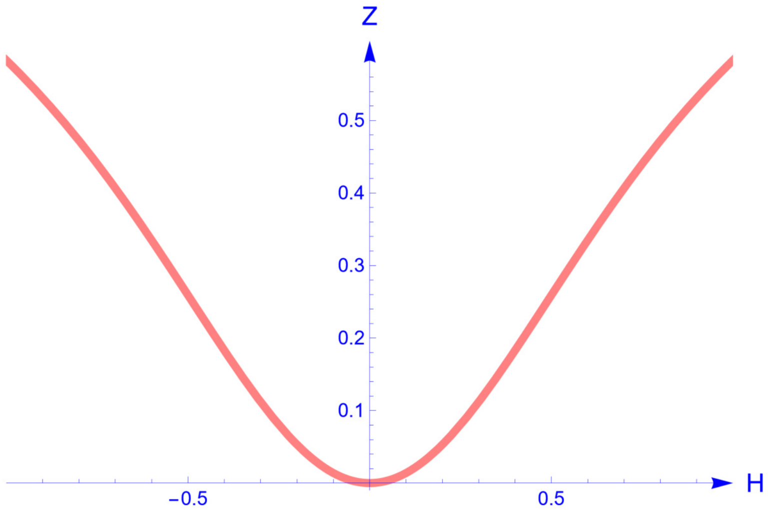

Figure 272. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(\sqrt{3}/2, 1/2\right)$. The slope is 0 in 0. We have plotted here the function $f(h\,u, h\,v)$

30.4 Alternative proof of continuity (optional)

Irrelevant. The function is not continuous.

30.5 Differentiability

The function is not continuous. So it is not differentiable.

30.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

30.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

30.8 Overview

$$f(x,y) = \begin{cases} \dfrac{x^2 y^2}{x^2 y^2 + (x-y)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	no
partial derivatives exist	yes
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 31.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} x^2\left(\sin\left(\frac{1}{x+y}\right)+2\right) & \text{if } x+y \neq 0, \\ 0 & \text{if } x+y = 0. \end{cases}$$

31.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| x^2\left(\sin\left(\frac{1}{x+y}\right)+2\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|x^2\left(\sin\left(\frac{1}{x+y}\right)+2\right)\right| &\leq x^2\left|\left(\sin\left(\frac{1}{x+y}\right)+2\right)\right| \\ &\leq x^2\left(\left|\sin\left(\frac{1}{x+y}\right)\right|+2\right) \\ &\leq 3\,x^2 \\ &\leq 3\sqrt{x^2+y^2}^{\,2}.\end{aligned}$$

It is sufficient to take $\delta = (\epsilon/3)^{1/2}$. We can find a δ, so we conclude that the function is continuous.

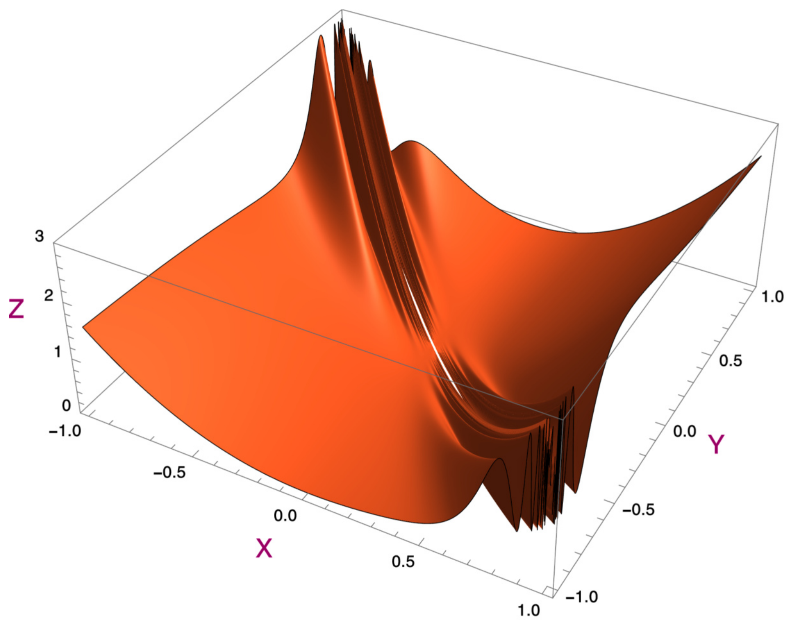

Figure 273. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

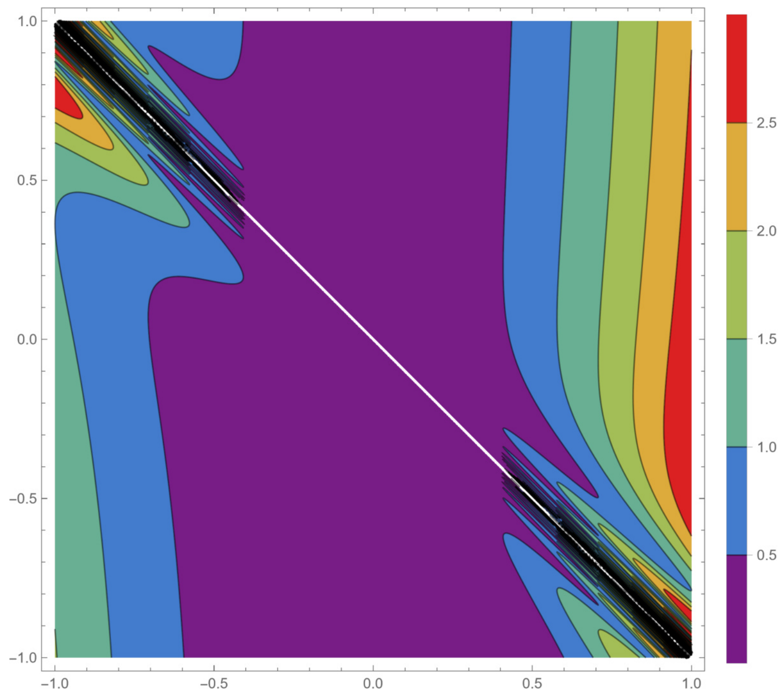

Figure 274. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

31.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = x^2\left(\sin\left(\frac{1}{x}\right)+2\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} h\left(\sin\left(\frac{1}{h}\right)+2\right)\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y)=0 & \text{if } y\neq 0;\\ 0 & \text{if } y=0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0)=0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0)=0.$$

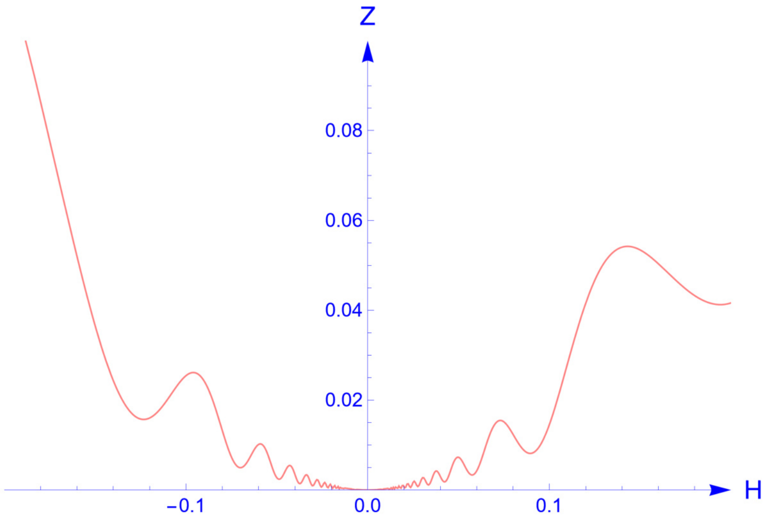

Figure 275. We see here a figure of the graph of the function restricted to the horizontal X-axis through $(0,0)$. We can imagine that this function has a tangent line in 0. We have plotted here the function $f(h,0)$.

31.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu,0+hv)-f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu,0+hv)-f(0,0)}{h} \\ &= \lim_{h\to 0} h\,u^2 \left(\sin\left(\frac{1}{hu+hv}\right)+2\right) \\ &= 0. \end{aligned}$$

So the directional derivatives do always exist.

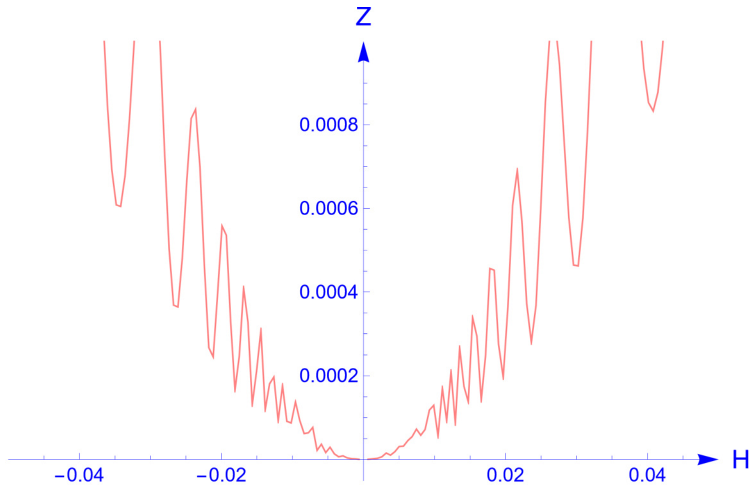

Figure 276. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$. One can imagine the existence of a tangent line in 0.

31.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Looking for a function definition of the partial derivatives.

We have to be able to define the partial derivatives in at least one neighbourhood around $(0,0)$. We have no problems with points that are in the interior of the definition domains of the classical functions. We have there the classical calculation rules for defining those functions and the partial derivatives always exist there and are even continuous.

But in our case, we have to investigate the points that are in exceptional subsets in the definition of the function.

In this case these are the points $(a, -a)$.

Let us look at a point $(a, -a)$ with $a \neq 0$. We are going to investigate the function in $(a, -a)$ in the X-direction. This function is defined by

$$f(a+h, -a) = \begin{cases} (a+h)^2 \left(\sin\left(\frac{1}{h}\right) + 2\right) & \text{if } h \neq 0, \\ 0 & \text{if } h = 0. \end{cases}$$

We see that this function is not continuous in $h = 0$, so the partial derivative to x does not exist if $a \neq 0$. Whatever the position of the green isolated point, see the graph on the following figure, would be on the Y-axis, it could never be a limit because there is an infinite set of accumulation points on the Y-axis. The conclusion is that the partial derivative $\frac{\partial f}{\partial x}(a, -a)$ does not exist for all a with $a \neq 0$. The graph of the function has on the Y-axis no unique accumulation point unless of course $a = 0$ and in that case the function is suddenly dramatically squeezed by the factor h^2. But we covered this exceptional case before because if a is 0, then this calculation is that of a partial derivative.

We conclude that there is no alternative proof possible following these lines explained at the start. There is no neighbourhood of $(0, 0)$ having all partial derivatives defined.

We consult a figure for this observation.

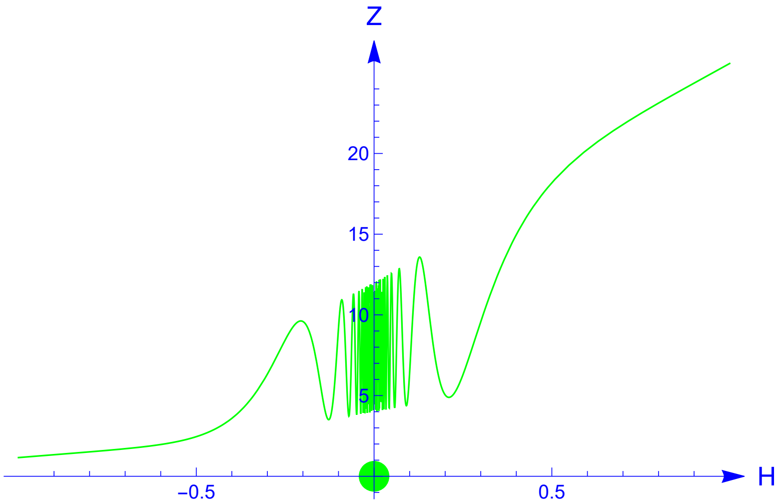

Figure 277. We see here a figure of the graph of the function restricted to the line through $(a, -a)$ with direction $(1, 0)$, this is the X-direction. We have drawn this figure with the value $a = 2$. This is a figure of the function with function definition $f(a + h, -a)$. There is no unique accumulation point possible on the Y-axis.

The partial derivatives do not all exist in any neighbourhood of $(0, 0)$. So the partial derivatives cannot be defined in any neighbourhood of $(0, 0)$. The conclusion is that an alternative proof following the lines described at the start of this section cannot be given. Other alternative proofs can of course exist.

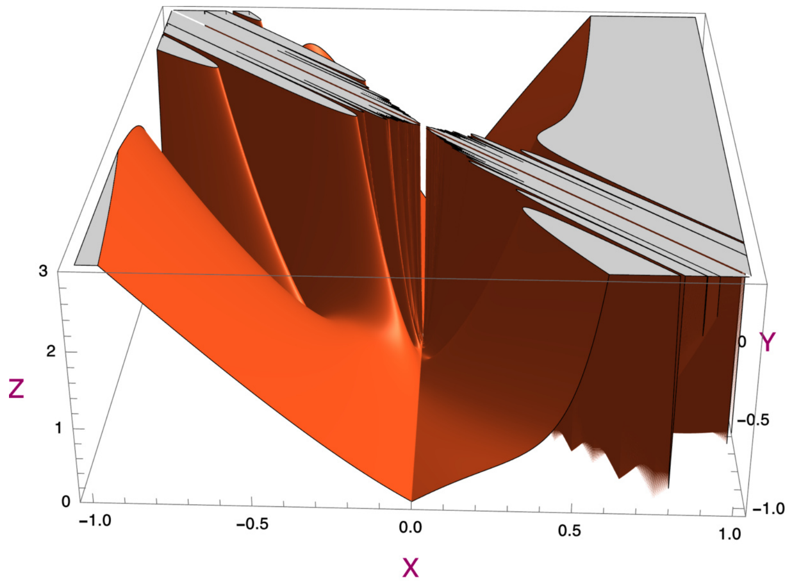

Figure 278. We see here the absolute value of the first partial derivative $\left|\frac{\partial f}{\partial x}\right|$.

31.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

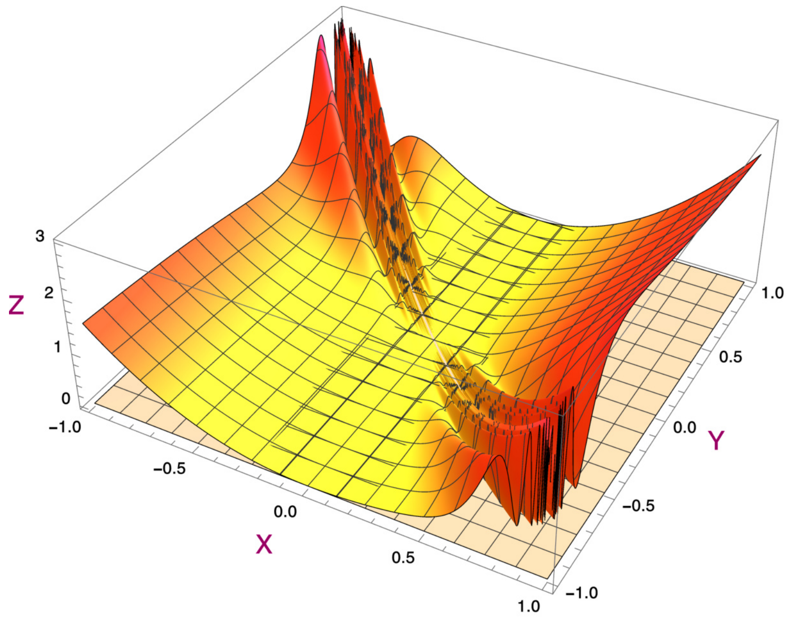

Figure 279. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function quite nicely.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

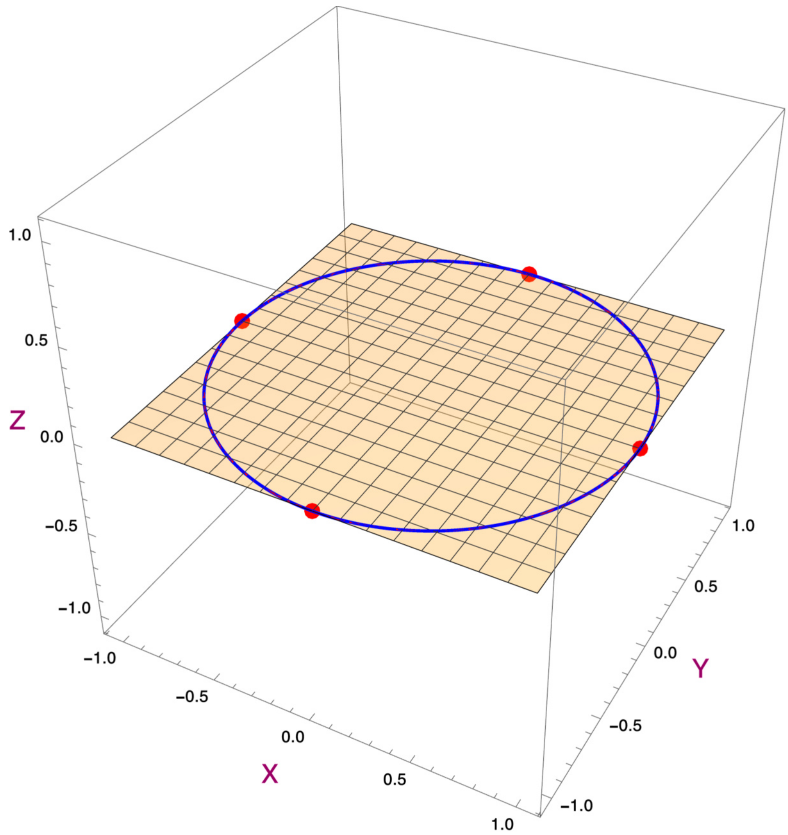

Figure 280. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h, k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h^2\left(\sin\left(\dfrac{1}{h+k}\right)+2\right)}{\sqrt{h^2+k^2}} & \text{if } (h,k) \neq (0,0) \text{ and } h+k \neq 0; \\ 0 & \text{if } (h,k) = (0,0) \text{ or } h+k = 0 \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k)-(0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k)-(0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left| \frac{h^2\left(\sin\left(\dfrac{1}{h+k}\right)+2\right)}{\sqrt{h^2+k^2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{h^2\left(\sin\left(\frac{1}{h+k}\right)+2\right)}{\sqrt{h^2+k^2}}\right| &\leq \frac{h^2\left(\left|\sin\left(\frac{1}{h+k}\right)\right|+2\right)}{\sqrt{h^2+k^2}}\\ &\leq \frac{h^2\,3}{\sqrt{h^2+k^2}}\\ &\leq \frac{3\sqrt{h^2+k^2}^2}{\sqrt{h^2+k^2}}\\ &\leq 3\sqrt{h^2+k^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon/3$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous.

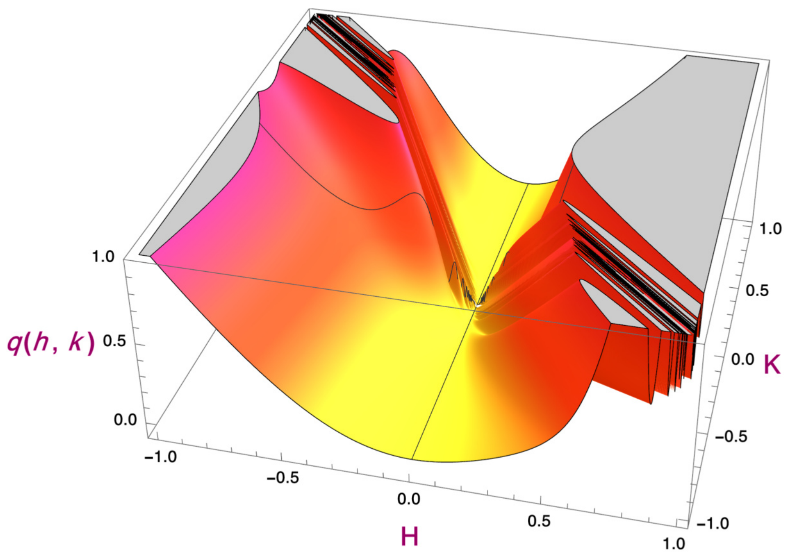

Figure 281. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

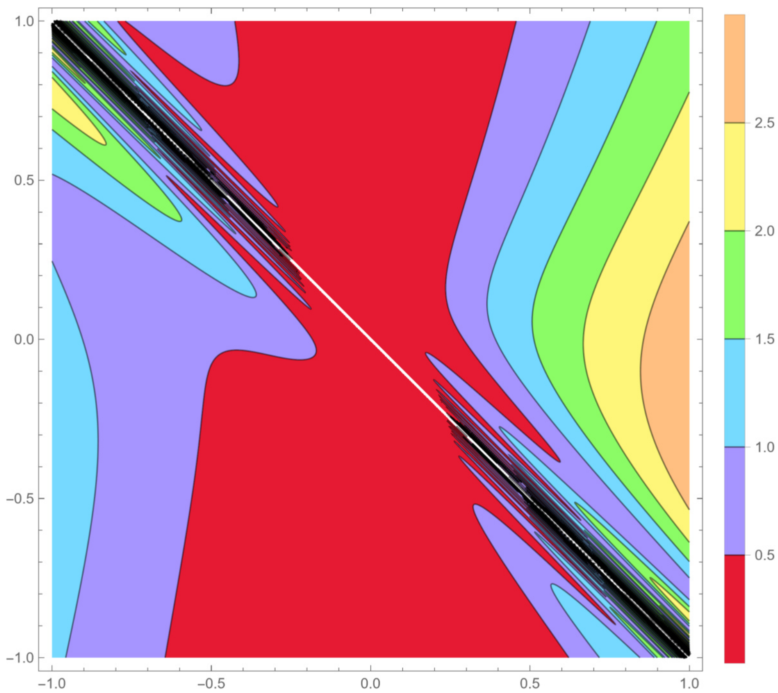

Figure 282. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

31.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\, u + \frac{\partial f}{\partial y}(0,0)\, v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the func-

tion $f(x,y)$ is locally Lipschitz continuous in $(0,0)$. We cite here the criterion that we will use.

A function is differentiable in (a,b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.
2. The directional derivatives have the following form: $D_{(u,v)}(a,b) = \nabla f(a,b)\cdot(u,v) = \frac{\partial f}{\partial x}(a,b)\,u + \frac{\partial f}{\partial y}(a,b)\,v$.
3. The function is locally Lipschitz continuous in (a,b). This means that there exists at least one neighbourhood of (a,b) and a number $K>0$ such that for all (x_1,y_1) and (x_2,y_2) in the neighbourhood, we have $|f(x_1,y_1)-f(x_2,y_2)| < K\,\|(x_1,y_1)-(x_2,y_2)\|$. Remark that the same K must be valid for all (x_1,y_1) and (x_2,y_2).

We want to show that the partial derivative to x is unbounded. If this is the case, then the function cannot be Lipschitz continuous because the slopes have to be bounded if a function is Lipschitz continuous.

We calculate the partial derivative to x in a neighbourhood of $(0,0)$ in points not on the line $x+y=0$. The derivative is there

$$\frac{\partial f}{\partial x}(x,y) = -\frac{x^2\cos\left(\frac{1}{x+y}\right)}{(x+y)^2} + 2\,x\,\sin\left(\frac{1}{x+y}\right) + 4\,x.$$

The second and third terms are continuous and bounded. So we take a look at the first term and call this $h(x,y)$.

$$h(x,y) = -\frac{x^2\cos\left(\frac{1}{x+y}\right)}{(x+y)^2}.$$

We think that this function h behaves very badly on curves tangent to $x+y=0$ in $(0,0)$. We take the curve $y=-x+x^2$. We calculate now

$$h(x,-x+x^2) = -\frac{\cos\left(\frac{1}{x^2}\right)}{x^2}.$$

We see that this curve is not bounded. We can be a little bit more explicit and define the sequence $x_n = \frac{1}{\sqrt{2\pi}\sqrt{n}}$, $n\in\mathbf{N}_0$, converging to 0. Then $h(x_n,-x_n+x_n^2) = -2\,n\,\pi$ proving the unboundedness.

So we cannot apply this alternative criterion for differentiability.

31.7 Continuity of the partial derivatives

We have seen in section 4 that the partial derivatives are not defined in any neighbourhood of $(0,0)$. So we cannot use this particular alternative criterion for continuity.

31.8 Overview

$$f(x,y) = \begin{cases} x^2 \left(\sin\left(\frac{1}{x+y} \right) + 2 \right) & \text{if } x+y \neq 0, \\ 0 & \text{if } x+y = 0. \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

Exercise 32.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} x \sin\left(\frac{1}{x}\right) + y \sin\left(\frac{1}{y}\right) & \text{if } x\,y \neq 0, \\ 0 & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

32.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| x \sin\left(\frac{1}{x}\right) + y \sin\left(\frac{1}{y}\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left| x \sin\left(\frac{1}{x}\right) + y \sin\left(\frac{1}{y}\right) \right| &\leq |x| \left| \sin\left(\frac{1}{x}\right) \right| + |y| \left| \sin\left(\frac{1}{y}\right) \right| \\ &\leq |x| + |y| \\ &\leq \sqrt{x^2+y^2} + \sqrt{x^2+y^2} \\ &\leq 2\sqrt{x^2+y^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon/2$. We can find a δ, so we conclude that the function is continuous.

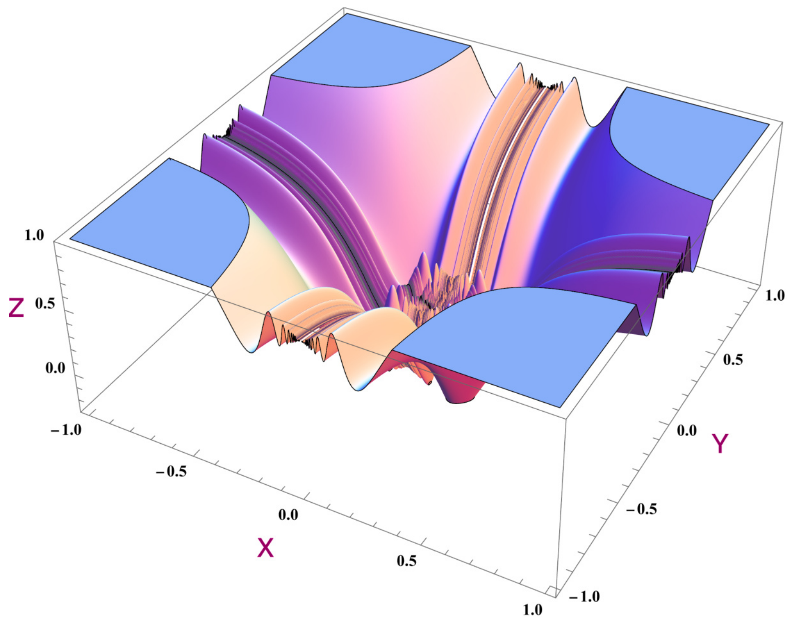

Figure 283. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

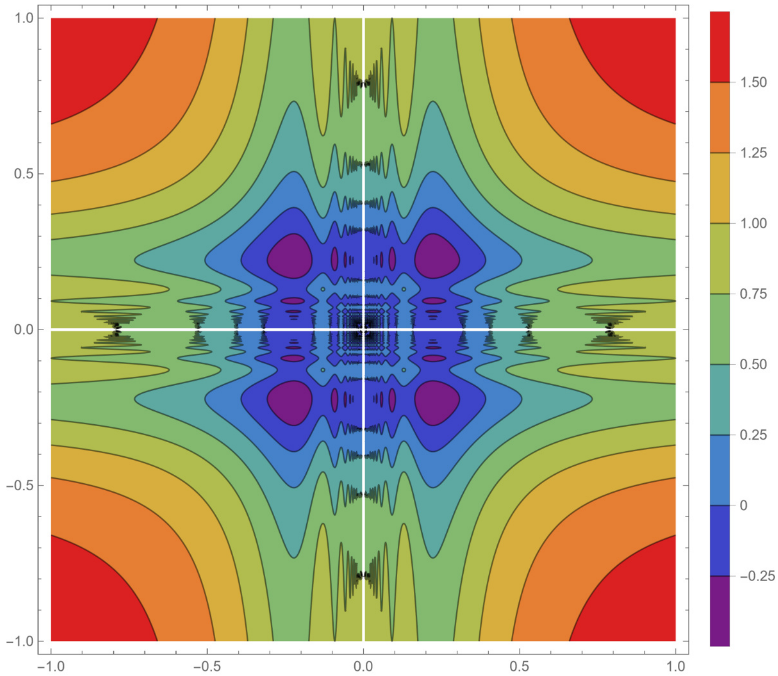

Figure 284. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

32.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).
We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

32.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}$$
$$= \lim_{h\to 0} u \sin\left(\frac{1}{hu}\right) + v \sin\left(\frac{1}{hv}\right).$$

This directional derivative does not exist.

So the directional derivatives do not always exist.

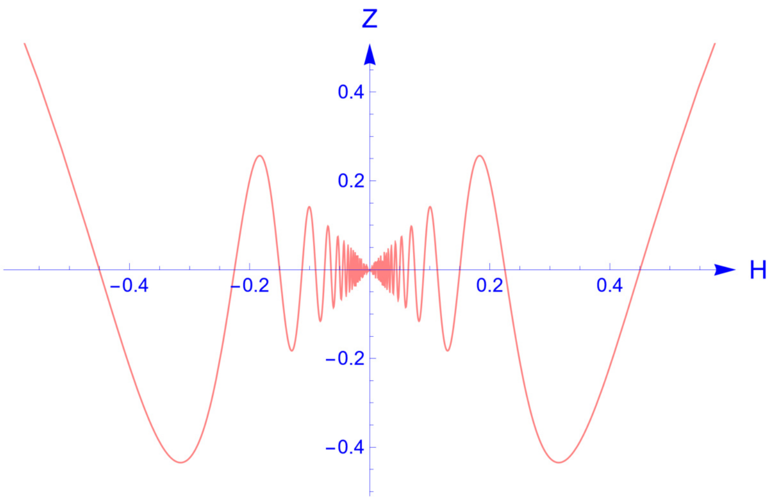

Figure 285. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$. We see that this function has continuous behaviour in 0. But it is not differentiable in 0. This is a classical example of non differentiability in one variable.

32.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we

have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Looking for a definition of the partial derivatives.

We have to be able to define the partial derivatives in at least one neighbourhood around $(0,0)$. We have no problems with points that are in the interior of the definition domains of the classical functions. We have there the classical calculation rules for defining those functions and the partial derivatives always exist there and are even continuous.

But in our case, we have to investigate the points that are in exceptional subsets in the definition of the function.

In this case these are the points $(a,0)$ and $(0,b)$.

Let us look at a point $(a,0)$ with $a \neq 0$. We are going to investigate the function in $(a,0)$ in the Y-direction. This function is defined by

$$f(a,h) = \begin{cases} a\sin\left(\frac{1}{a}\right) + h\sin\left(\frac{1}{h}\right) & \text{if } h \neq 0, \\ 0 & \text{if } h = 0. \end{cases}$$

We see that this function is continuous but not differentiable. The conclusion is that the partial derivative $\frac{\partial f}{\partial y}(a,0)$ does not exist for all a with $a \neq 0$.

We consult a figure for this observation.

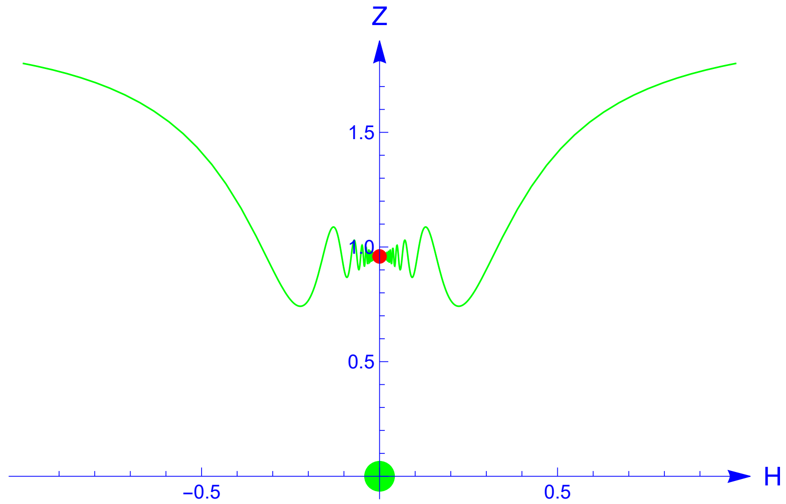

Figure 286. We see here a figure of the graph of the function restricted to the line through $(a, 0)$ with direction $(0, 1)$, this is the Y-direction. This function is not continuous. Thus it is not differentiable. The figure is drawn with $a = 2$.

The partial derivatives do not all exist in any neighbourhood of $(0, 0)$. So the partial derivatives cannot be defined in any neighbourhood of $(0, 0)$. The conclusion is that an alternative proof following the lines described at the start of this section cannot be given. Other alternative proofs can of course exist.

We note also that a function can be perfectly differentiable, even as the partial derivatives are not defined in any *neighbourhood* of $(0, 0)$. It is though necessary that the partial derivative in the *point* $(0, 0)$ itself does exist in order that the function is differentiable in $(0, 0)$. From this viewpoint, the existence of the partial derivatives in a neighbourhood of $(0, 0)$ is pure luxury.

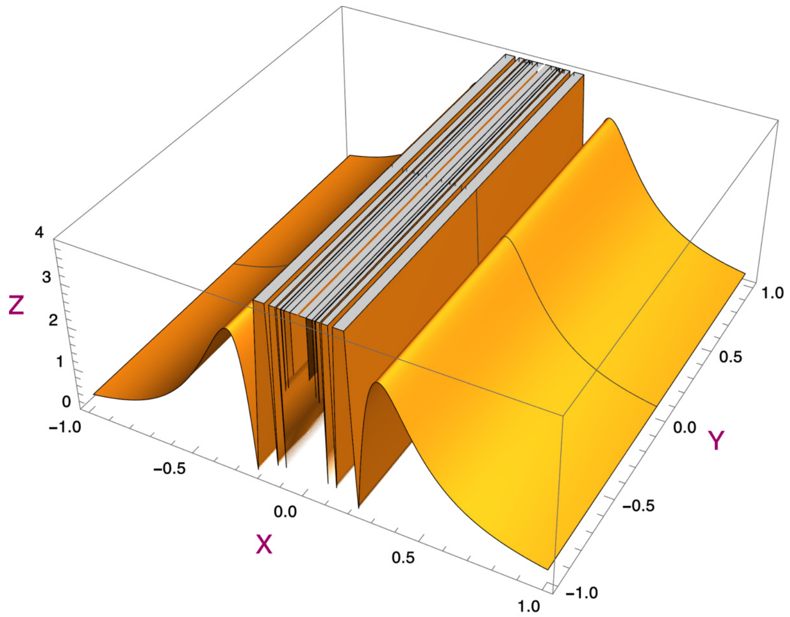

Figure 287. We see here the absolute value of the first partial derivative $\left|\dfrac{\partial f}{\partial x}\right|$. We can observe the unboundedness from this picture.

32.5 Differentiability

There are directional derivatives that do not exist. Thus the function is not differentiable. So it is futile to continue.

32.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

32.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

32.8 Overview

$$f(x,y)=\begin{cases} x\sin\left(\dfrac{1}{x}\right)+y\sin\left(\dfrac{1}{y}\right) & \text{if } xy\neq 0,\\ 0 & \text{if } x=0 \text{ or } y=0.\end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 33.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} x^2 \sin^2\left(\frac{y}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

33.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| x^2 \sin^2\left(\frac{y}{x}\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned} \left| x^2 \sin^2\left(\frac{y}{x}\right) \right| &\leq x^2 \left| \sin^2\left(\frac{y}{x}\right) \right| \\ &\leq x^2 \\ &\leq \sqrt{x^2+y^2}^2. \end{aligned}$$

It is sufficient to take $\delta = \sqrt{\epsilon}$. We can find a δ, so we conclude that the function is continuous.

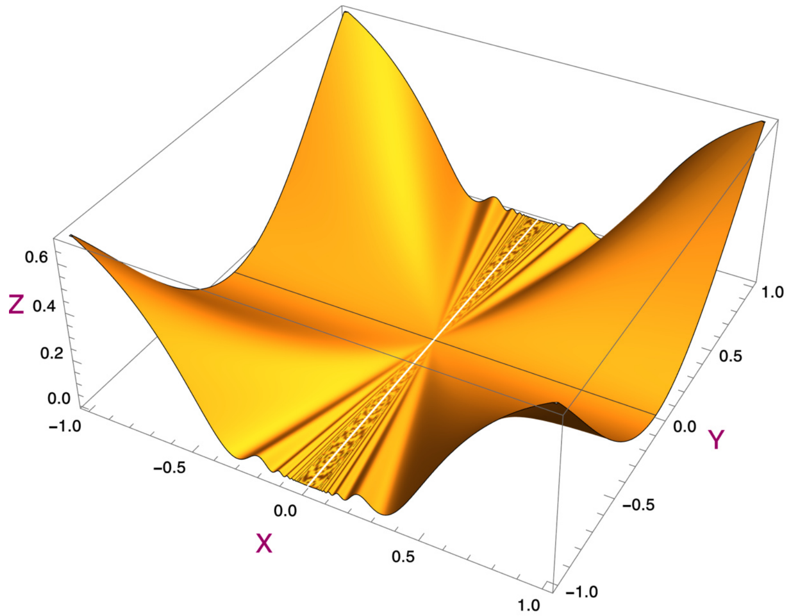

Figure 288. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

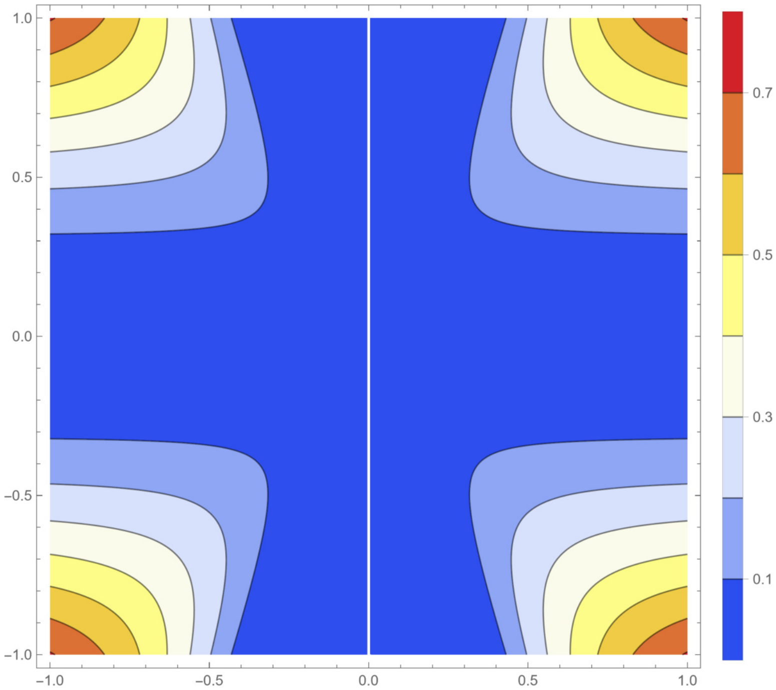

Figure 289. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0, 0)$.

33.2 Partial derivatives

Discussion of the partial derivative to *x* in (0,0).

We observe then that

$$f|_{y=0}(x, y) = \begin{cases} f(x, 0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y)=0 & \text{if } y\neq 0;\\ 0 & \text{if } y=0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0)=0 \quad\text{and}\quad \frac{\partial f}{\partial y}(0,0)=0.$$

33.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} h\,u^2 \sin^2\left(\frac{v}{u}\right) \\ &= 0. \end{aligned}$$

Remark that $u \neq 0$. But we covered that case already while discussing the partial derivatives.

So the directional derivatives do always exist.

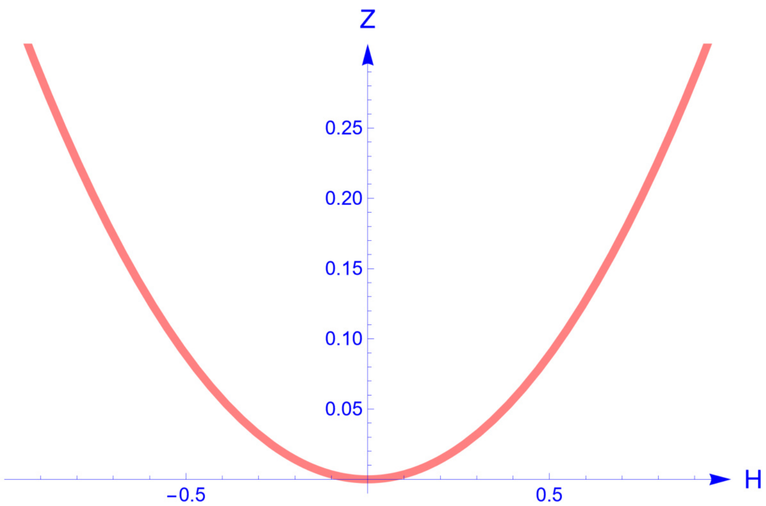

Figure 290. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

33.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Looking for a definition of the partial derivatives. We have to be able to define the partial derivatives in at least one neighbourhood around $(0,0)$. We have no problems with points that are in the interior of the definition domains of the classical functions. We have there the classical calculation rules for defining those functions and the partial derivatives always exist there and are even continuous.

But in our case, we have to investigate the points that are in exceptional subsets in the definition of the function.

In this case these are the points $(0,b)$.

Let us look at a point $(0,b)$ with $b \neq 0$. We are going to investigate the function in $(0,b)$ in the X-direction. There is no problem in the Y-direction. The function is there identically zero. This function is defined by

$$f(h,b) = \begin{cases} h^2 \sin^2\left(\frac{b}{h}\right) & \text{if } h \neq 0, \\ 0 & \text{if } h = 0. \end{cases}$$

We see that this function is obviously differentiable.
We have indeed

$$\frac{\partial f}{\partial x}(0,b) = \lim_{h\to 0} \frac{f(h,b) - f(0,b)}{h} = \lim_{h\to 0} h \sin^2\left(\frac{b}{h}\right) = 0.$$

We can use the squeeze method for this calculation. The conclusion is that the partial derivative $\frac{\partial f}{\partial x}(0,b)$ does exist and equals zero for all b.

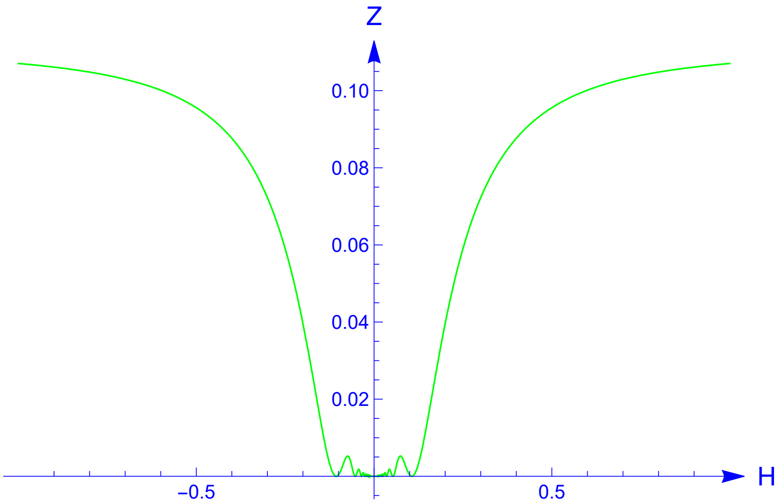

Figure 291. We see here a figure of the graph of the function restricted to the line through $(0, b)$ with direction $(1, 0)$, this is the X-direction. We have drawn the function $f(h, b)$ We see that this function is continuous and differentiable in $h = 0$. We have drawn this figure with the value $b = 1/3$. This is a figure of the function with function definition $f(h, b)$.

Remember that we already calculated the partial derivatives in $(0, 0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x, y) = \begin{cases} 2\sin\left(\frac{y}{x}\right)\left(x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x, y) = \begin{cases} x\sin\left(\frac{2y}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

$$\begin{aligned}
\left|\frac{\partial f}{\partial x}(x,y)\right| &\leq \left|2\,\sin\left(\frac{y}{x}\right)\left(x\,\sin\left(\frac{y}{x}\right)-y\cos\left(\frac{y}{x}\right)\right)\right| \\
&\leq \left|2\,\sin\left(\frac{y}{x}\right)\right|\left|\left(x\,\sin\left(\frac{y}{x}\right)-y\cos\left(\frac{y}{x}\right)\right)\right| \\
&\leq 2\left(|x|\left|\sin\left(\frac{y}{x}\right)\right|+|y|\left|\cos\left(\frac{y}{x}\right)\right|\right) \\
&\leq 2\left(|x|+|y|\right) \\
&\leq 2\left(\sqrt{x^2+y^2}+\sqrt{x^2+y^2}\right) \\
&\leq 4\sqrt{x^2+y^2} \\
&\leq 4.
\end{aligned}$$

We have for the last inequality chosen the neighbourhood defined by $\sqrt{x^2+y^2} < 1$.

Let us try to prove that $\left|\frac{\partial f}{\partial y}\right|$ is bounded.

$$\begin{aligned}
\left|\frac{\partial f}{\partial y}(x,y)\right| &\leq \left|x\,\sin\left(\frac{2y}{x}\right)\right| \\
&\leq |x| \\
&\leq \sqrt{x^2+y^2} \\
&\leq 1.
\end{aligned}$$

We have for the last inequality chosen the neighbourhood defined by $\sqrt{x^2+y^2} < 1$.

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

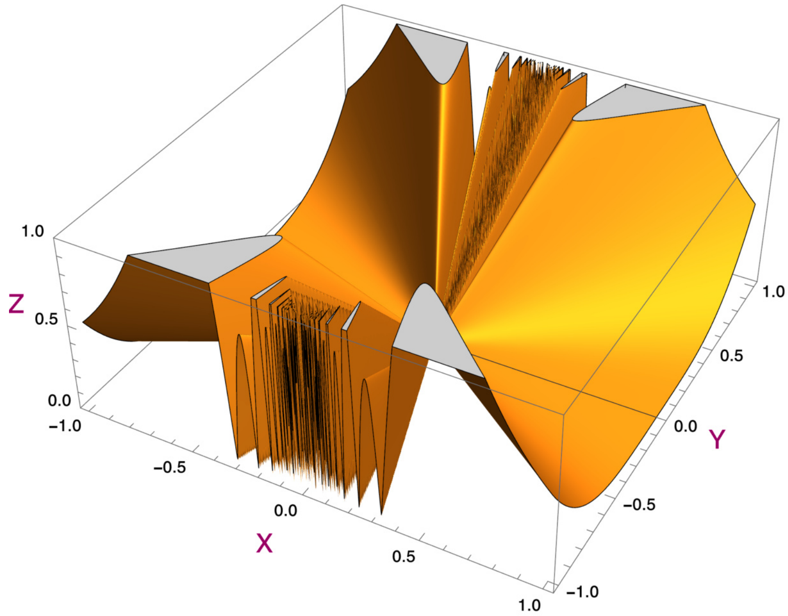

Figure 292. We see here the absolute value of the first partial derivative $\left|\dfrac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

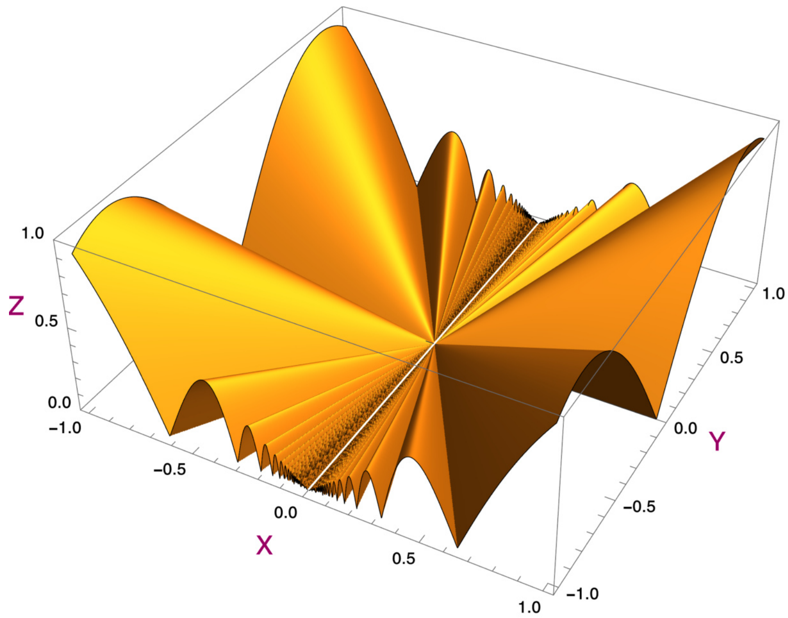

Figure 293. We see here the absolute value of the second partial derivative $\left|\dfrac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture.

33.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

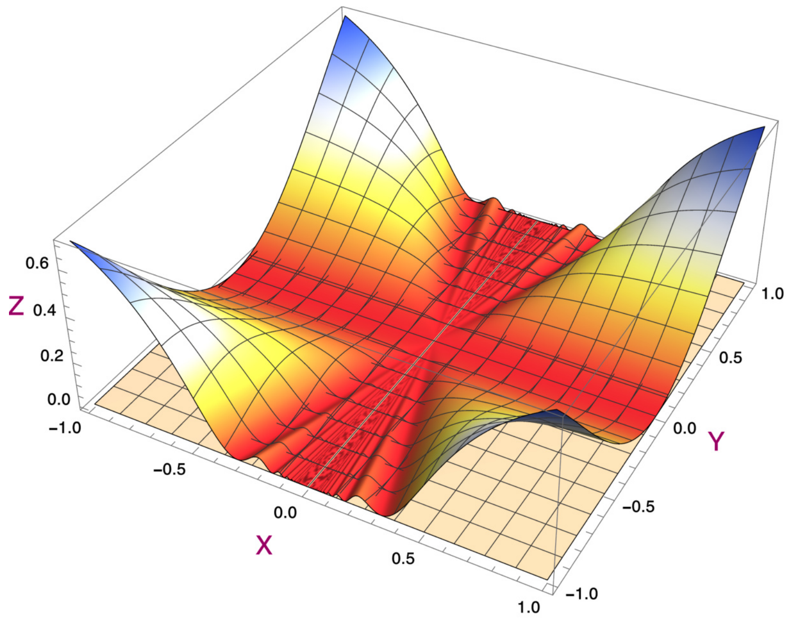

Figure 294. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\, x + \frac{\partial f}{\partial y}(0,0)\, y$, which is graphically the candidate tangent plane and the function $f(x, y)$. We see here that the candidate tangent plane fits the function very nicely.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we

know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

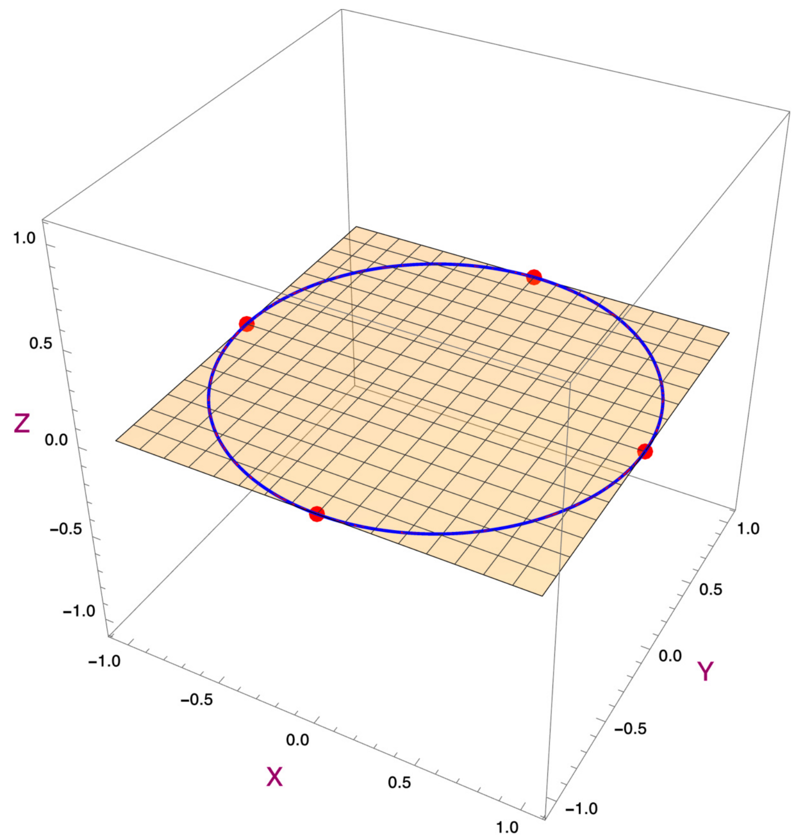

Figure 295. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h^2\,\sin^2\left(\dfrac{k}{h}\right)}{\sqrt{h^2+k^2}} & \text{if } (h,k) \neq (0,0) \text{ and } h \neq 0; \\ 0 & \text{if } (h,k) = (0,0) \text{ or } h = 0 \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left| \frac{h^2\,\sin^2\left(\frac{k}{h}\right)}{\sqrt{h^2+k^2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{h^2\sin^2\left(\frac{k}{h}\right)}{\sqrt{h^2+k^2}}\right| &\leq \frac{h^2}{\sqrt{h^2+k^2}}\\ &\leq \frac{\sqrt{h^2+k^2}^2}{\sqrt{h^2+k^2}}\\ &\leq \sqrt{h^2+k^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous.

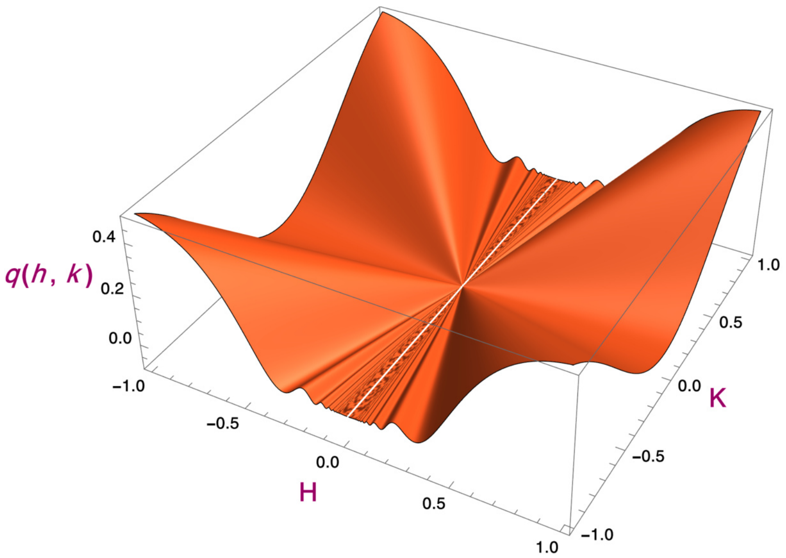

Figure 296. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

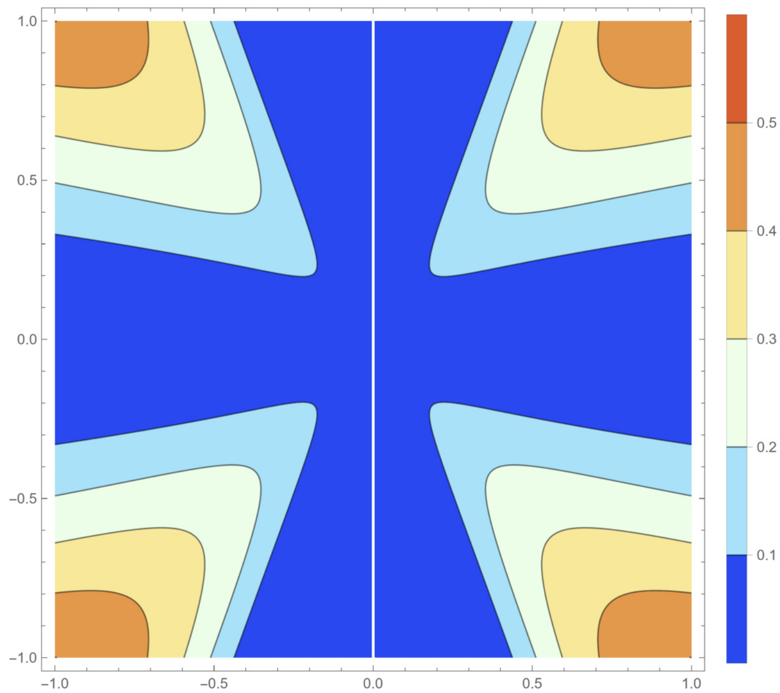

Figure 297. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

33.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\, u + \frac{\partial f}{\partial y}(0,0)\, v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the func-

tion $f(x,y)$ is locally Lipschitz continuous in $(0,0)$. We cite here the criterion that we will use.

A function is differentiable in (a,b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.
2. The directional derivatives have the following form: $D_{(u,v)}(a,b) = \nabla f(a,b)\cdot(u,v) = \frac{\partial f}{\partial x}(a,b)\,u + \frac{\partial f}{\partial y}(a,b)\,v$.
3. The function is locally Lipschitz continuous in (a,b). This means that there exists at least one neighbourhood of (a,b) and a number $K > 0$ such that for all (x_1,y_1) and (x_2,y_2) in the neighbourhood, we have $|f(x_1,y_1) - f(x_2,y_2)| < K\,||(x_1,y_1) - (x_2,y_2)||$. Remark that the same K must be valid for all (x_1,y_1) and (x_2,y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be "locally flat" and thus "locally linear" which is partly expressed by this Lipschitz continuity condition.

Let us try to prove the Lipschitz condition. We are going to use a well known method.

$$\begin{aligned}
&|f(x_1,y_1) - f(x_2,y_2)| \\
&\qquad = |f(x_1,y_1) - f(x_1,y_2) + f(x_1,y_2) - f(x_2,y_2)| \\
&\qquad \le |f(x_1,y_1) - f(x_1,y_2)| + |f(x_1,y_2) - f(x_2,y_2)|.
\end{aligned}$$

We focus now on the first term $|f(x_1,y_1) - f(x_1,y_2)|$. We fix now x_1 and look upon $f(x_1,y)$ as a function in one variable y. So it springs to mind that we can estimate this by using Lagrange's intermediate value theorem. So we have

$$|f(x_1,y_1) - f(x_1,y_2)| = \left|\frac{\partial f}{\partial y}(x_1,\xi)\right|\;|y_1 - y_2|$$

where ξ is a number lying in the open interval (y_1, y_2) if e.g. $y_1 \leq y_2$. Remark that we already have proven that the partial derivatives are bounded in a neighbourhood, we can take that bound found for $\frac{\partial f}{\partial y}$, say M_2 and that neighbourhood in this case also. So $\left|\frac{\partial f}{\partial y}(x_1, \xi)\right| \leq M_2$. We work in a completely analogous way for the second term $|f(x_1, y_2) - f(x_2, y_2)|$. We have in that case $\left|\frac{\partial f}{\partial x}(\xi, y_2)\right| \leq M_1$ where ξ is now a number in the open interval (x_1, x_2) if e.g. $x_1 \leq x_2$.

We take up our inequality again

$$
\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)| \\
&\qquad \leq |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\qquad \leq |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)| \\
&\qquad \leq M_2\,|(y_1 - y_2)| + M_1\,|(x_1 - x_2)| \\
&\qquad \leq M_2\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + M_1\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&\qquad \leq (M_1 + M_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.
\end{aligned}
$$

So this function is locally Lipschitz continuous. We have an alternative proof for the differentiability.

33.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

We know that the partial derivative to x exists and is equal to

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} 2\sin\left(\frac{y}{x}\right)\left(x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We want to see if it is continuous or not.

We will make a distinction of three cases. We consider the function **first** for $x \neq 0$. **Then** we consider it in the point $(0,0)$. The **third** case is considering the function in $(0,b)$ with $b \neq 0$.

If $x \neq 0$, then we have a function composed of classical functions which are known to be infinitely differentiable. So in this first case there is no problem.

Now we are secondly going to consider the point $(0,0)$.
Discussion of the continuity of the first partial derivative in $(0,0)$.

We are investigating now the continuity in the point $(0,0)$. We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove the inequality $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if the inequality $||(x,y)-(0,0)|| < \delta$ holds, then it follows that $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$
\begin{aligned}
\left|\frac{\partial f}{\partial x}(x,y)\right| &\leq \left|2\,\sin\left(\frac{y}{x}\right)\left(x\,\sin\left(\frac{y}{x}\right)-y\cos\left(\frac{y}{x}\right)\right)\right| \\
&\leq \left|2\,\sin\left(\frac{y}{x}\right)\right|\,\left|\left(x\,\sin\left(\frac{y}{x}\right)-y\cos\left(\frac{y}{x}\right)\right)\right| \\
&\leq 2\left(|x|\,\left|\sin\left(\frac{y}{x}\right)\right|+|y|\,\left|\cos\left(\frac{y}{x}\right)\right)\right| \\
&\leq 2\,(|x|+|y|) \\
&\leq 2\left(\sqrt{x^2+y^2}+\sqrt{x^2+y^2}\right) \\
&\leq 4\sqrt{x^2+y^2}.
\end{aligned}
$$

It is sufficient to take $\delta = \epsilon/4$. We can find a δ, so we conclude that the function is continuous in $(0,0)$.

We start considering the third case. We are left with proving the continuity of $\frac{\partial f}{\partial x}$ in a neighbourhood of $(0,b)$ with $b \neq 0$. To make the calculations easier, let us first do the translation transformation from $(0,b)$ to $(0,0)$. We transform by the translation $x = u$ and $y = b+v$ and investigate the function in $(u,v) = (0,0)$.

We have then the function

$$
g(u,v) = \begin{cases} 2\,(b+v)\,\sin\left(\frac{b+v}{u}\right)\,\cos\left(\frac{b+v}{u}\right) - 2\,u\,\sin^2\left(\frac{b+v}{u}\right) & \text{if } u \neq 0, \\ 0 & \text{if } u = 0. \end{cases}
$$

The second term in the main definition is certainly continuous. We drop that term and investigate the first term. By rewriting the first term by expanding $b+v$. we have

$$2\,(b+v)\,\sin\left(\frac{b+v}{u}\right)\,\cos\left(\frac{b+v}{u}\right)$$
$$= 2\,b\,\sin\left(\frac{b+v}{u}\right)\,\cos\left(\frac{b+v}{u}\right) + 2\,v\,\sin\left(\frac{b+v}{u}\right)\cos\left(\frac{b+v}{u}\right).$$

The second term in the last equation is certainly continuous, so we are left with investigating the first term

$$2\,b\,\sin\left(\frac{b+v}{u}\right)\cos\left(\frac{b+v}{u}\right).$$

We use a trigonometric identity

$$2\,b\,\sin\left(\frac{b+v}{u}\right)\cos\left(\frac{b+v}{u}\right) = b\,\sin\left(\frac{2(b+v)}{u}\right).$$

This term is certainly discontinuous. It is a classical example of a discontinuous function. We can be more explicit. The sequence $\left(u_n = \frac{b\,(n+1)}{\pi\,n^2}, v_n = \frac{b}{n}\right)$, $n \in \mathbf{N}_0$, converges to $(0,0)$ and makes this term converge to 0. But the sequence $\left(u_n = \frac{4\,b\,(n+1)}{\pi\,n\,(4\,n+1)}, v_n = \frac{b}{n}\right)$, $n \in \mathbf{N}_0$, converges to $(0,0)$ and makes this term converge to b. But $b \neq 0$.

We conclude that we cannot use this particular criterion for the differentiability of the function.

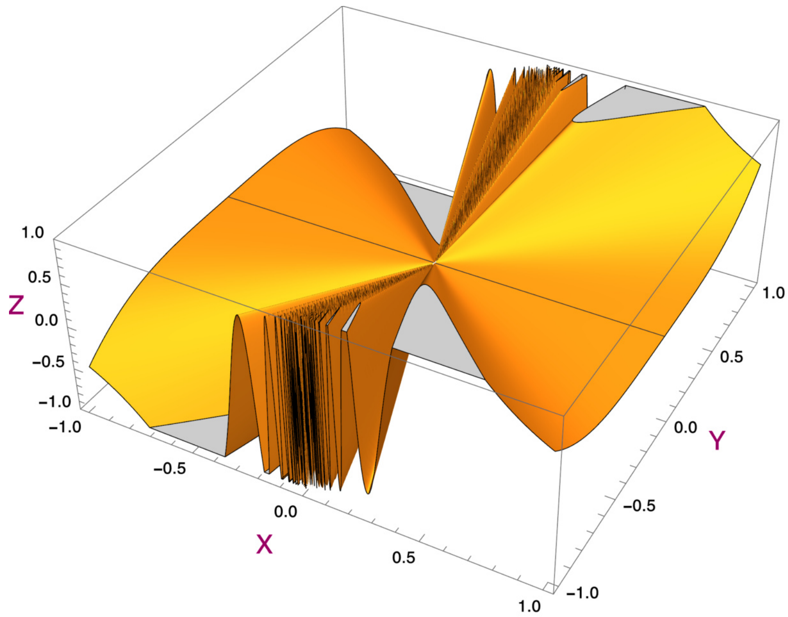

Figure 298. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial x}(x, y)$. This looks like a continuous function in $(0, 0)$.

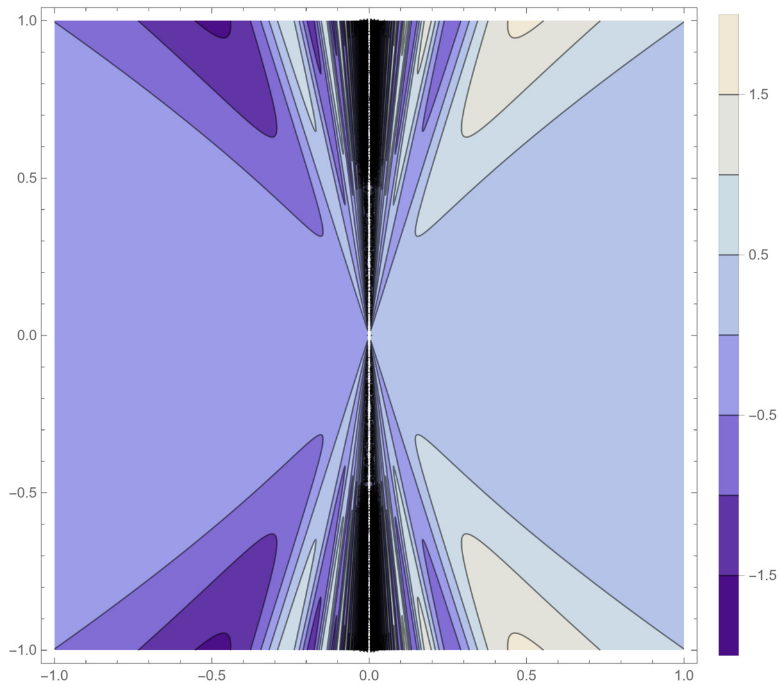

Figure 299. We see here a figure of the contour plot of the $\frac{\partial f}{\partial x}(x, y)$. Only level curves of level around 0 come close to $(0, 0)$.

33.8 Overview

$$f(x, y) = \begin{cases} x^2 \sin^2\left(\frac{y}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

33.9 One step further

We want to know if this function is further uneventful from the point of view of differentiability. Let us take a look at the second order partial derivative twice to y.

$$\frac{\partial^2 f}{\partial y^2} = 2\cos\left(\frac{2y}{x}\right).$$

Let us take a look of a three dimensional plot of this second order partial derivative of the function.

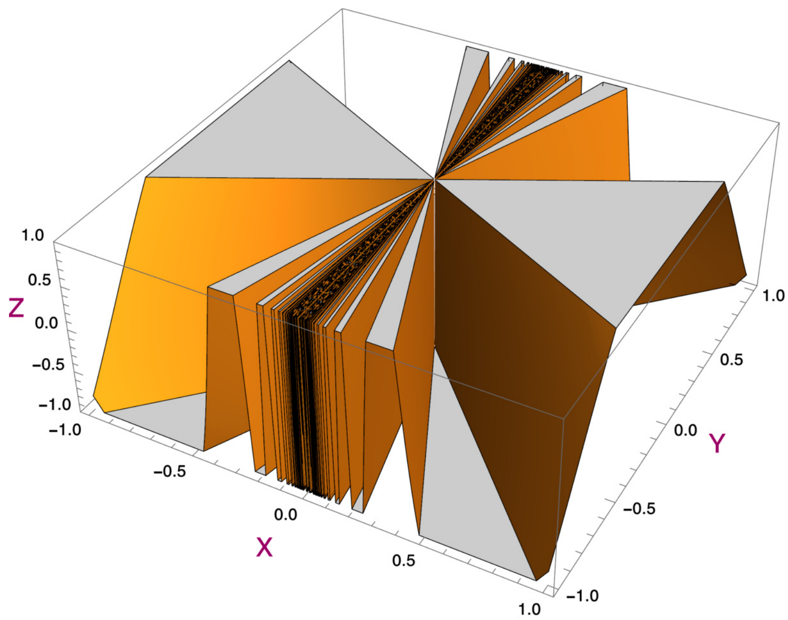

Figure 300. We see here a figure of the second order partial derivative $\frac{\partial^2 f}{\partial y^2}(x, y)$. It seems quite improbable that this second order partial derivative is continuous. We stop however our investigations here and leave this to the initiative of the interested reader.

Exercise 34.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{\sin^2(x+y)}{|x|+|y|} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

34.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{\sin^2(x+y)}{|x|+|y|} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left|\frac{\sin^2(x+y)}{|x|+|y|}\right| &\leq \frac{(x+y)^2}{\sqrt{x^2+y^2}} \\
&\leq \frac{(|x|+|y|)^2}{\sqrt{x^2+y^2}} \\
&\leq \frac{\left(\sqrt{x^2+y^2}+\sqrt{x^2+y^2}\right)^2}{\sqrt{x^2+y^2}} \\
&\leq \frac{\left(2\sqrt{x^2+y^2}\right)^2}{\sqrt{x^2+y^2}} \\
&\leq 4\sqrt{x^2+y^2}.
\end{aligned}$$

For the steps in the inequalities we have used that $\sqrt{x^2+y^2} \leq |x|+|y|$ so that $1/(|x|+|y|) \leq 1/\sqrt{x^2+y^2}$. By multiplying with $\sin^2(x+y)$, we have that $\sin^2(x+y)/(|x|+|y|) \leq \sin^2(x+y)/\sqrt{x^2+y^2}$.

It is sufficient to take $\delta = \epsilon/4$. We can find a δ, so we conclude that the function is continuous.

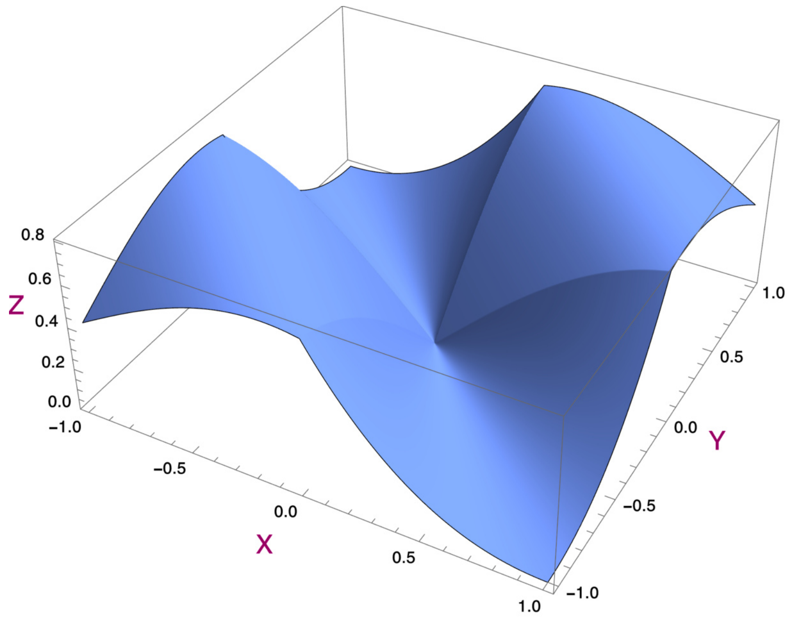

Figure 301. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

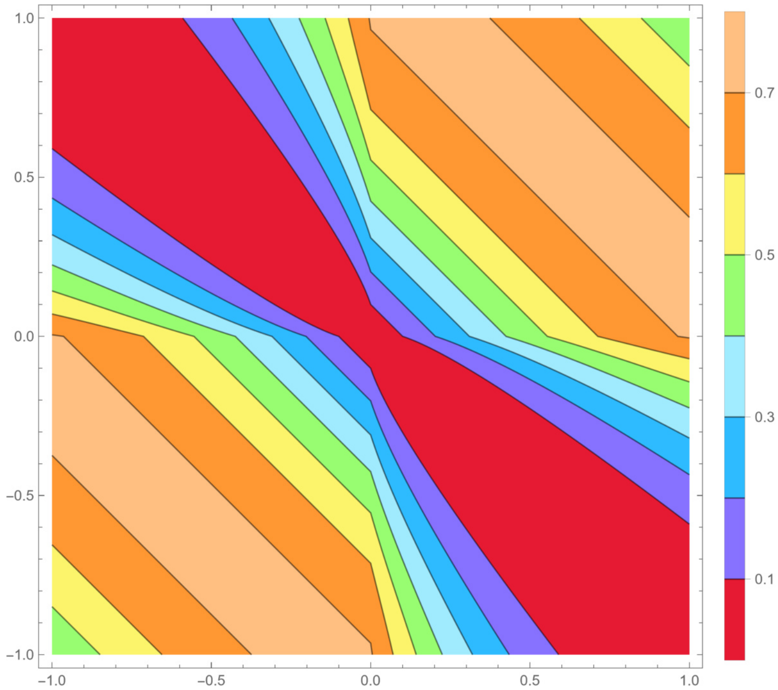

Figure 302. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0, 0)$.

34.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x, y) = \begin{cases} f(x, 0) = \dfrac{\sin^2(x)}{|x|} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$
$$= \lim_{h\to 0} \frac{\sin^2(h)}{h\,|h|}$$
$$= \lim_{h\to 0} \frac{\sin(h)}{|h|}.$$

So the partial derivative to x does not exist.

Because the function is symmetric, we have in a completely similar way that the partial derivative to y does not exist.

We conclude that the partial derivative to x does not exist. and that the partial derivative to y does not exist.

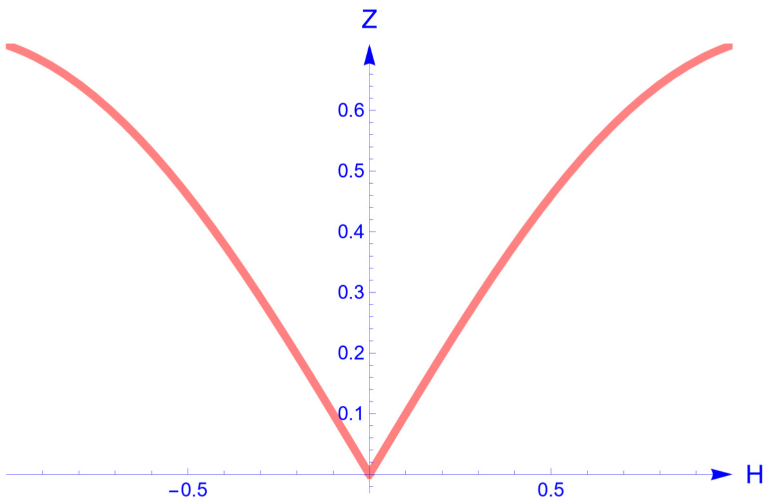

Figure 303. We see here a figure of the graph of the function restricted to the horizontal X-axis through $(0,0)$. We observe the non differentiability of the function. We have plotted here the function $f(h,0)$.

34.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u, v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h \to 0} \frac{f(0 + hu, 0 + hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h \to 0} \frac{f(0 + hu, 0 + hv) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{\sin^2(h(u+v))}{h(|hu| + |hv|)} \\ &= \lim_{h \to 0} \frac{\sin^2(h(u+v))}{h^2(u+v)^2} \frac{h(u+v)^2}{|hu| + |hv|} \\ &= \lim_{h \to 0} \frac{h(u+v)^2}{|hu| + |hv|} \\ &= \lim_{h \to 0} \frac{h(u+v)^2}{|h|(|u| + |v|)}. \end{aligned}$$

This last limit does not exist unless $u + v = 0$.

So the directional derivatives do not always exist.

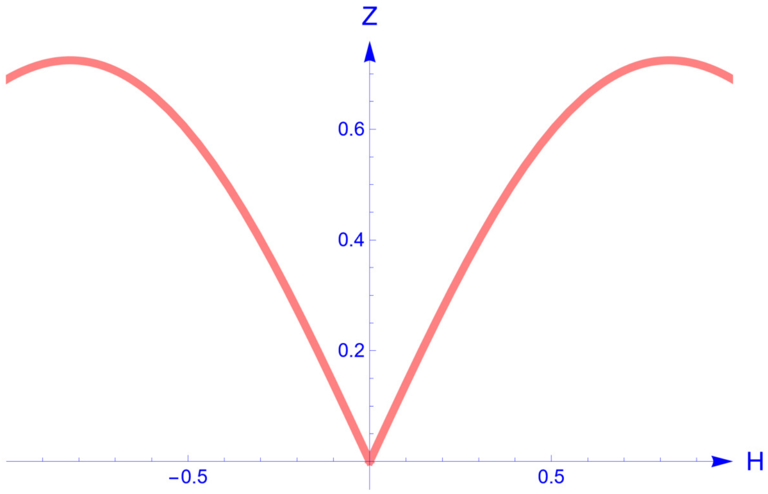

Figure 304. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$. We see that the directional derivative does not exist.

34.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

The partial derivatives do not exist. So this alternative criterion cannot be applied.

34.5 Differentiability

We see that at least one of the directional derivatives does not exist. So the function is not differentiable.

34.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

34.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

34.8 Overview

$$f(x,y) = \begin{cases} \dfrac{\sin^2(x+y)}{|x|+|y|} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	no
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 35.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = |xy|.$$

35.1 Continuity

We can argue that this function is composed of functions that are continuous. So there is nothing more to prove. We will argue with an ϵ-δ approach anyway.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\big|\,|xy| - 0\,\big| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}||x\,y|| &\le |x|\,|y| \\ &\le \sqrt{x^2+y^2}\,\sqrt{x^2+y^2} \\ &\le \sqrt{x^2+y^2}^{\,2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon^{1/2}$. We can find a δ, so we conclude that the function is continuous.

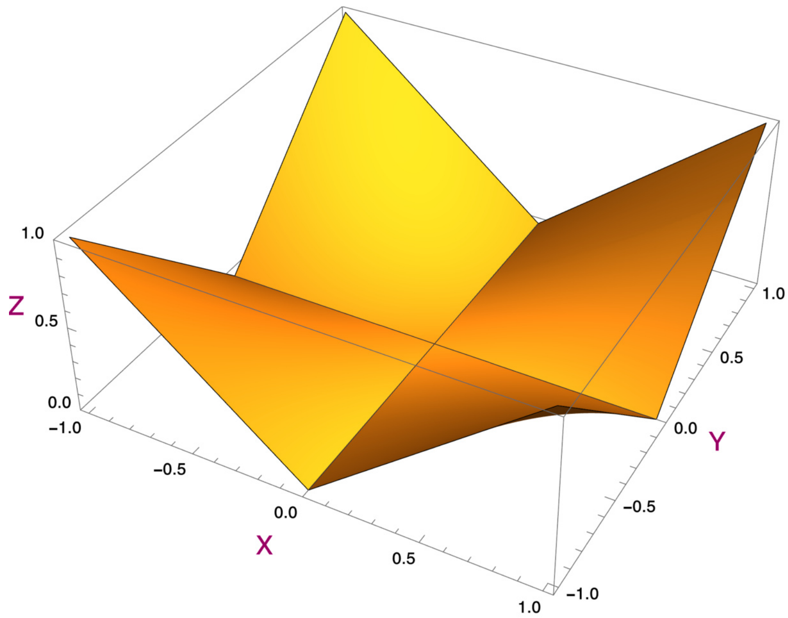

Figure 305. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

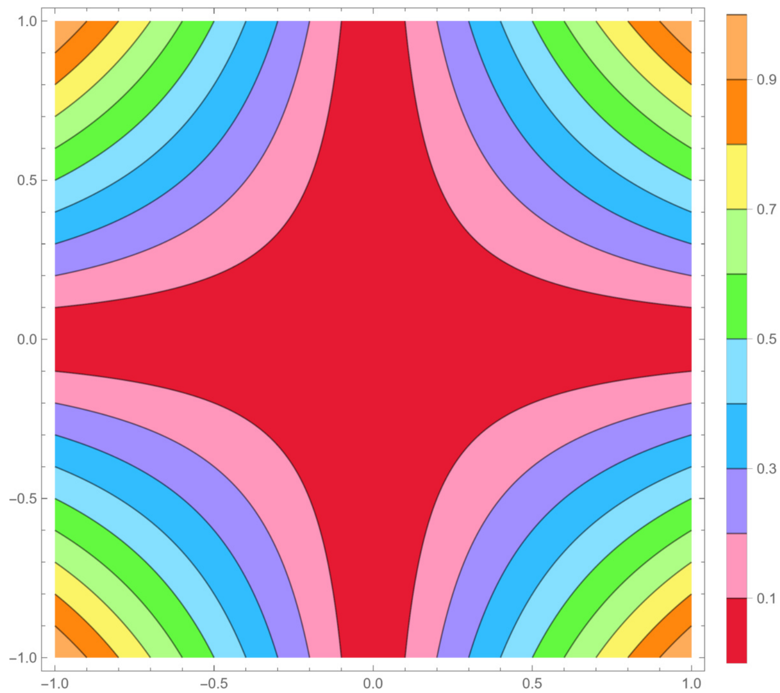

Figure 306. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

35.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f\big|_{y=0}(x,y) = 0.$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

We can do a completely similar calculation for the partial derivative to y.

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

35.3 Directional derivatives

Let (u, v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2 + v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u, v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h \to 0} \frac{f(0 + h\,u, 0 + h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h \to 0} \frac{f(0 + h\,u, 0 + h\,v) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{|h^2\,u\,v|}{h} \\ &= 0. \end{aligned}$$

So the directional derivatives do always exist.

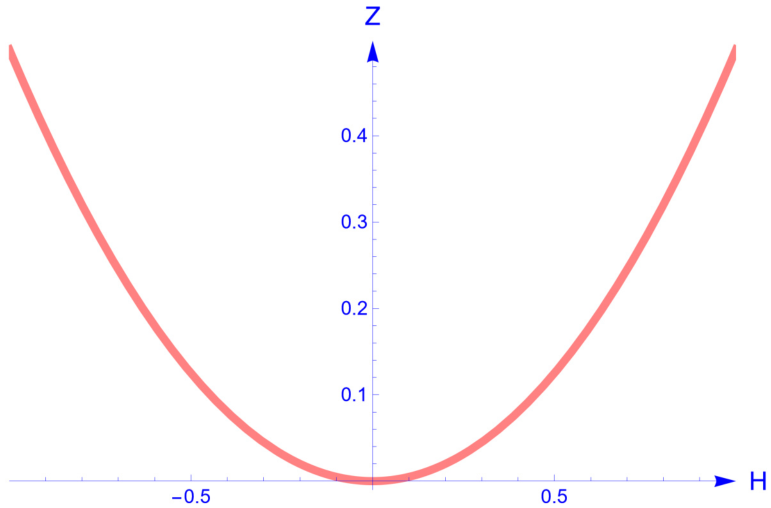

Figure 307. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$.

35.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

We want to investigate the existence of the partial derivatives not in the point $(0,0)$ but on points of the X and Y axes close to $(0,0)$.

Let us take a point $(a,0)$ with $a \neq 0$ and investigate the behaviour of the function in the vertical direction. In order to do that, we will look at the function $f(a,h)$ and check the continuity and differentiability of the function.

We calculate the function

$$f(a,h) = \begin{cases} |a\,h| & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

We will look for the partial derivative and we calculate the limit

$$\lim_{h \to 0} \frac{|a\,h|}{h}.$$

We see that this limit does not exist if $a \neq 0$. So the partial derivative does not exist.

We remark that the partial derivative does not always exist in any neighbourhood of $(0,0)$. So we cannot use this criterion for our alternative proof.

We illustrate in the next picture the behaviour of the function.

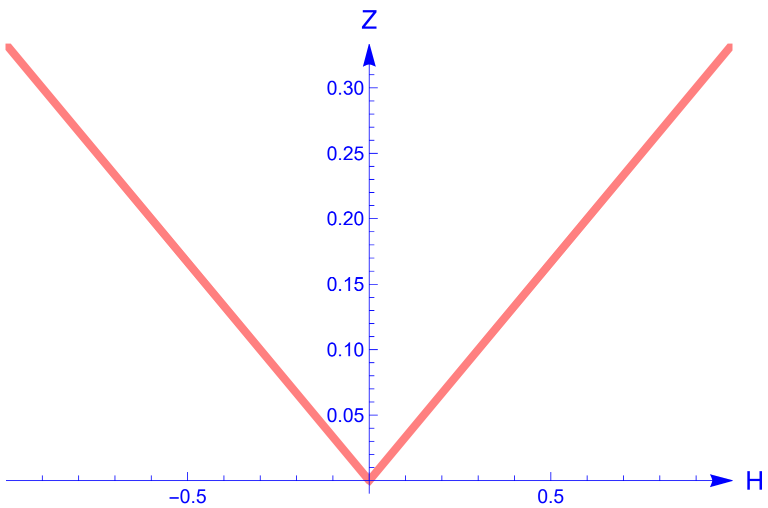

Figure 308. We see here a two dimensional figure of the graph of the function $f(a,h)$. We have drawn the function here for the value $a = 1/2$ which is exemplary for the values of a close to 0. This does not look like a differentiable function. We have plotted here the function $f(a,h)$.

35.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

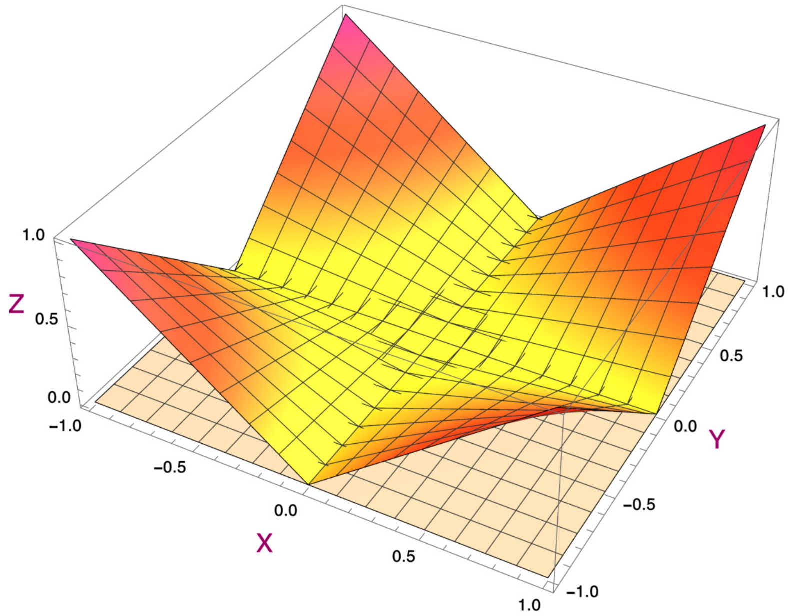

Figure 309. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function not so convincingly. There remain at least doubts.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u,v,D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

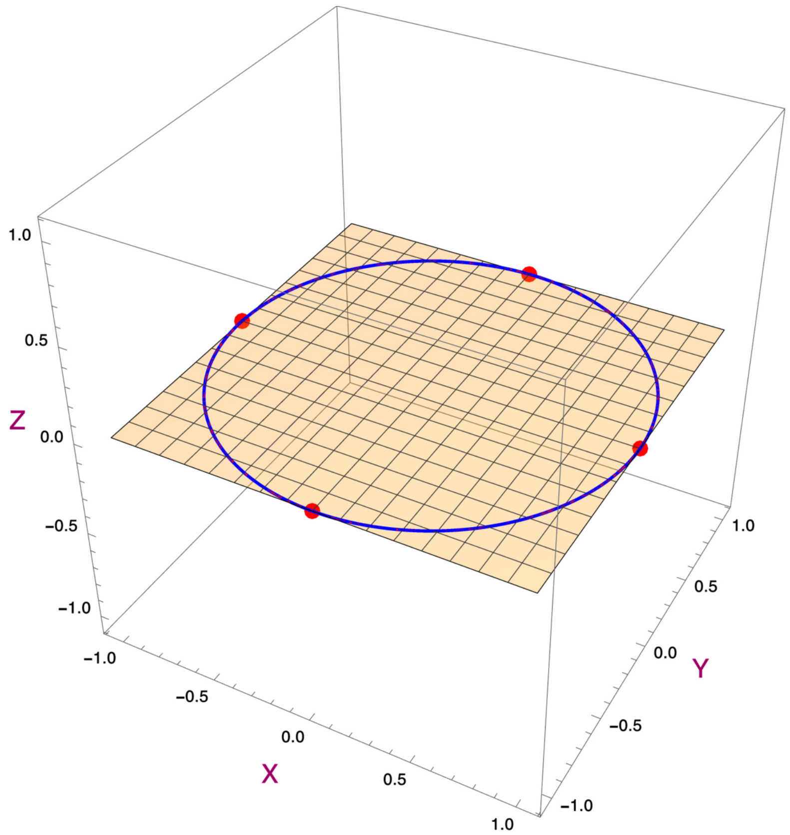

Figure 310. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h, k)$ in $(0, 0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{|h\,k|}{\sqrt{h^2+k^2}} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left| \frac{|h\,k|}{\sqrt{h^2+k^2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{|h\,k|}{\sqrt{h^2+k^2}}\right| &\leq \frac{|h|\,|k|}{\sqrt{h^2+k^2}} \\ &\leq \frac{\sqrt{h^2+k^2}\,\sqrt{h^2+k^2}}{\sqrt{h^2+k^2}} \\ &\leq \sqrt{h^2+k^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous.

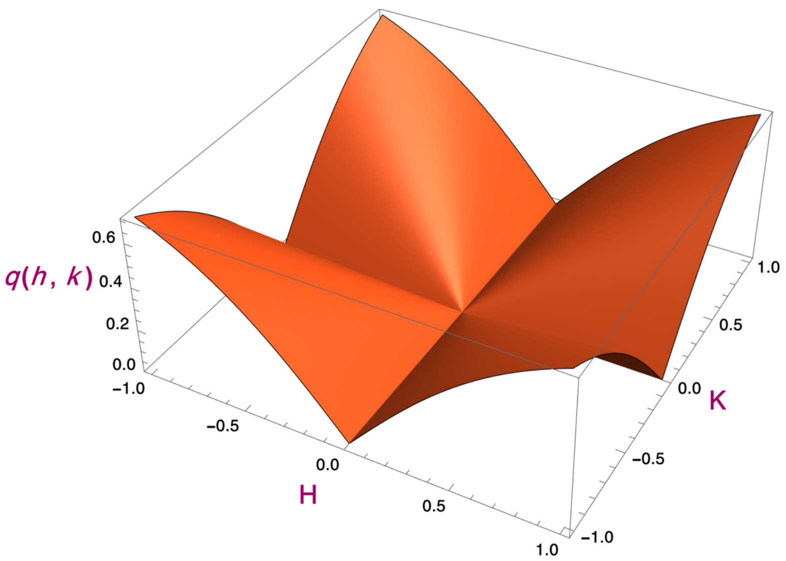

Figure 311. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

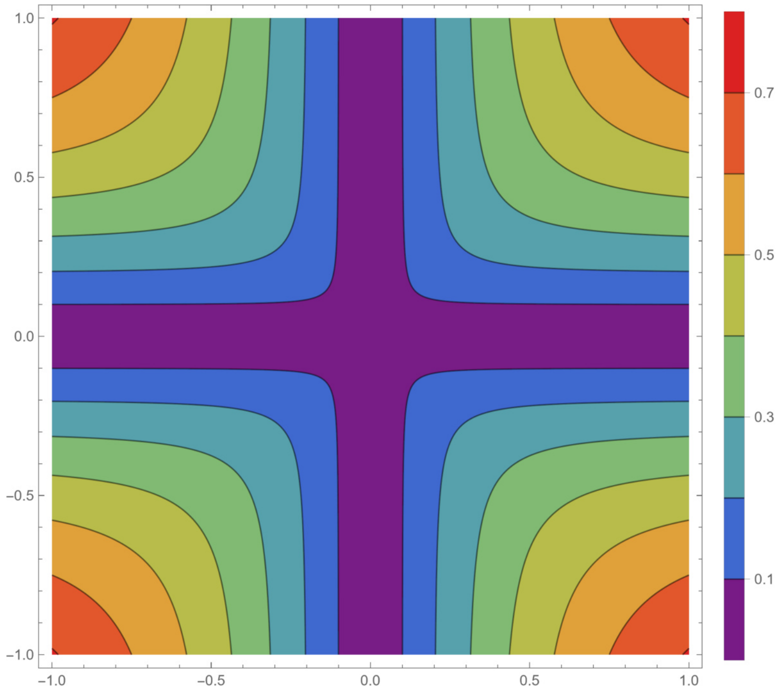

Figure 312. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

35.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\, u + \frac{\partial f}{\partial y}(0,0)\, v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the func-

tion $f(x, y)$ is locally Lipschitz continuous in $(0, 0)$. We cite here the criterion that we will use.

A function is differentiable in (a, b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.
2. The directional derivatives have the following form: $D_{(u,v)}(a, b) = \nabla f(a, b) \cdot (u, v) = \frac{\partial f}{\partial x}(a, b)\, u + \frac{\partial f}{\partial y}(a, b)\, v$.
3. The function is locally Lipschitz continuous in (a, b). This means that there exists at least one neighbourhood of (a, b) and a number $K > 0$ such that for all (x_1, y_1) and (x_2, y_2) in the neighbourhood, we have $|f(x_1, y_1) - f(x_2, y_2)| < K \left\|(x_1, y_1) - (x_2, y_2)\right\|$. Remark that the same K must be valid for all (x_1, y_1) and (x_2, y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be “locally flat" and thus “locally linear" which is partly expressed by this Lipschitz continuity condition.

Let us try to prove the Lipschitz condition. We are going to use a well known method.

$$\begin{aligned}
||x_1\, y_1| - |x_2\, y_2|| &\le |x_1\, y_1 - x_2\, y_2\,| \\
&\le |x_1\, y_1 - x_1\, y_2 + x_1\, y_2 - x_2\, y_2| \\
&\le |x_1\, y_1 - x_1\, y_2| + |x_1\, y_2 - x_2\, y_2| \\
&\le |x_1|\,|y_1 - y_2| + |y_2|\,|x_1 - x_2| \\
&\le |y_1 - y_2| + |x_1 - x_2| \\
&\le \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&\le 2\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.
\end{aligned}$$

We have chosen here the restriction to the neighbourhood defined by $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} < 1$ so that $|x_1| < 1$ and $|y_2| < 1$. We conclude that f is locally Lipschitz with a Lipschitz constant $K = 2$.

We have thus an alternative proof for the differentiability.

35.7 Continuity of the partial derivatives

We have seen in 35.4 that not all the partial derivatives exist in any neighbourhood of $(0, 0)$. So this criterion cannot be used.

35.8 Overview

$$f(x, y) = |x\,y|.$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

Exercise 36.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} y \sin\left(\frac{x}{\sqrt{|y|}}\right) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

36.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| y \sin\left(\frac{x}{\sqrt{|y|}}\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left| y \sin\left(\frac{x}{\sqrt{|y|}}\right)\right| &\le |y| \left|\sin\left(\frac{x}{\sqrt{|y|}}\right)\right| \\ &\le |y| \\ &\le \sqrt{x^2+y^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function is continuous.

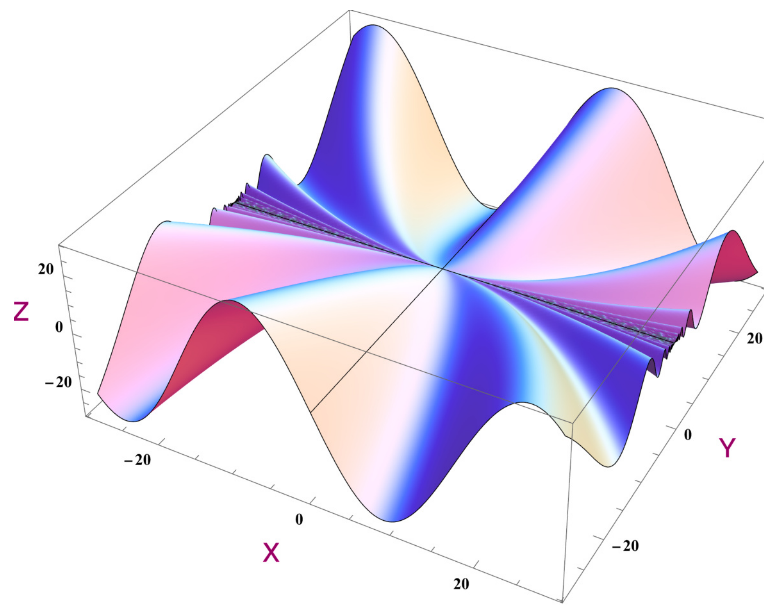

Figure 313. We see here a three dimensional figure of the graph of the function. This is a more global view of the function.

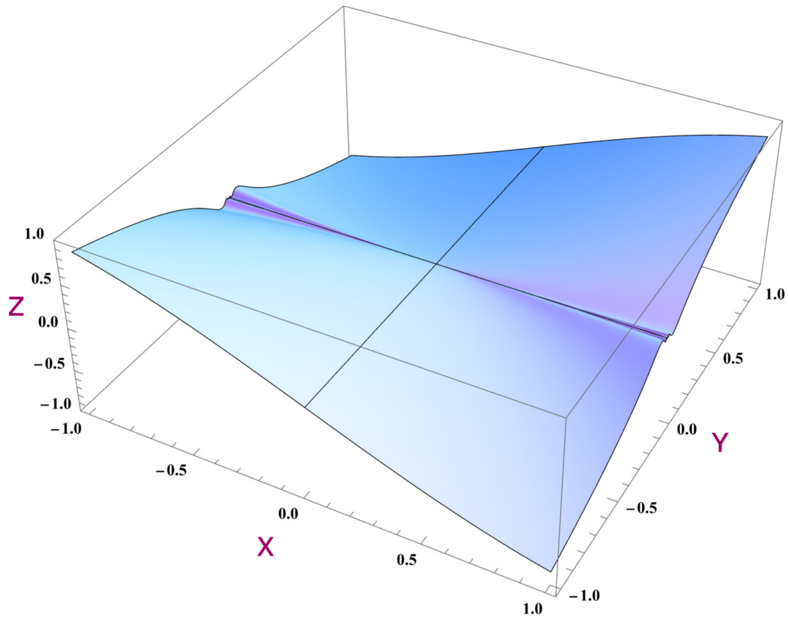

Figure 314. We see here a three dimensional figure of the graph of the function. This is a more local view of the function. This looks like a continuous function.

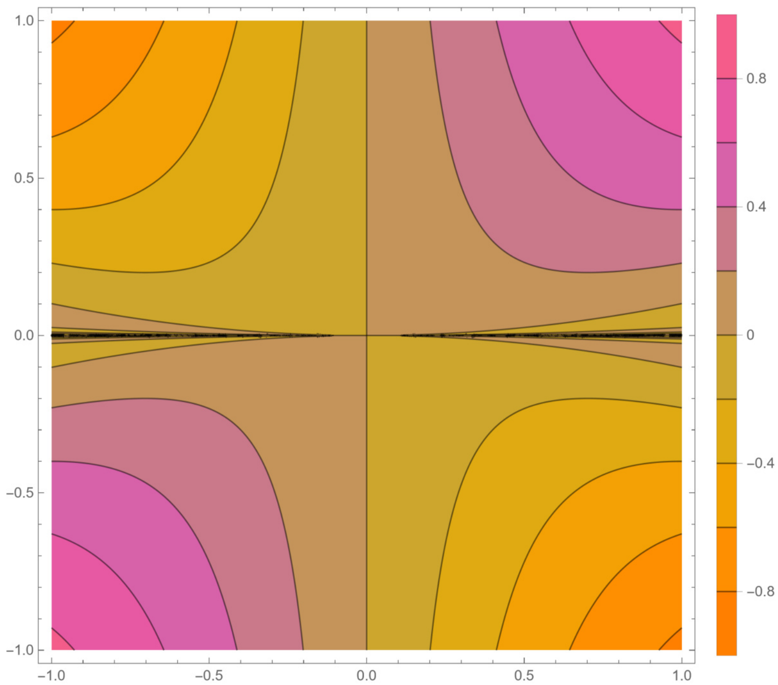

Figure 315. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

36.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

36.3 Directional derivatives

Let (u, v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2 + v^2} = 1$. Then the directional derivative in $(0, 0)$ in the direction (u, v), notation $D_{(u,v)}(0, 0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0 + hu, 0 + hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1, 0)$ and $(0, 1)$.

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0 + hu, 0 + hv) - f(0,0)}{h} \\ &= \lim_{h\to 0} v \sin\left(\frac{hu}{\sqrt{|hv|}}\right) \\ &= \lim_{h\to 0} v \sin\left(\frac{\operatorname{sgn}(h)\sqrt{|h|}\,u}{\sqrt{|v|}}\right) \\ &= 0. \end{aligned}$$

This calculation is valid if $v \neq 0$. Remark that we already discussed the case $v = 0$ in the part on the partial derivatives.

So the directional derivatives do always exist.

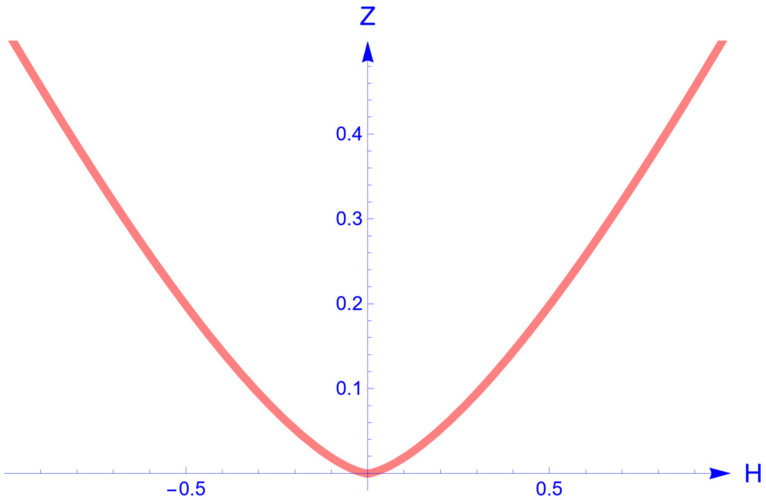

Figure 316. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$. Remark the slope in 0.

36.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

We want to investigate the existence of the partial derivatives not in the point $(0,0)$ but on points of the X-axis close to $(0,0)$.

Let us take a point $(a,0)$ and investigate the behaviour of the function in the vertical direction. In order to do that, we will look at the function $f(a,h)$ and see if the function is continuous and differentiable.

We calculate the function

$$f(a,h) = \begin{cases} h \sin\left(\frac{a}{\sqrt{|h|}}\right) & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

We will look for the partial derivative and we calculate for the limit

$$\lim_{h \to 0} \frac{h \sin\left(\frac{a}{\sqrt{|h|}}\right)}{h} = \lim_{h \to 0} \sin\left(\frac{a}{\sqrt{h}}\right).$$

We see that this limit does not exist if $a \neq 0$. So the partial derivative does not exist.

We illustrate in the next picture the behaviour of the function.

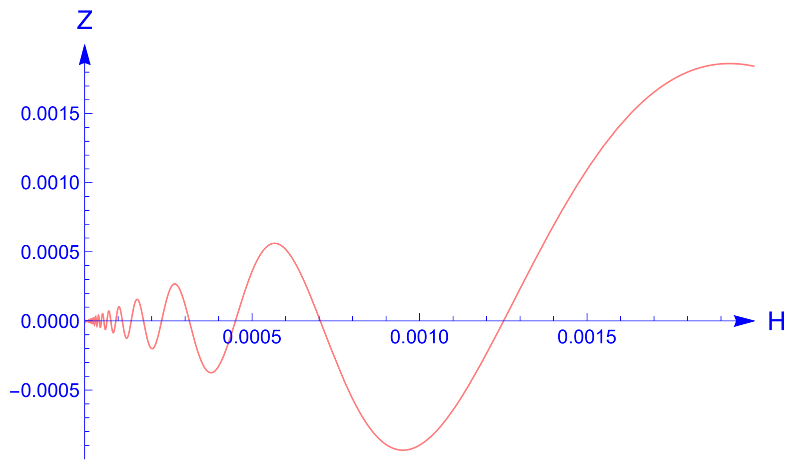

Figure 317. We see here a two dimensional figure of the graph of the function $f(a, h)$. We have drawn the function here for the value $a = 1/3$ which is representative for the values of a close to 0. This is not a differentiable function. This is the classical example of a continuous function in one variable that is continuous but not differentiable.

We draw the conclusion that it is pointless to investigate this function further in the neighbourhood of $(0, 0)$ because some of the partial derivatives do not exist in any neighbourhood of $(0, 0)$. The partial derivative to y in $(a, 0)$ with $a \neq 0$ does not exist.

36.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

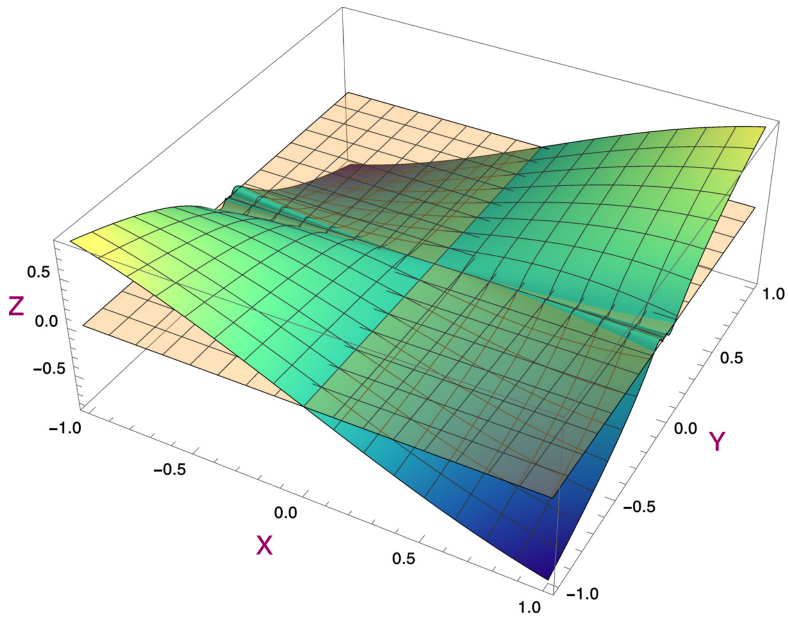

Figure 318. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function very nicely.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

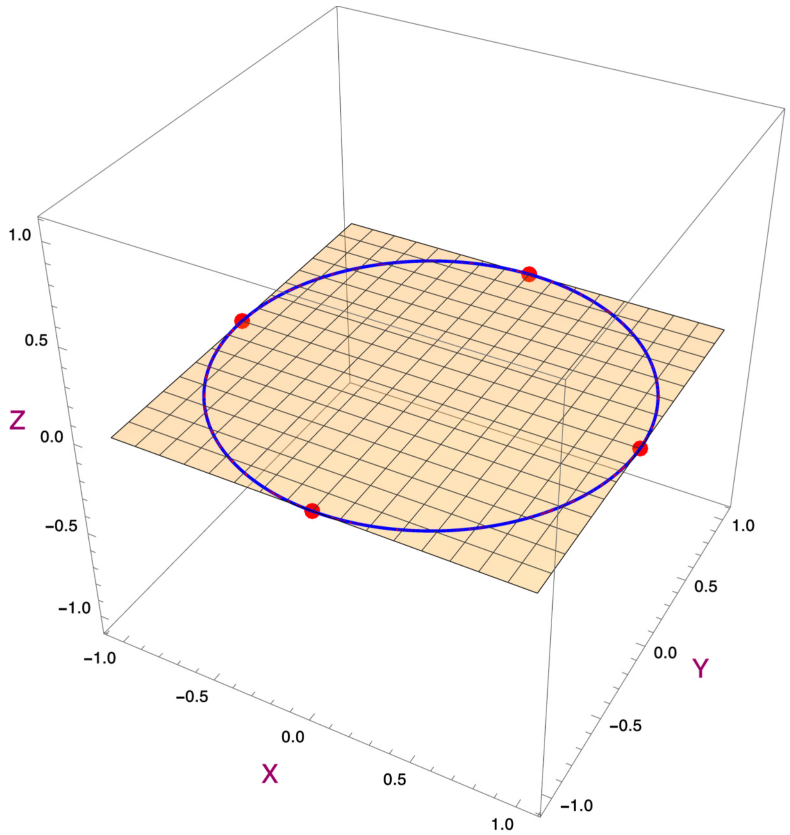

Figure 319. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h, k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{k\sin\left(\dfrac{h}{\sqrt{|k|}}\right)}{\sqrt{h^2+k^2}} & \text{if } k \neq 0 \text{ and } (h,k) \neq (0,0), \\ 0 & \text{if } k = 0 \text{ or } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left| \frac{k\,\sin\left(\frac{h}{\sqrt{|k|}}\right)}{\sqrt{h^2+k^2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have

the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{k \sin\left(\frac{h}{\sqrt{|k|}}\right)}{\sqrt{h^2+k^2}} \right| &\le \frac{|k| \left|\sin\left(\frac{h}{\sqrt{|k|}}\right)\right|}{\sqrt{h^2+k^2}} \\
&\le \frac{|k| \left(\frac{|h|}{\sqrt{|k|}}\right)}{\sqrt{h^2+k^2}} \\
&\le \frac{\sqrt{|k|}\,|h|}{\sqrt{h^2+k^2}} \\
&\le \frac{\sqrt{|k|}\sqrt{h^2+k^2}}{\sqrt{h^2+k^2}} \\
&\le \sqrt{\sqrt{h^2+k^2}} \\
&\le \sqrt{h^2+k^2}^{1/2}.
\end{aligned}$$

It is sufficient to take $\delta = \epsilon^2$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous.

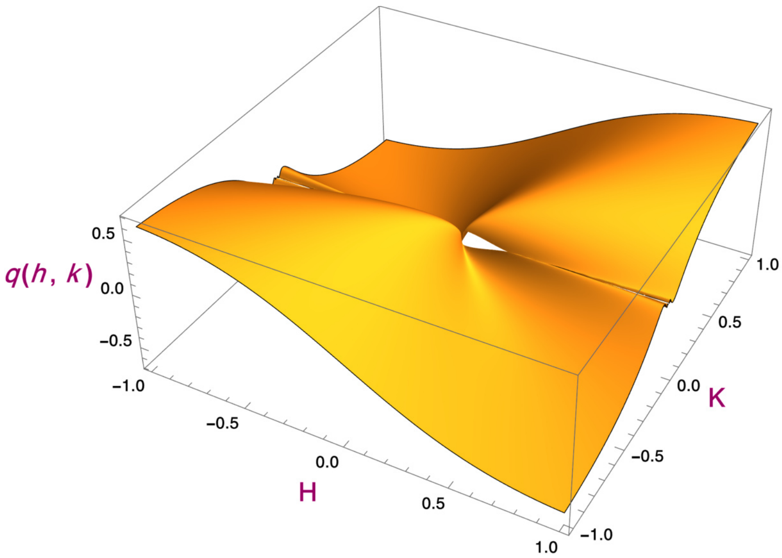

Figure 320. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

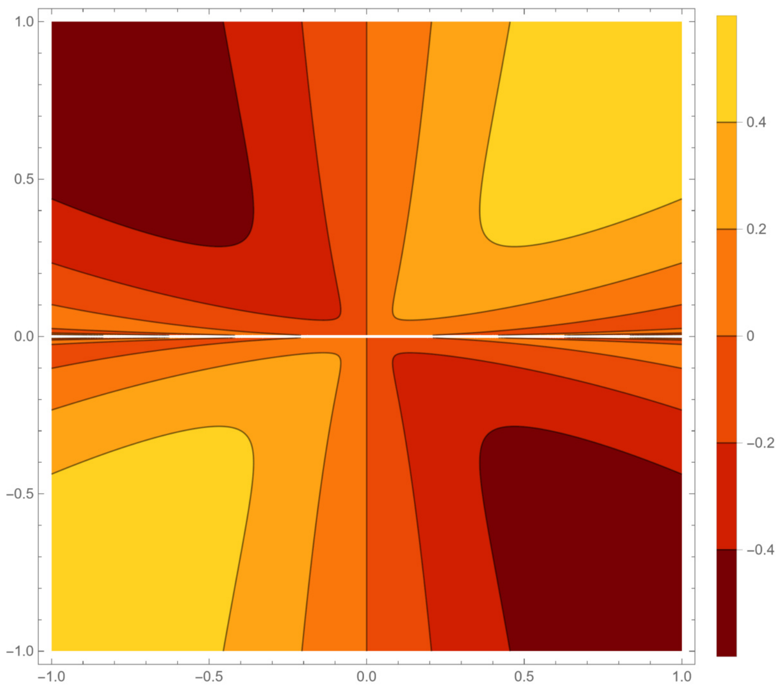

Figure 321. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

36.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\,u + \frac{\partial f}{\partial y}(0,0)\,v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the function $f(x,y)$ is locally Lipschitz continuous in $(0,0)$. We cite here the criterion that we will use.

A function is differentiable in (a,b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.
2. The directional derivatives have the following form: $D_{(u,v)}(a,b) = \nabla f(a,b) \cdot (u,v) = \frac{\partial f}{\partial x}(a,b)\,u + \frac{\partial f}{\partial y}(a,b)\,v$.
3. The function is locally Lipschitz continuous in (a,b). This means that there exists at least one neighbourhood of (a,b) and a number $K > 0$ such that for all (x_1,y_1) and (x_2,y_2) in the neighbourhood, we have $|f(x_1,y_1) - f(x_2,y_2)| < K\,||(x_1,y_1) - (x_2,y_2)||$. Remark that the same K must be valid for all (x_1,y_1) and (x_2,y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be “locally flat" and thus “locally linear" which is partly expressed by this Lipschitz continuity condition.

We will try to prove that the partial derivative to y is not bounded if $y \geq 0$ in any neighbourhood of $(0,0)$. This will be sufficient to conclude that the function is not Lipschitz continuous in $(0,0)$.

Let us take a look at the partial derivative to y. This is equal, where it exists, to

$$\frac{\partial f}{\partial y}(x,y) = \sin\left(\frac{x}{\sqrt{y}}\right) - \frac{x\cos\left(\frac{x}{\sqrt{y}}\right)}{2\sqrt{y}}.$$

We see that the first term is bounded. We concentrate on the second term and call that $h(x,y)$.

$$h(x,y) = -\frac{x\cos\left(\frac{x}{\sqrt{y}}\right)}{2\sqrt{y}}.$$

Let us now take $y = x^3$ with $x > 0$. Then we have

$$h(x,x^3) = -\frac{x\cos\left(\frac{x}{\sqrt{x^3}}\right)}{2\sqrt{x^3}} = -\frac{\cos\left(\frac{1}{\sqrt{x}}\right)}{2\sqrt{x}}.$$

It is clear that this function is not bounded. To be more specific, let us take the sequence $x_n = \frac{1}{4\pi^2 n^2}$, $n \in \mathbf{N}_0$, that converges to 0. We have then $h(x_n, x_n^3) = -n\pi$. This is clearly not bounded.

So the function f is not Lipschitz continuous and we conclude that an alternative proof following these lines is not possible.

36.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

Some of the partial derivatives do not exist in any neighbourhood of (0.0). Consult section 4. An alternative proof following this criterion is not possible.

36.8 Overview

$$f(x,y) = \begin{cases} y \sin\left(\dfrac{x}{\sqrt{|y|}}\right) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

Exercise 37.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \sin(y)\ \mathrm{sgn}(\sin(x)).$$

37.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left|\sin(y)\ \mathrm{sgn}(\sin(x)) - 0\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\sin(y)\ \mathrm{sgn}(\sin(x))\right| &\le |\sin(y)| \\ &\le |y| \\ &\le \sqrt{x^2+y^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function is continuous.

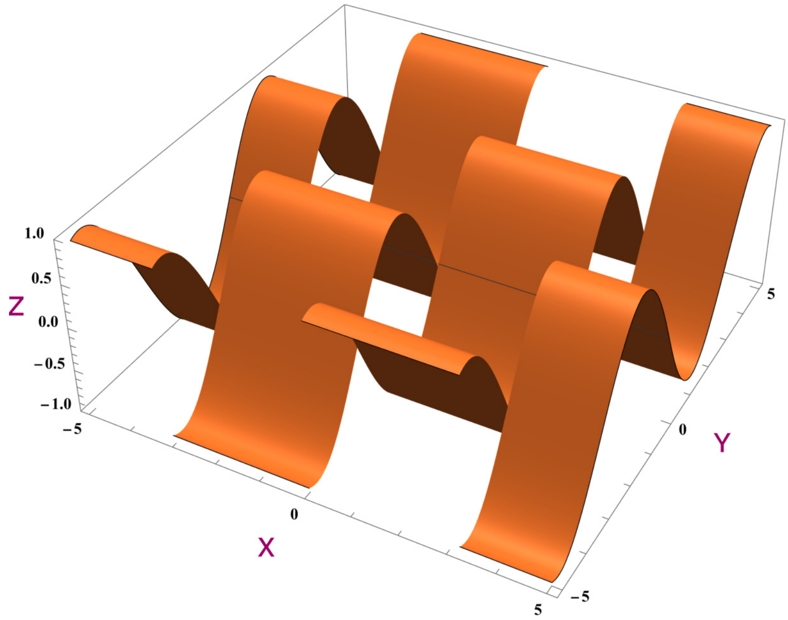

Figure 322. We see here a three dimensional figure of the graph of the function. We have here a more global view of the function.

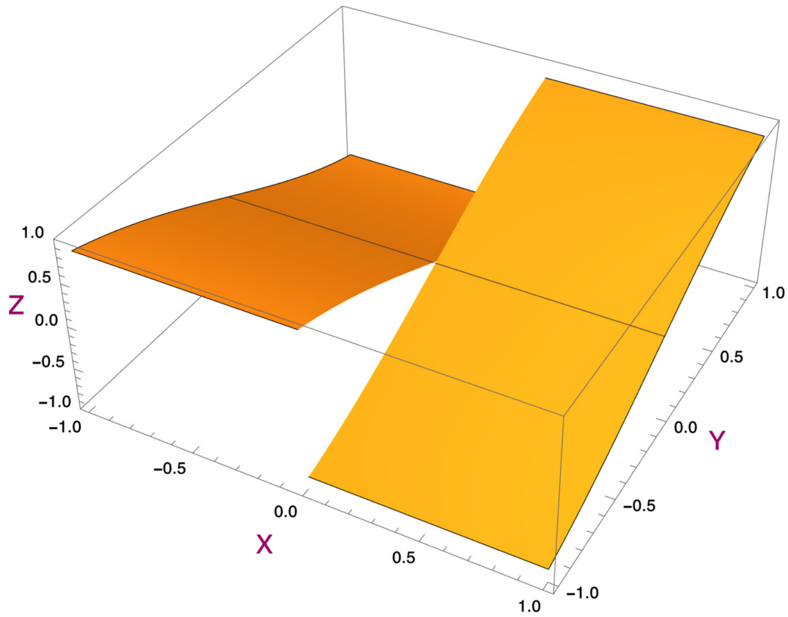

Figure 323. We see here a three dimensional figure of the graph of the function. We have here a more local view of the function.

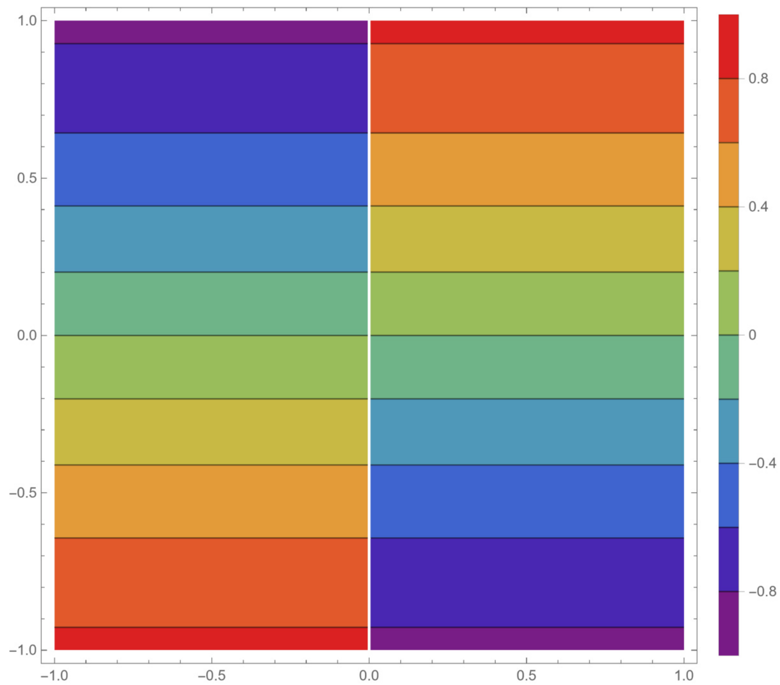

Figure 324. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0, 0)$.

37.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We will restrict in the following the function to the region $[-1, 1] \times [-1, 1]$ in order to work more easily with the following function definition

$$f(x, y) = \begin{cases} -\sin(y) & \text{if } -1 \leq x < 0,\ -1 \leq y \leq 1; \\ \sin(y) & \text{if } 0 < x \leq 1,\ -1 \leq y \leq 1; \\ 0 & \text{if } x = 0. \end{cases}$$

We observe then that

$$f|_{y=0}(x, y) = \begin{cases} f(x, 0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} 0 = 0.$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} 0 = 0.$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

37.3 Directional derivatives

Let (u, v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2 + v^2} = 1$. Then the directional derivative in $(0, 0)$ in the direction (u, v), notation $D_{(u,v)}(0, 0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1, 0)$ and $(0, 1)$.

In the case of our function we have to calculate the following limit
So let us try to write down he function $f(0 + hu, 0 + hv)$. In the case that $u > 0$, we have if we restrict h also to the interval $[-1, 1]$

$$f(0+hu, 0+hv)$$
$$= \begin{cases} \operatorname{sgn}(\sin(hu))\ \sin(hv) = -\sin(hv) & \text{if } -1 \le h < 0; \\ \operatorname{sgn}(\sin(hu))\ \sin(hv) = \sin(hv) & \text{if } 0 \le h \le 1. \end{cases}$$

In the case that $u < 0$, we have if we restrict h also to the interval $[-1, 1]$

$$f(0+hu, 0+hv)$$
$$= \begin{cases} \operatorname{sgn}(\sin(hu))\ \sin(hv) = \sin(hv) & \text{if } -1 \le h < 0; \\ \operatorname{sgn}(\sin(hu))\ \sin(hv) = -\sin(hv) & \text{if } 0 \le h \le 1. \end{cases}$$

So, if $u > 0$, we have

$$
\begin{aligned}
D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{\operatorname{sgn}(\sin(h\,u))\ \sin(h\,v)}{h} \\
&= \lim_{h\to 0} \frac{\operatorname{sgn}(\sin(h\,u))\ \sin(h\,v)}{h\,v}\, v \\
&= \lim_{h\to 0} \operatorname{sgn}(\sin(h\,u))\, v.
\end{aligned}
$$

So there is no limit if $v \neq 0$. But we covered that case before.

Some of the directional derivatives do not exist.

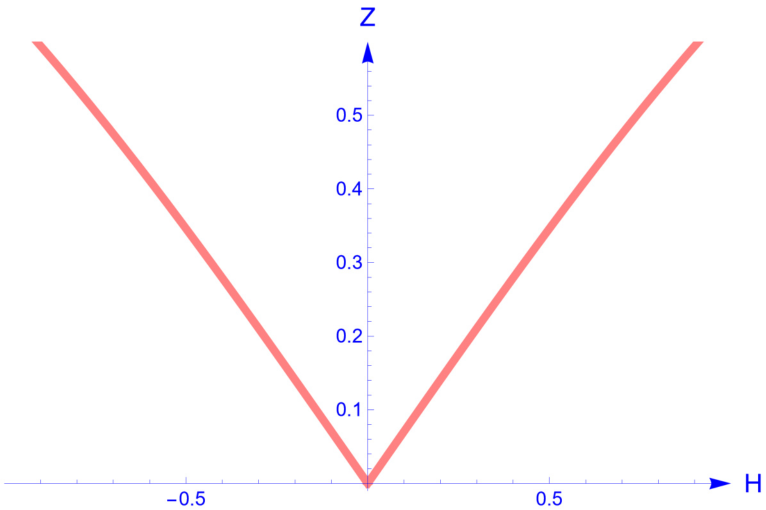

Figure 325. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

37.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

The problem is here that the partial derivatives have to exist in a neighbourhood of $(0,0)$. We see that all points on the Y-axis, $(0,0)$ excluded, have no derivative to x because the function is in the X-direction in these points not continuous. One can easily see this from the three dimensional figure of the function. So the derivatives in those points do not exist in any neighbourhood of $(0,0)$.

37.5 Differentiability

At least one of the directional derivatives does not exist, thus the function is not differentiable.

37.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

37.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

37.8 Overview

$$f(x, y) = \sin(y)\ \mathrm{sgn}(\sin(x))$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 38.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{\sin\left(4x\sqrt{|y|}\right)}{\sqrt{|xy|}} & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0. \end{cases}$$

38.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{\sin\left(4x\sqrt{|y|}\right)}{\sqrt{|xy|}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left|\frac{\sin\left(4\,x\,\sqrt{|y|}\right)}{\sqrt{|x\,y|}}\right| &\le \frac{\left|\sin\left(4\,x\,\sqrt{|y|}\right)\right|}{\sqrt{|x\,y|}} \\
&\le \frac{\left|4\,x\,\sqrt{|y|}\right|}{\sqrt{|x\,y|}} \\
&\le \frac{4\,|x|\,\sqrt{|y|}}{\sqrt{|x\,y|}} \\
&\le \frac{4\,|x|}{\sqrt{|x|}} \\
&\le 4\sqrt{|x|} \\
&\le 4\sqrt{\sqrt{x^2+y^2}} \\
&\le 4\sqrt{x^2+y^2}^{1/2}.
\end{aligned}$$

It is sufficient to take $\delta = (\epsilon/4)^2$. We can find a δ, so we conclude that the function is continuous.

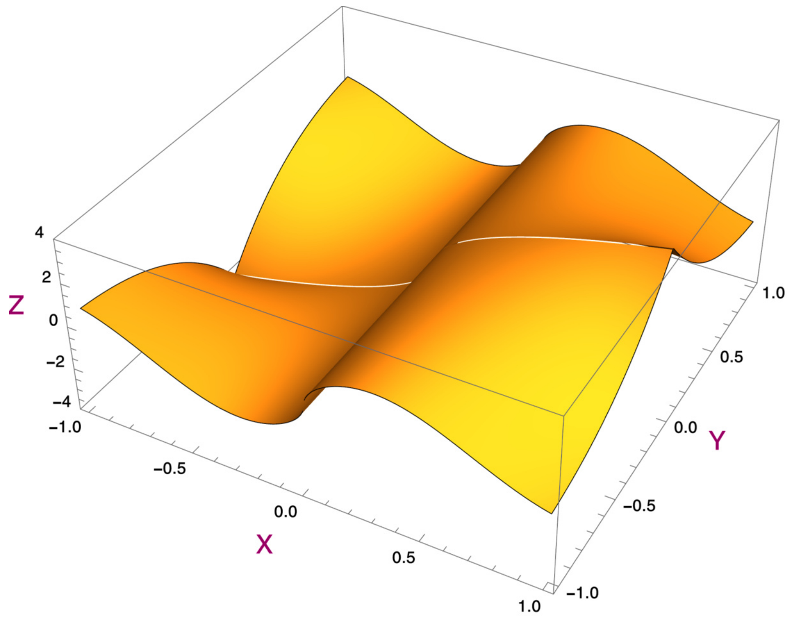

Figure 326. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

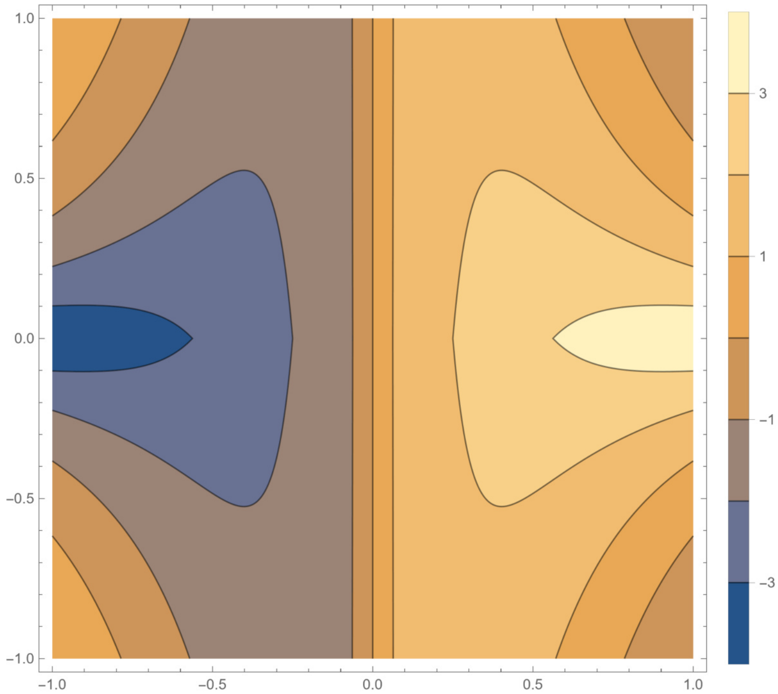

Figure 327. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

38.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y)=0 & \text{if } y\neq 0;\\ 0 & \text{if } y=0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

38.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\sin\left(4\,h\,u\sqrt{|h\,v|}\right)}{h\sqrt{|h^2\,u\,v|}} \\ &= \lim_{h\to 0} \frac{\sin\left(4\,h\,u\sqrt{|h\,v|}\right)}{4\,h\,u\sqrt{|h\,v|}} \frac{4\,h\,u\sqrt{|h\,v|}}{h\sqrt{|h^2\,u\,v|}} \\ &= \lim_{h\to 0} \frac{4\,u}{\sqrt{|h\,u|}}. \end{aligned}$$

This limit is not finite if $u \neq 0$. But we covered that case before.

So some of the directional derivatives do not exist.

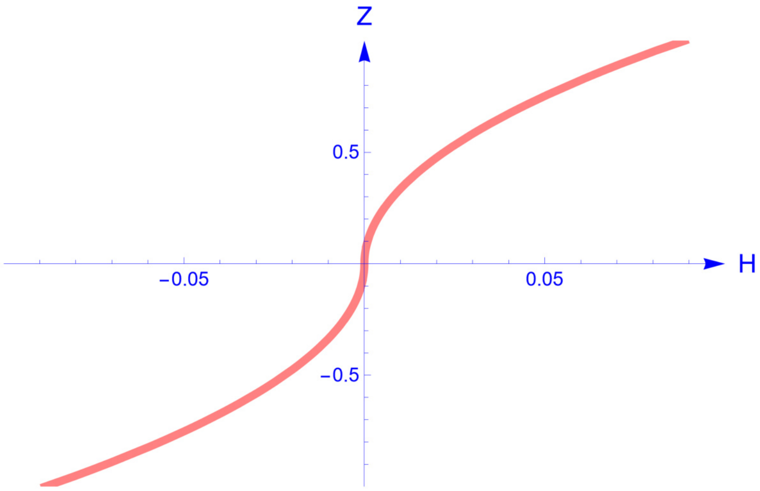

Figure 328. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$. We see a vertical tangent in 0. This causes the non existence of the directional derivative.

38.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

We want to investigate the existence of the partial derivatives not in the point $(0,0)$ but on points of the X and Y axes close to $(0,0)$.

Let us take a point $(a,0)$ and investigate the behaviour of the function in the vertical direction. In order to do that, we will look at the function $f(a,h)$ and see if the function is continuous and differentiable. We calculate for $a > 0$ the function

$$f(a,h) = \begin{cases} \dfrac{\sin\left(4a\sqrt{|h|}\right)}{\sqrt{a\sqrt{|h|}}} & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

Let us calculate the limit.

$$\lim_{h\to 0} \frac{\sin\left(4a\sqrt{|h|}\right)}{\sqrt{a\sqrt{|h|}}} = \lim_{h\to 0} \frac{\sin\left(4a\sqrt{|h|}\right)}{\left(4a\sqrt{|h|}\right)} \frac{\left(4a\sqrt{|h|}\right)}{\sqrt{a\sqrt{|h|}}}$$

$$= 4\sqrt{a}.$$

We see that this function is not continuous if $a \neq 0$ and not differentiable.

We illustrate in the next picture the behaviour of the function.

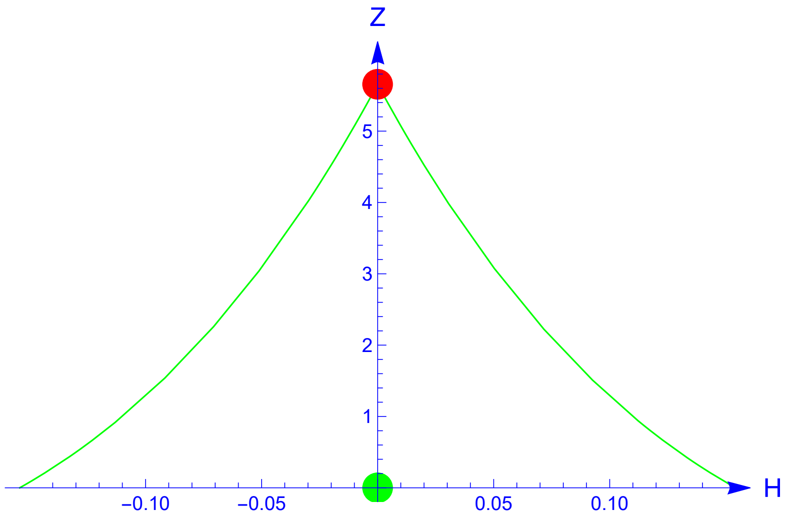

Figure 329. We see here a figure of the graph of the function $f(a,h)$. We have drawn the function here for the value $a = 1/3$ which is representative for the values of a close to 0. This does not look like a differentiable function. We have plotted here the function $f(a,h)$.

We draw the conclusion that it is pointless to investigate this function further in the neighbourhood of $(0,0)$ because the partial derivatives do not all exist in any neighbourhood of $(0,0)$. The partial derivative to y in $(a,0)$ does not exist.

38.5 Differentiability

Not all directional derivatives exist. So the function is not differentiable.

38.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

38.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

38.8 Overview

$$f(x,y) = \begin{cases} \dfrac{\sin\left(4x\sqrt{|y|}\right)}{\sqrt{|xy|}} & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0. \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 39.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \max\{x,y\}.$$

39.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\Big| \max\{x,y\}| - 0 \Big| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\Big| \max\{x,y\} \Big| &\leq \max\{|x|,|y|\} \\
&\leq \max\{\sqrt{x^2+y^2}, \sqrt{x^2+y^2}\} \\
&\leq \sqrt{x^2+y^2}.
\end{aligned}$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function is continuous.

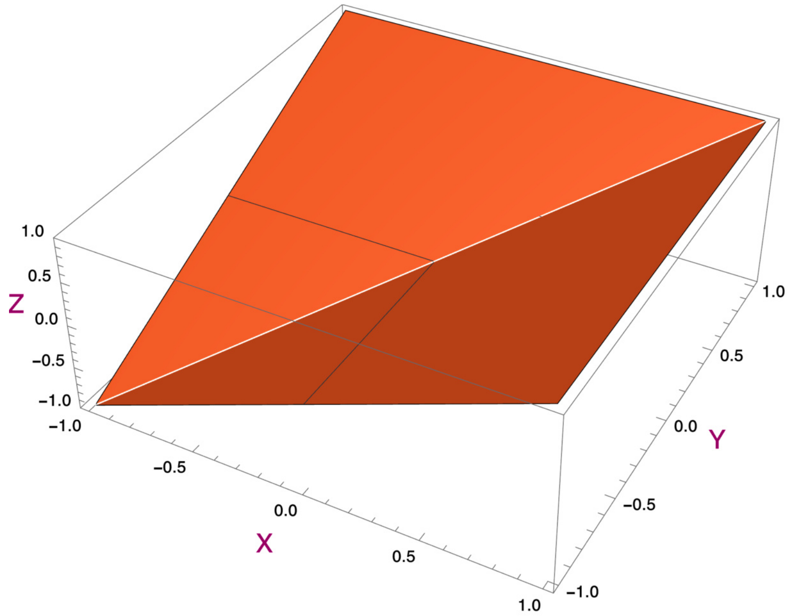

Figure 330. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

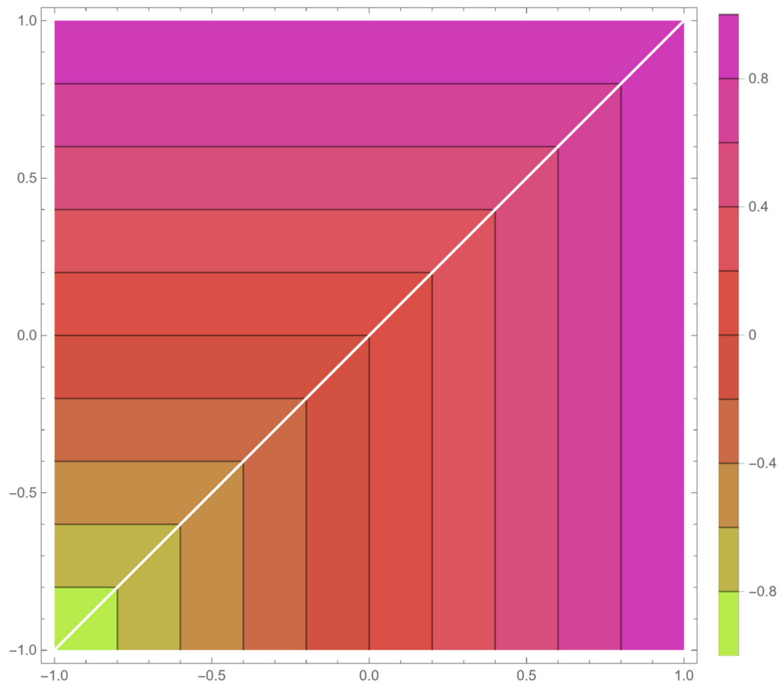

Figure 331. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

39.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We know that

$$\max\{x,y\} = \frac{x+y}{2} + \frac{|x-y|}{2}.$$

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = \dfrac{x}{2} + \dfrac{|x|}{2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{\frac{h}{2} + \frac{|h|}{2}}{h}$$
$$= \lim_{h\to 0} \frac{1}{2} + \frac{\operatorname{sgn}(h)}{2}.$$

So the partial derivative to x does not exist.

Discussion of the partial derivative to y in (0,0).
We know that

$$\max\{x, y\} = \frac{x+y}{2} + \frac{|x-y|}{2}.$$

We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = \frac{y}{2} + \frac{|-y|}{2} & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{h}{2} + \frac{|-h|}{2}}{h}$$
$$= \lim_{h\to 0} \frac{1}{2} - \frac{\operatorname{sgn}(-h)}{2}.$$

So the partial derivative to y does not exist.

We conclude that the partial derivative to x does not exist and that the partial derivative to y does not exist.

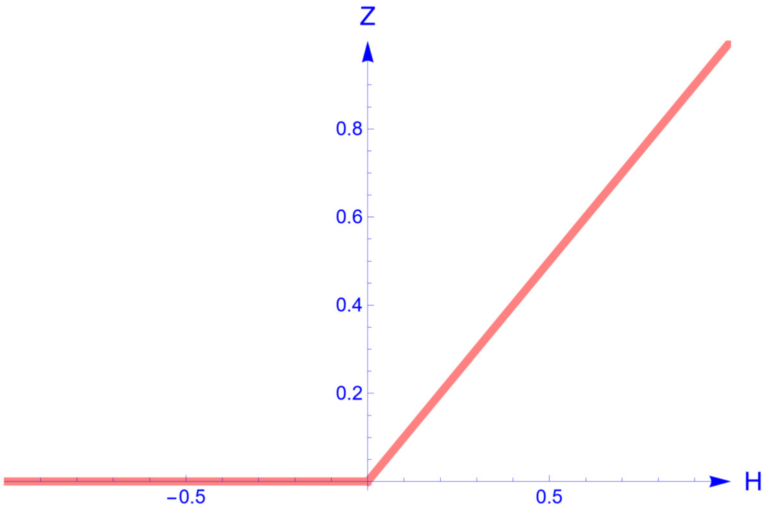

Figure 332. We see here a figure of the graph of the function restricted to the horizontal X-axis through $(0,0)$. The derivative does not exist. We have plotted here the function $f(h,0)$.

39.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0)=\lim_{h\to 0}\frac{f(0+hu,0+hv)-f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0}\frac{f(0+hu,0+hv)-f(0,0)}{h} \\ &= \lim_{h\to 0}\frac{|h(u-v)|+h(u+v)}{2h} \\ &= \lim_{h\to 0}\frac{\operatorname{sgn}(h)|(u-v)|+(u+v)}{2}. \end{aligned}$$

This limit does not exist if $u - v \neq 0$.

Some of the directional derivatives do not exist.

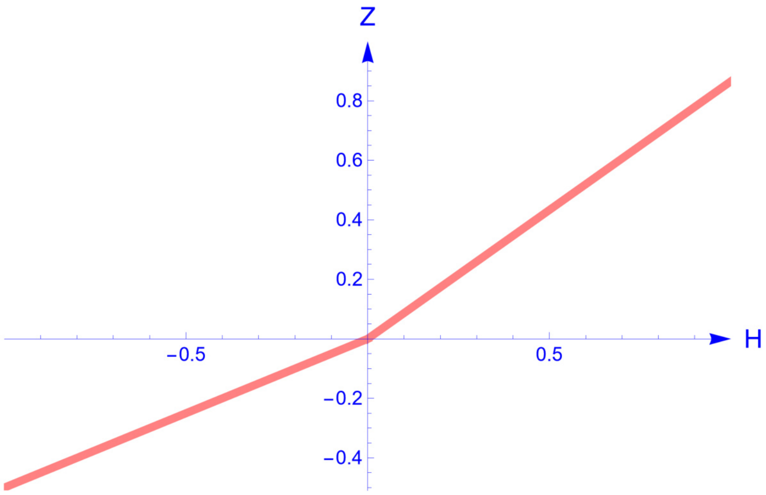

Figure 333. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(\sqrt{3}/2, 1/2\right)$. The derivative in 0 does not exist. There is no tangent line in 0. We have plotted here the function $f(h\,u, h\,v)$.

39.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

The partial derivatives do not exist. An alternative proof following this criterion does not exist.

39.5 Differentiability

At least one of the directional derivatives does not exist, thus the function is not differentiable.

39.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

39.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

39.8 Overview

$$f(x, y) = \max\{x, y\} = \frac{x + y}{2} + \frac{|x - y|}{2}.$$

continuous	yes
partial derivatives exist	no
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 40.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = y\sqrt{|x|}.$$

40.1 Continuity

We can argue that this function is composed of functions that are continuous. So there is nothing more to prove. We will argue with an ϵ-δ approach anyway.
We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| y\sqrt{|x|} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned} \left| y\sqrt{|x|} \right| &\leq |y|\sqrt{|x|} \\ &\leq \sqrt{x^2+y^2}\sqrt{\sqrt{x^2+y^2}} \\ &\leq \sqrt{x^2+y^2}^{3/2}. \end{aligned}$$

It is sufficient to take $\delta = \epsilon^{2/3}$. We can find a δ, so we conclude that the function is continuous.

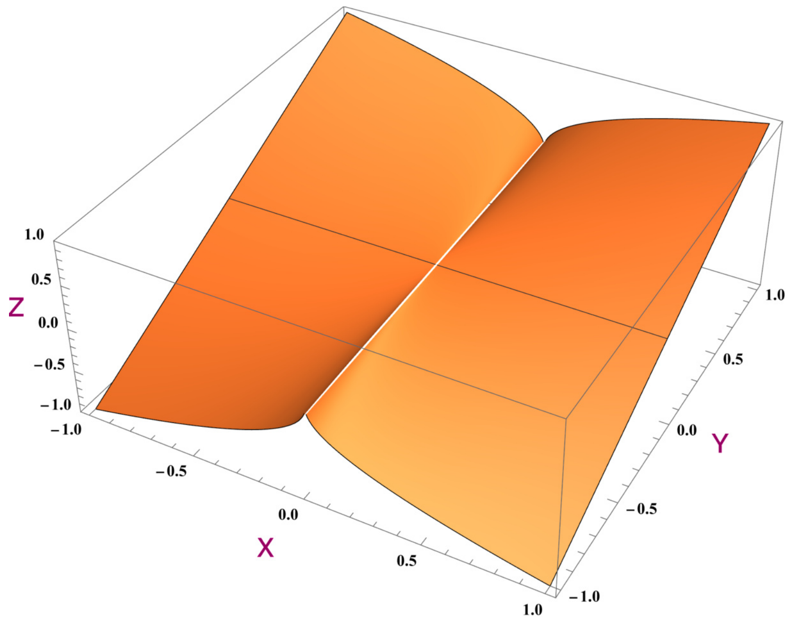

Figure 334. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

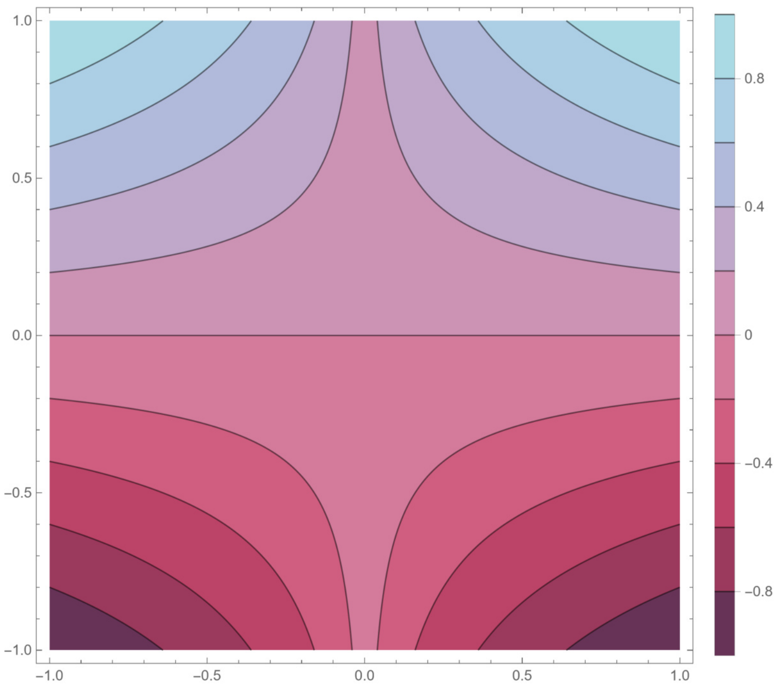

Figure 335. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

40.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f\big|_{y=0}(x,y) = f(x,0) = 0.$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = f(0,y) = 0.$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

40.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0}\frac{f(0+hu,0+hv)-f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} v\sqrt{|h\,u|} \\ &= 0. \end{aligned}$$

So the directional derivatives do always exist.

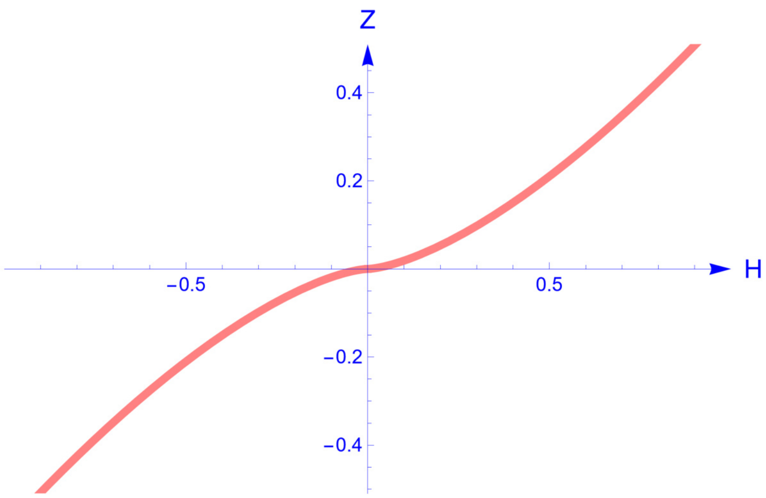

Figure 336. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

40.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

We want to answer the question whether the derivatives do exist in a neighbourhood of $(0,0)$.

Let us take a point $(0,b)$ and investigate the behaviour of the function in the horizontal direction. In order to do that, we will look at the function $f(h,b)$ and see if the function is continuous and differentiable.

We calculate the function

$$f(h,b) = \begin{cases} \sqrt{h}\,b & \text{if } h > 0; \\ \sqrt{-h}\,b & \text{if } h < 0; \\ 0 & \text{if } h = 0. \end{cases}$$

We see that this function is is not differentiable.

We illustrate in the next picture the behaviour of the function.

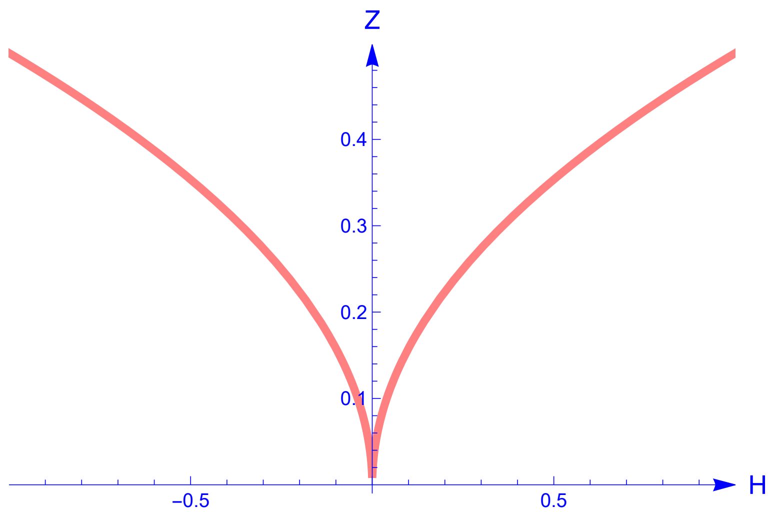

Figure 337. We see here a two dimensional figure of the graph of the function $f(h,b)$. We have drawn the function here for the value $b = 1/2$ which is exemplary for the values of b close to 0. This function does not look like a differentiable function.

We draw the conclusion that it is pointless to investigate this function further in the neighbourhood of $(0,0)$ because the partial derivatives do not exist in any neighbourhood of $(0,0)$.

40.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

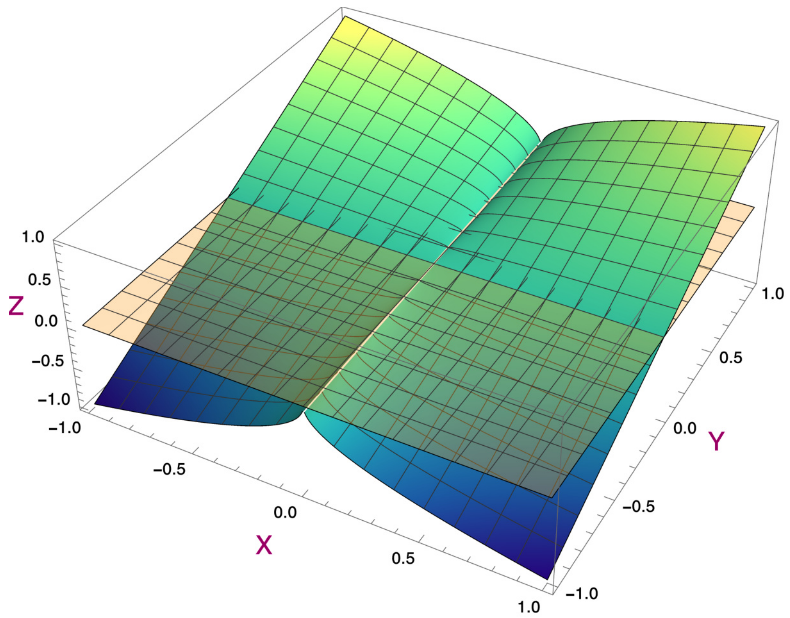

Figure 338. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x, y)$. We see here that the candidate tangent plane fits the function quite nicely.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we

know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

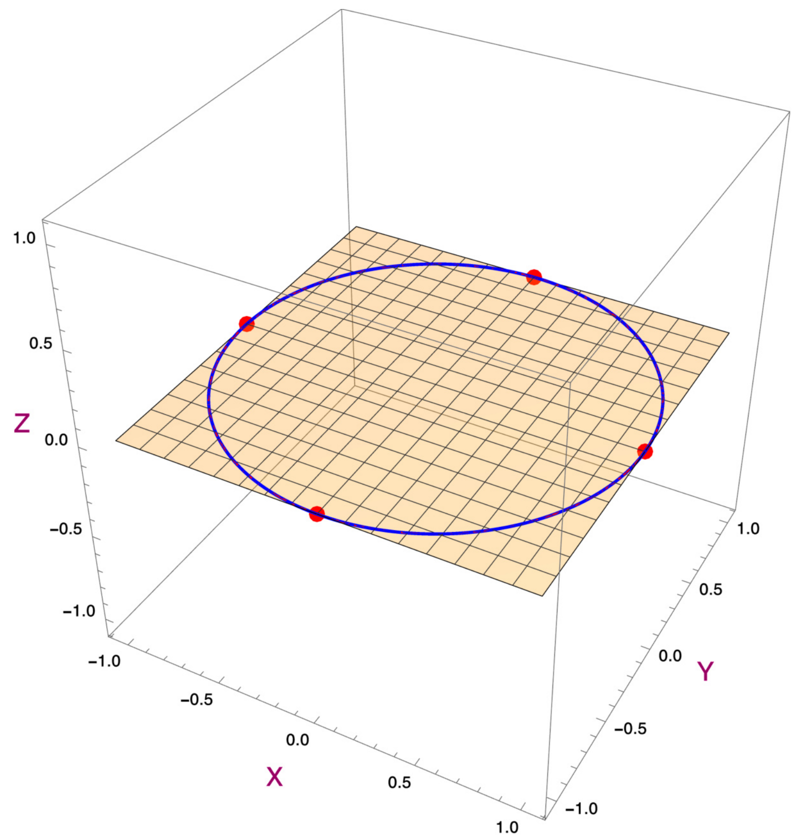

Figure 339. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{k\sqrt{|h|}}{\sqrt{h^2+k^2}} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left| \frac{k\sqrt{|h|}}{\sqrt{h^2+k^2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{k\sqrt{|h|}}{\sqrt{h^2+k^2}}\right| &\leq \frac{|k|\sqrt{|h|}}{\sqrt{h^2+k^2}} \\ &\leq \frac{\sqrt{h^2+k^2}\sqrt{\sqrt{h^2+k^2}}}{\sqrt{h^2+k^2}} \\ &\leq \sqrt{h^2+k^2}^{1/2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon^2$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous.

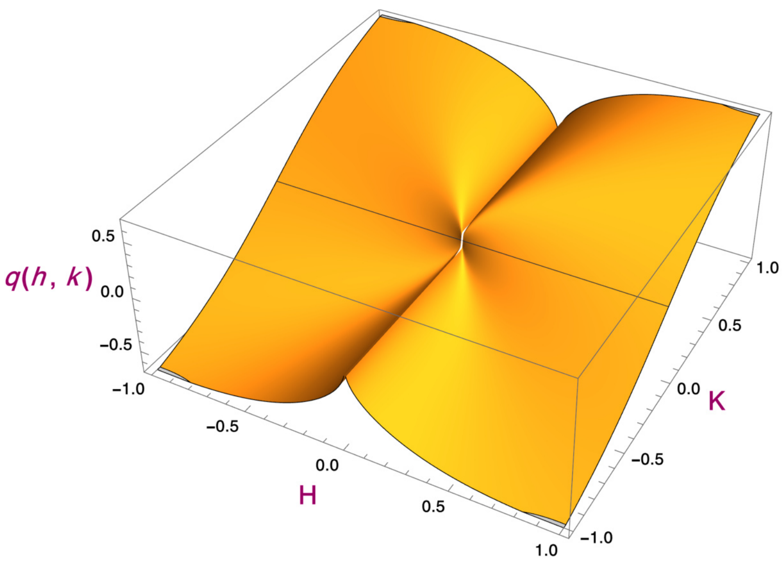

Figure 340. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

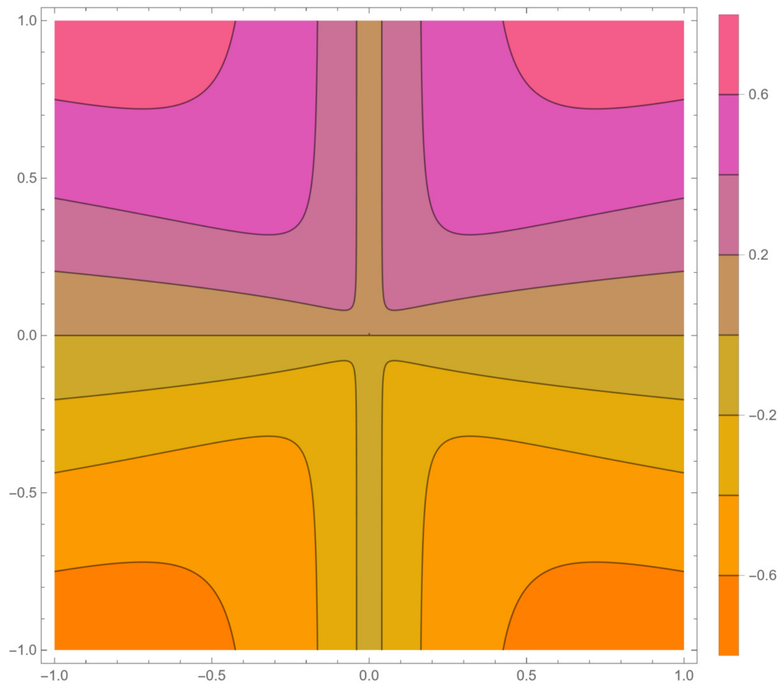

Figure 341. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

40.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

As we have seen in 40.4, we have the square root function in the domain which is the standard example of non Lipschitz continuity. We have there e.g. at the right hand side of $h = 0$ small intervals $[h_1, h_2]$ on which the difference quotients can be arbitrarily large. This is caused geometrically by the vertical tangent on the vertical axis. So the function is not Lipschitz. An alternative proof following this criterion is not possible.

40.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability. The partial derivatives do not all exist in any neighbourhood of $(0,0)$. So we cannot find an alternative proof.

40.8 Overview

$$f(x, y) = y\sqrt{|x|}.$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

Exercise 41.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} x \sin\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

41.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| x \sin\left(\frac{1}{x^2+y^2}\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left| x \sin\left(\frac{1}{x^2+y^2}\right)\right| &\leq |x| \left|\sin\left(\frac{1}{x^2+y^2}\right)\right| \\ &\leq |x| \\ &\leq \sqrt{x^2+y^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function is continuous.

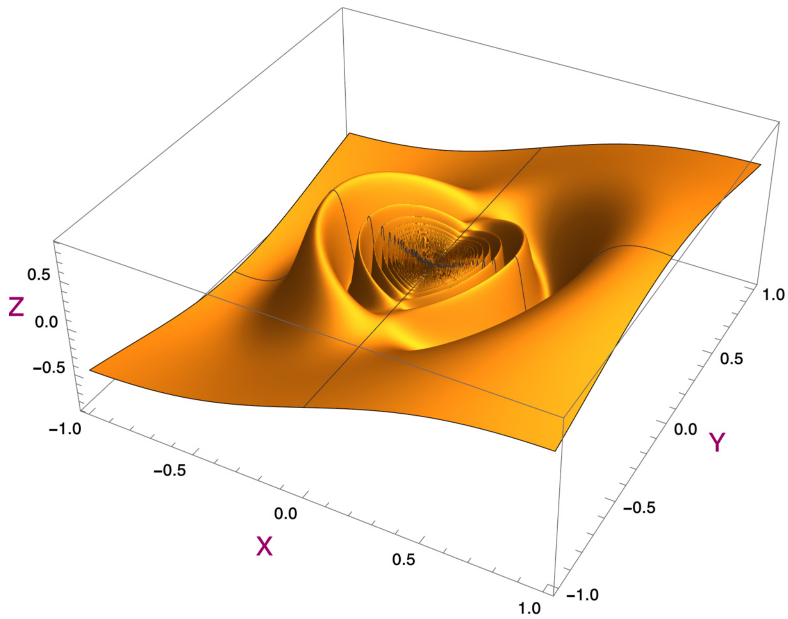

Figure 342. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

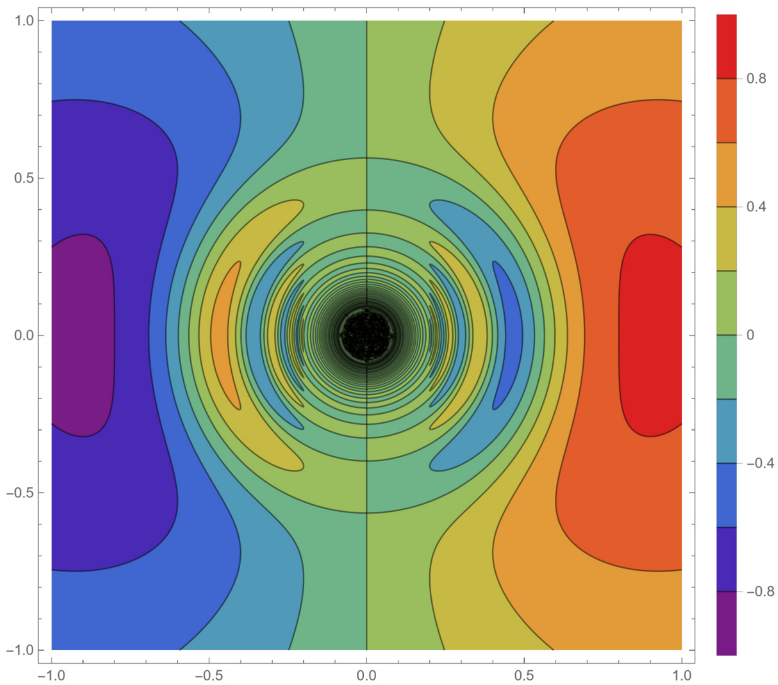

Figure 343. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

41.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).
We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = x \sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \sin\left(\frac{1}{h^2}\right). \end{aligned}$$

This limit does not exist.

So the partial derivative to x does not exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial y}(0,0) = 0.$$

and that the partial derivative to x does not exist.

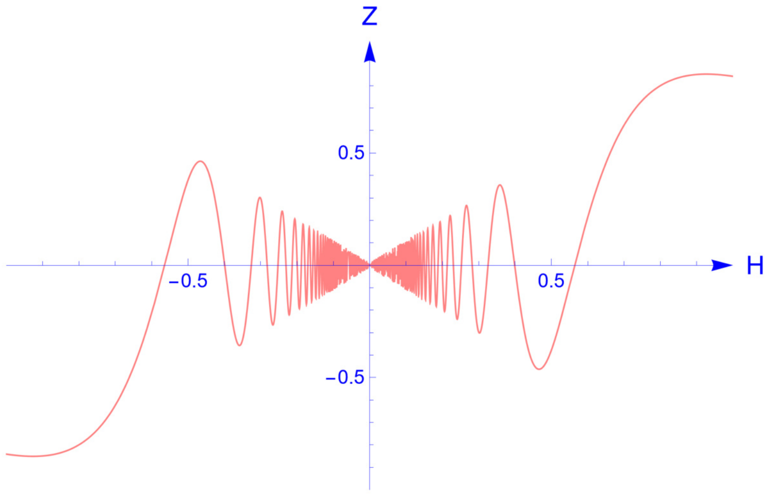

Figure 344. We see here a figure of the graph of the function restricted to the horizontal X-axis through $(0,0)$. We see that the function is continuous. It is a classical example of a function that is not differentiable. We have plotted here the function $f(h,0)$.

41.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu,0+hv)-f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu,0+hv)-f(0,0)}{h} \\ &= \lim_{h\to 0} u \sin\left(\frac{1}{h^2\,(u^2+v^2)}\right). \end{aligned}$$

This limit does not exist.

So the directional derivatives do not always exist.

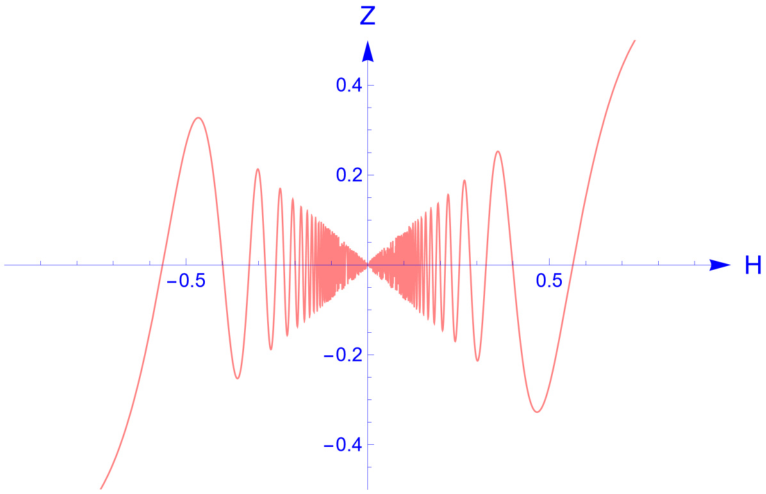

Figure 345. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$. This is a classical example of a function that is continuous but the derivative does not exist.

41.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

The partial derivatives do not exist in any neighbourhood of $(0,0)$. So an alternative proof following this criterion is not possible.

41.5 Differentiability

We have that at least one of the directional derivatives does not exist, then the function is not differentiable. So it is futile to continue.

41.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

This section is **irrelevant** for this exercise, because the function is not differentiable

41.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable

41.8 Overview

$$f(x,y)=\begin{cases} x\sin\left(\dfrac{1}{x^2+y^2}\right) & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

continuous	yes
partial derivatives exist	no
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 42.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0, 0)$.

$$f(x, y) = \max\{|x|, |y|\} = \frac{|x| + |y|}{2} + \frac{||x| - |y||}{2}.$$

42.1 Continuity

We can reason that this function is composed of continuous functions. But we also try to reason by the definition only.
We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x, y) - f(0, 0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x, y) - (0, 0)|| < \delta$ it follows that $|f(x, y) - f(0, 0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0, 0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{|x| + |y|}{2} + \frac{||x| - |y||}{2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{|x|+|y|}{2}+\frac{||x|-|y||}{2}\right| &\leq \frac{|x|+|y|}{2}+\frac{||x|+|y||}{2}\\ &\leq |x|+|y|\\ &\leq \sqrt{x^2+y^2}+\sqrt{x^2+y^2}\\ &\leq 2\sqrt{x^2+y^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon/2$. We can find a δ, so we conclude that the function is continuous.

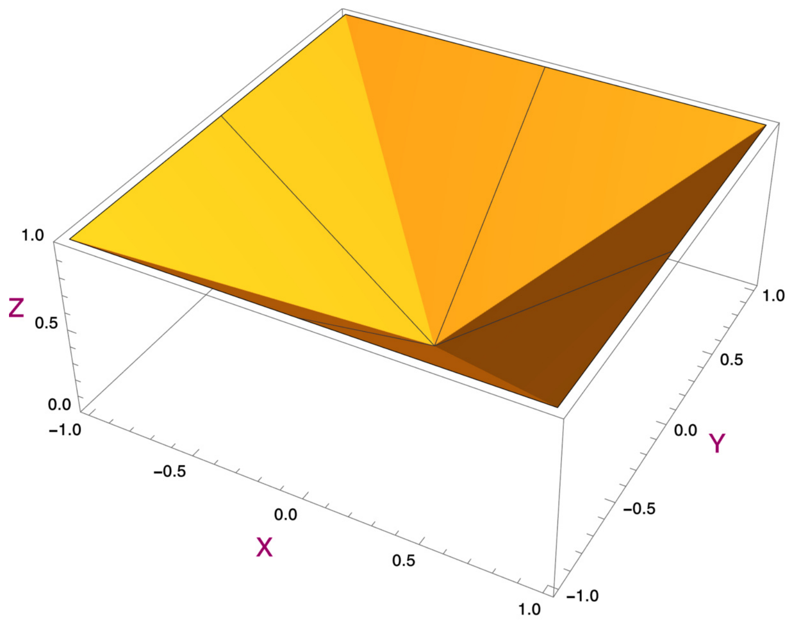

Figure 346. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

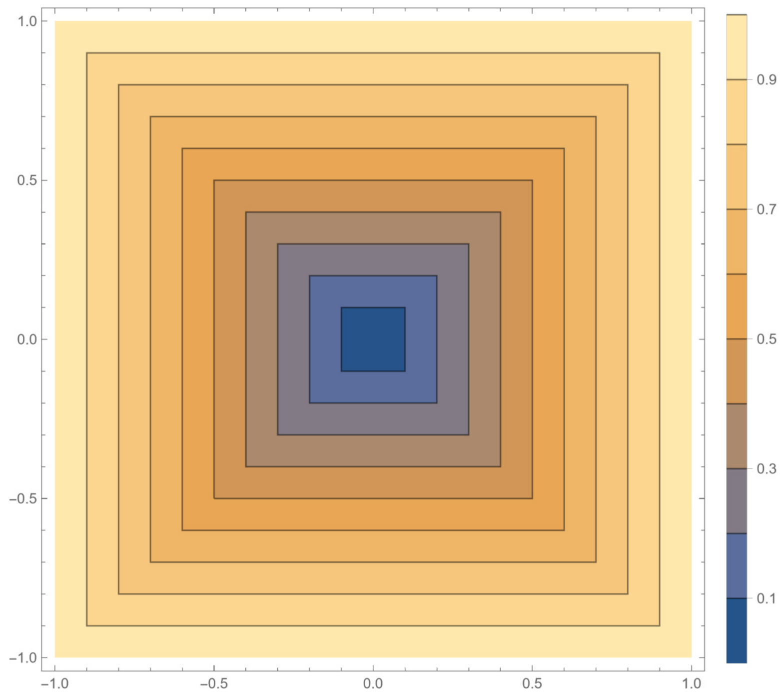

Figure 347. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

42.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = |x| & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0}\frac{|h|}{h}\\ &= \lim_{h\to 0}\operatorname{sgn}(h).\end{aligned}$$

So the partial derivative to x does not exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = |y| & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0}\frac{|h|}{h}\\ &= \lim_{h\to 0}\operatorname{sgn}(h).\end{aligned}$$

So the partial derivative to y does not exist.

We conclude that the partial derivative to x does not exist and the partial derivative to y does not exist.

42.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{||hu| - |hv|| + |hu| + |hv|}{2h} \\ &= \lim_{h\to 0} \frac{|h|\,||u| - |v|| + |h|\,|u| + |v|}{2h} \\ &= \lim_{h\to 0} \operatorname{sgn}(h)\,\frac{||u| - |v|| + |u| + |v|}{2}. \end{aligned}$$

This limit does not exist.

So the directional derivatives do not always exist.

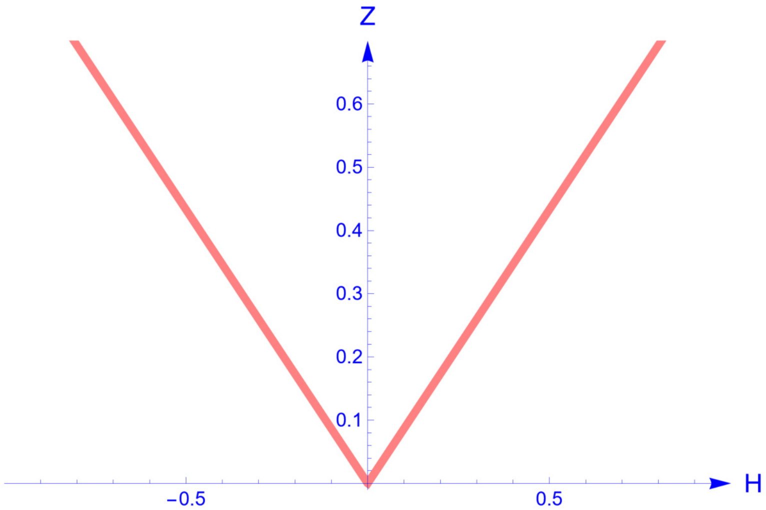

Figure 348. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(\sqrt{3}/2, 1/2\right)$. This function is not differentiable. The directional derivative does not exist. We have plotted here the function $f(hu, hv)$.

42.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

The partials do not exist in all neighbourhoods of $(0,0)$. An alternative proof following this criterion does not exist.

42.5 Differentiability

At least one of the directional derivatives does not exist, so the function is not differentiable.

42.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

This section is **irrelevant** for this exercise, because the function is not differentiable

42.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

42.8 Overview

$$f(x,y) = \max\{|x|,|y|\} = \frac{|x|+|y|}{2} + \frac{||x|-|y||}{2}.$$

continuous	yes
partial derivatives exist	no
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 43.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{x+y}\right) & \text{if } x+y \neq 0, \\ 0 & \text{if } x+y = 0. \end{cases}$$

43.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \left(x^2+y^2\right)\sin\left(\frac{1}{x+y}\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned} \left| \left(x^2 + y^2\right) \sin\left(\frac{1}{x+y}\right) \right| &\le \left(x^2 + y^2\right) \left| \sin\left(\frac{1}{x+y}\right) \right| \\ &\le x^2 + y^2 \\ &\le \sqrt{x^2 + y^2}^{\,2}. \end{aligned}$$

It is sufficient to take $\delta = \epsilon^{1/2}$. We can find a δ, so we conclude that the function is continuous.

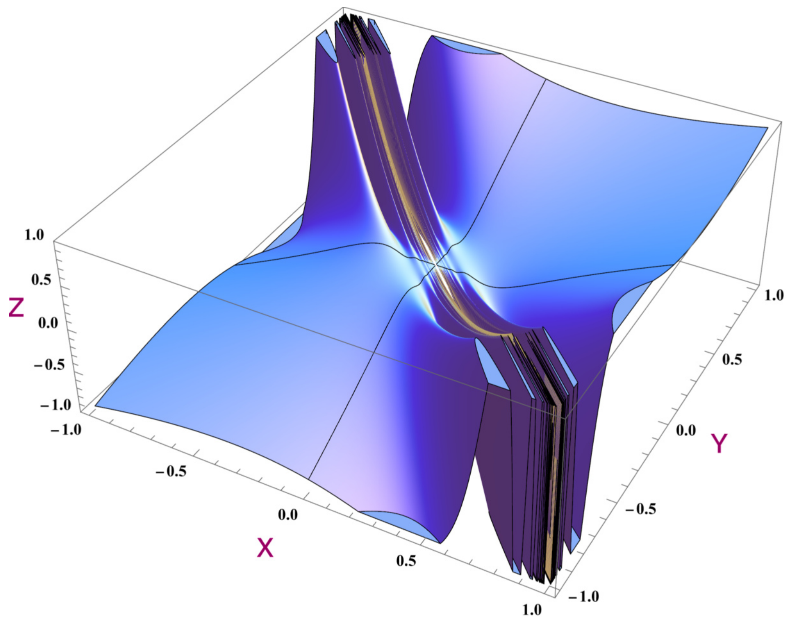

Figure 349. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

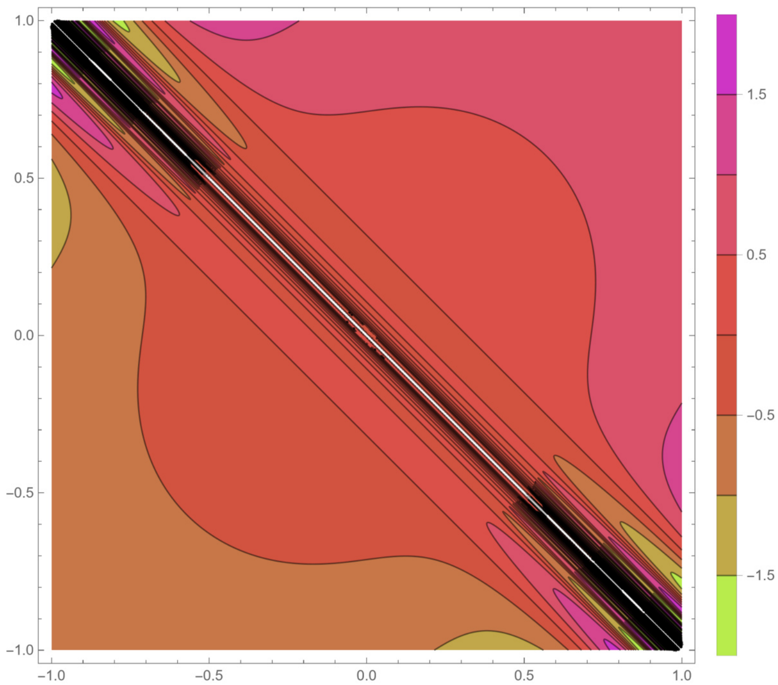

Figure 350. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

43.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f\big|_{y=0}(x,y) = \begin{cases} f(x,0) = x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} h\sin\left(\frac{1}{h}\right)\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f\,|_{x=0}(x,y) = \begin{cases} f(0,y) = y^2\sin\left(\frac{1}{y}\right) & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} h\,\sin\left(\frac{1}{h}\right)\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

43.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\ &= \lim_{h\to 0} h \left(u^2+v^2\right) \sin\left(\frac{1}{hu+hv}\right) \\ &= 0. \end{aligned}$$

This limit is 0 if $u+v \neq 0$. If $u+v=0$, we have differentiability by the definition which says that the function f is zero if $x+y=0$.
So the directional derivatives do always exist.

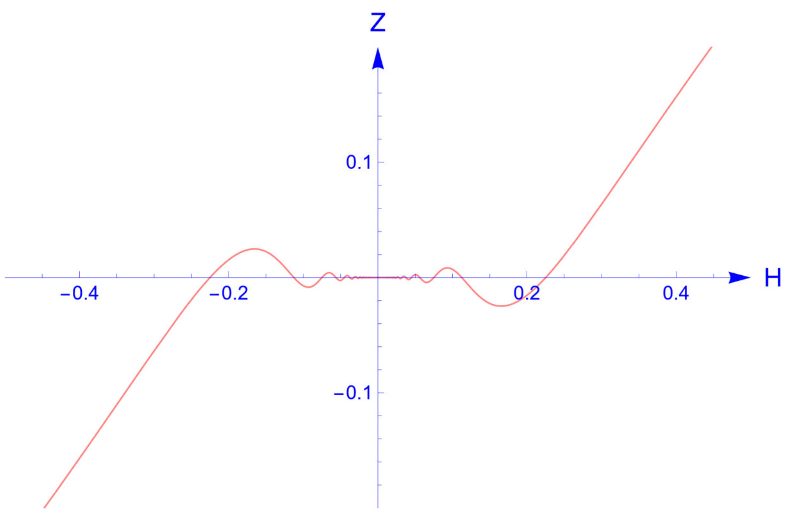

Figure 351. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$. This function is continuous and differentiable.

43.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

We want to investigate the existence of the partial derivatives not in the point $(0,0)$ but on points on the line defined by $x+y=0$ close to $(0,0)$. Let us take a point $(a,-a)$ and investigate the behaviour of the function in the vertical direction. In order to do that, we will look at the function $f(a,-a+h)$ and see if the function is continuous and differentiable. We will look for the partial derivative and we calculate the limit

$$\begin{aligned}
\lim_{h\to 0}\frac{f(a,-a+h)}{h} &= \lim_{h\to 0}\frac{(a^2+(h-a)^2)\,\sin\left(\frac{1}{h}\right)}{h} \\
&= \lim_{h\to 0} 2\,a^2\,\sin\left(\frac{1}{h}\right) - 2\,a\,h\,\sin\left(\frac{1}{h}\right) + h^2\,\sin\left(\frac{1}{h}\right) \\
&= \lim_{h\to 0} 2\,a^2\,\sin\left(\frac{1}{h}\right).
\end{aligned}$$

This limit does not exist if $a \neq 0$. We can be more explicit. Let us define $g(h) = 2\,a^2\,\sin\left(\frac{1}{h}\right)$. We define the sequence $h_n = \frac{1}{2\,\pi\,n}$ which converges to 0. Then $g(h_n) = 0$. But now we define $h_n = \frac{2}{\pi\,(4\,n+1)}$ which converges to 0. Then $g(h_n) = 2\,a^2$. This is impossible if the limit exists.

We conclude that the partial derivative to y of points on $x+y=0$ do not exist.

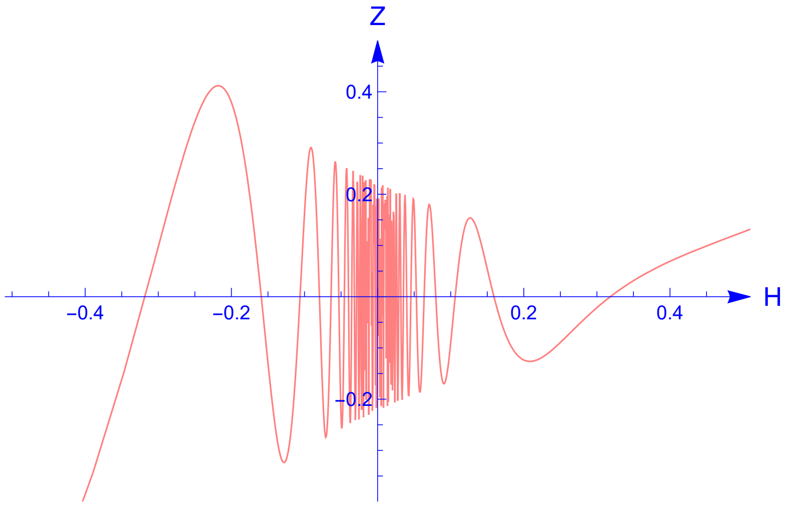

Figure 352. We see here a two dimensional figure of the graph of the function $f(a, -a+h)$. We have drawn the function here for the value $a = 1/3$ which is exemplary for the values of a close to 0. This is not a continuous nor a differentiable function.

So we cannot give an alternative proof following this criterion.

43.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

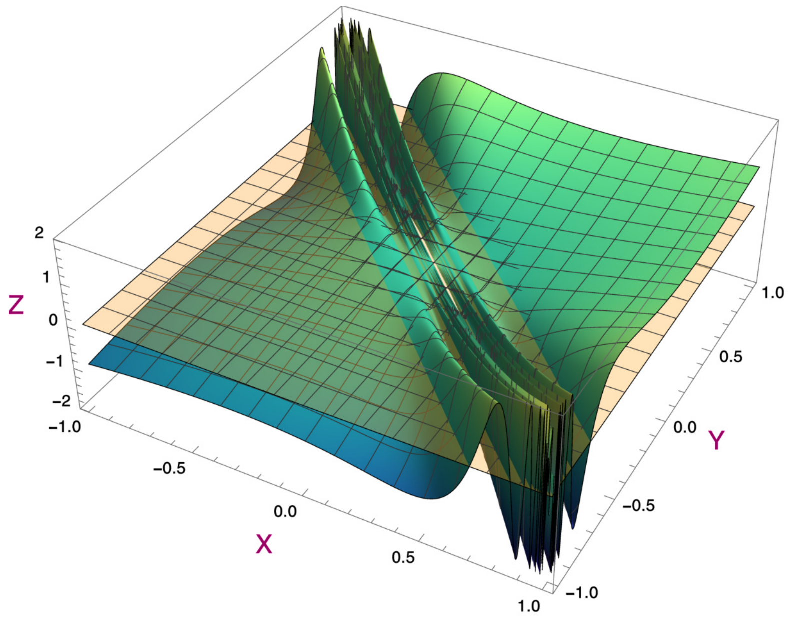

Figure 353. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function quite nicely. But further calculations will be necessary.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

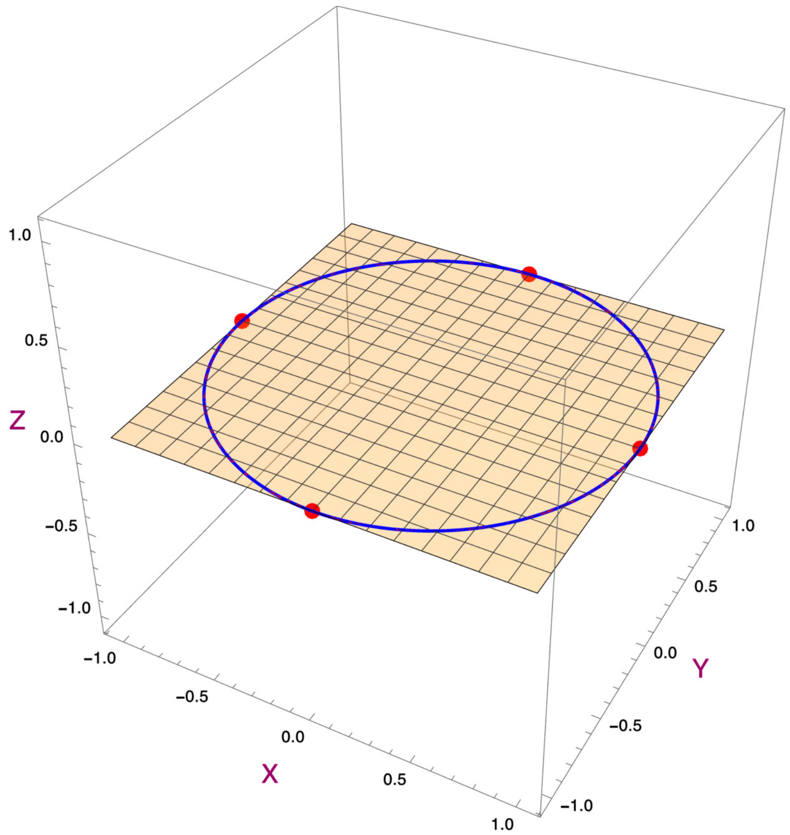

Figure 354. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \sqrt{h^2+k^2}\,\sin\left(\frac{1}{h+k}\right) & \text{if } (h,k) \neq (0,0) \text{ and } h+k \neq 0; \\ 0 & \text{if } (h,k) = (0,0) \text{ or } h+k = 0 \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left|\sqrt{h^2+k^2}\,\sin\left(\frac{1}{h+k}\right) - 0\right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\sqrt{h^2+k^2}\,\sin\left(\frac{1}{h+k}\right)\right| &\le \sqrt{h^2+k^2}\,\left|\sin\left(\frac{1}{h+k}\right)\right| \\ &\le \sqrt{h^2+k^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous.

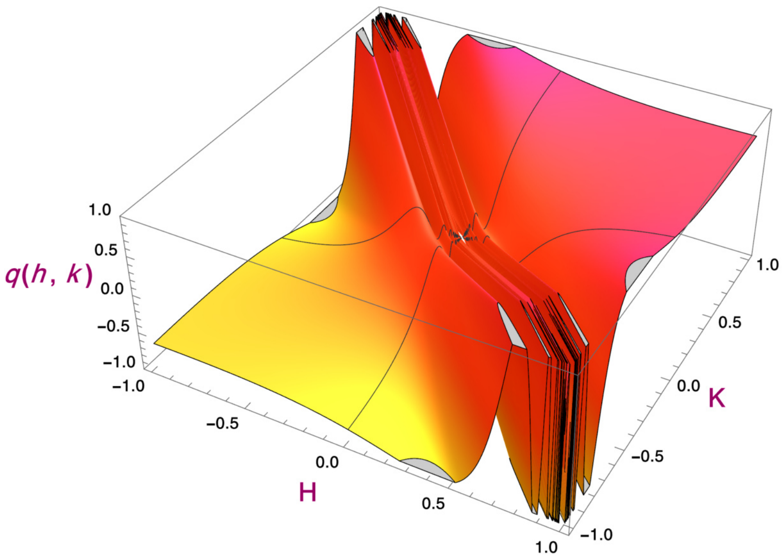

Figure 355. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

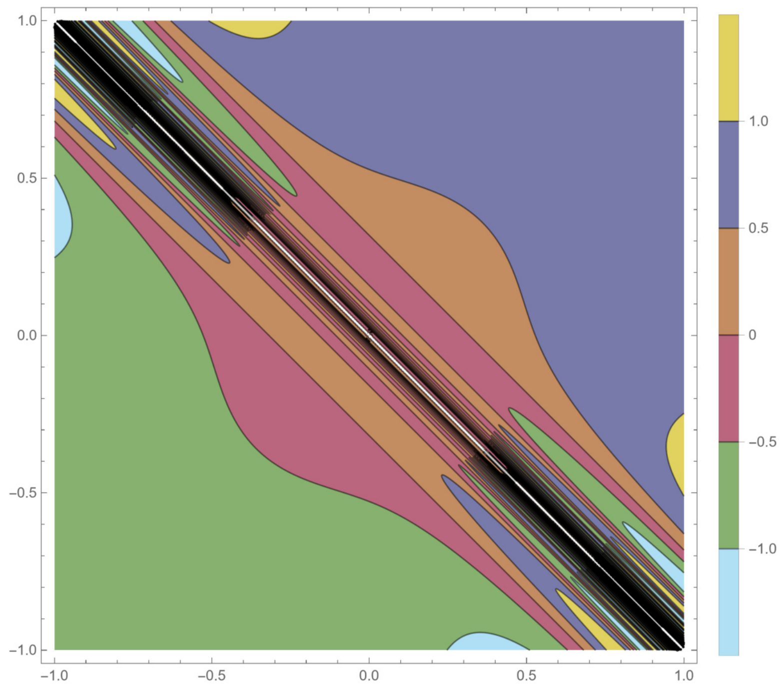

Figure 356. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

43.6 Alternative proof of differentiability (optional)

We have seen that we have that the function in points $(-a,a)$ is not continuous. So they cannot be Lipschitz continuous.

43.7 Continuity of the partial derivatives

The function has non existing directional derivatives in any neighbourhood of $(0,0)$. We cannot use this criterion.

43.8 Overview

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{x+y}\right) & \text{if } x+y \neq 0, \\ 0 & \text{if } x+y = 0. \end{cases}$$

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{x+y}\right) & \text{if } x+y \neq 0, \\ 0 & \text{if } x+y = 0. \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

Exercise 44.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{x^4+y^4}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

44.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \left(x^2+y^2\right) \sin\left(\frac{1}{x^4+y^4}\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\left| \left(x^2+y^2\right) \left|\sin\left(\frac{1}{x^4+y^4}\right)\right| \right| \le \left(x^2+y^2\right) \left|\sin\left(\frac{1}{x^4+y^4}\right)\right|$$
$$\le x^2+y^2$$
$$\le \sqrt{x^2+y^2}^2 .$$

It is sufficient to take $\delta = \epsilon^{1/2}$. We can find a δ, so we conclude that the function is continuous.

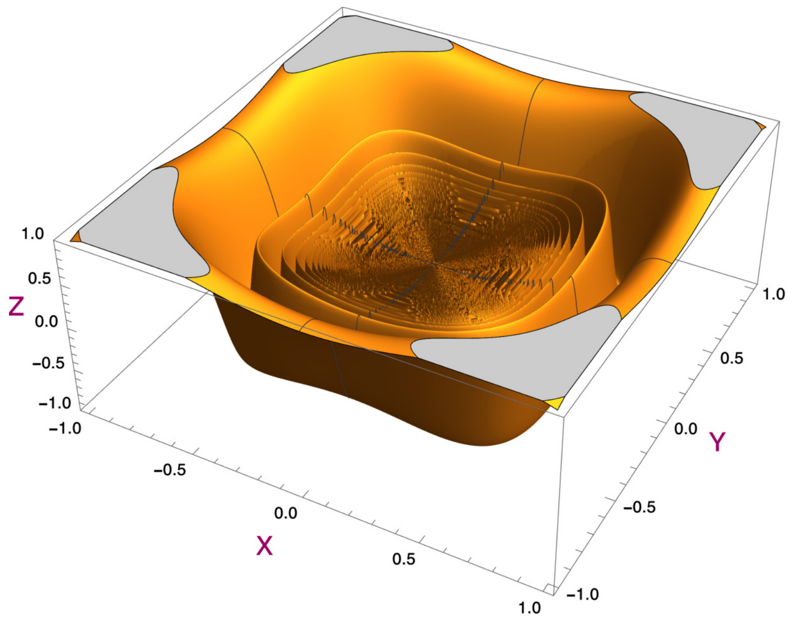

Figure 357. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

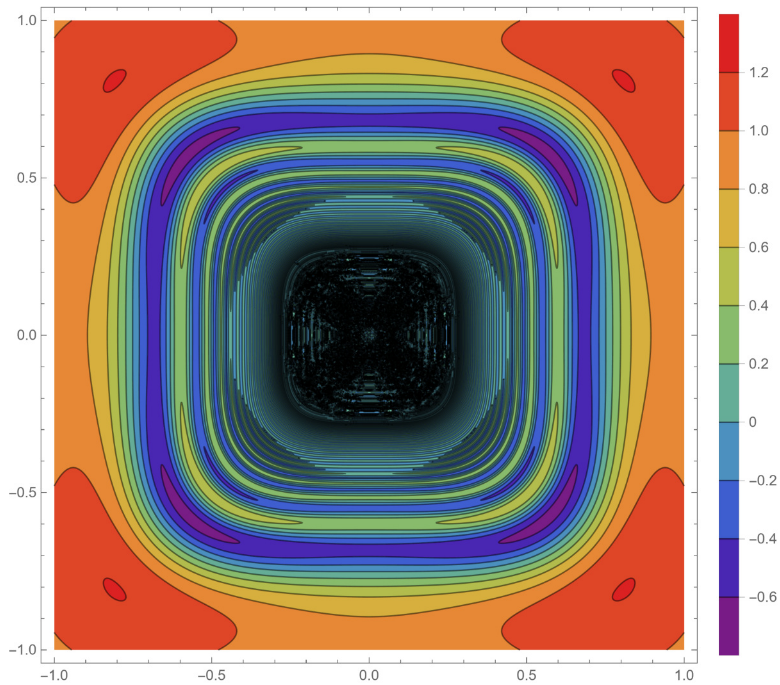

Figure 358. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

44.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f\,|_{y=0}(x,y) = \begin{cases} f(x,0) = x^2 \sin\left(\dfrac{1}{x^4}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} h\,\sin\left(\frac{1}{h^4}\right)\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = y^2\,\sin\left(\frac{1}{y^4}\right) & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} h\,\sin\left(\frac{1}{h^4}\right)\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

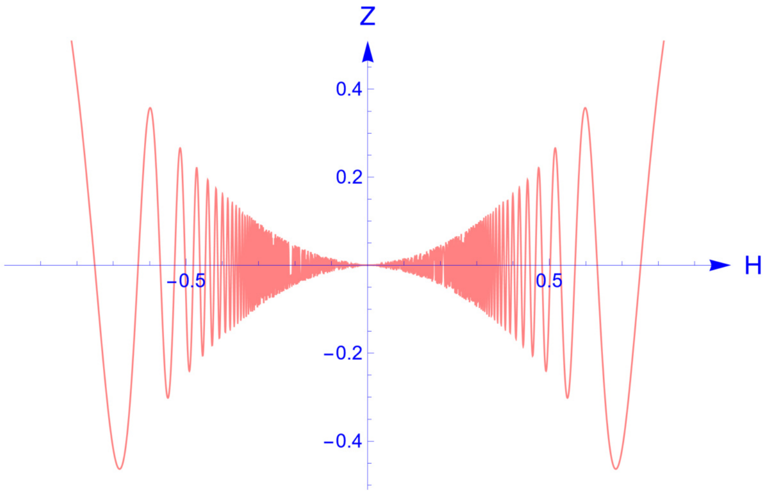

Figure 359. We see here a figure of the graph of the function restricted to the horizontal X-axis through $(0,0)$. This function is continuous and differentiable. We have plotted here the function $f(h,0)$.

44.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu,0+hv)-f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu,0+hv)-f(0,0)}{h} \\ &= \lim_{h\to 0} h\left(u^2+v^2\right)\sin\left(\frac{1}{h^4\left(u^4+v^4\right)}\right) \\ &= 0. \end{aligned}$$

So the directional derivatives do always exist.

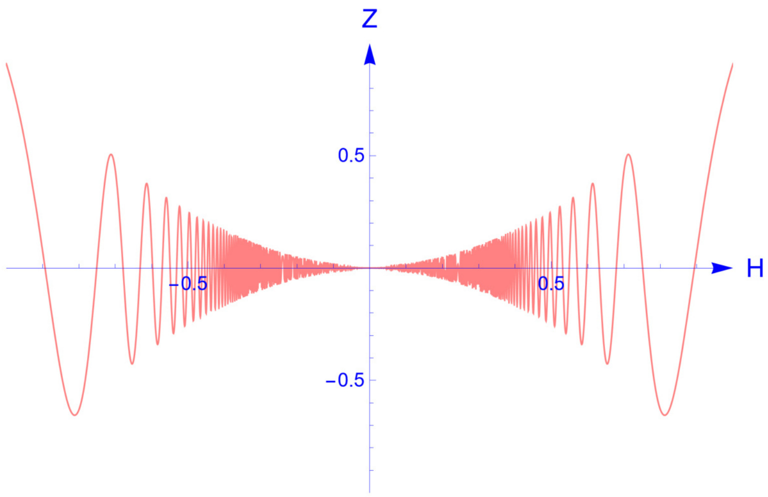

Figure 360. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$.

44.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} 2x \sin\left(\frac{1}{x^4+y^4}\right) - \frac{4x^3(x^2+y^2)\cos\left(\frac{1}{x^4+y^4}\right)}{(x^4+y^4)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We do not think we can apply this criterion because the partial derivative to x is not bounded in any neighbourhood of $(0,0)$. Let us analyse this further.

The first term in the main definition is continuous and so locally bounded. The second term looks suspicious. Let us call this main term $h(x,y)$. We have then

$$h(x,y) = -\frac{4x^3\,(x^2+y^2)\cos\left(\frac{1}{x^4+y^4}\right)}{(x^4+y^4)^2}.$$

Let us restrict by taking $y = \lambda\, x$ and $\lambda = 1$. We have then

$$h(x,x) = -\frac{2\cos\left(\frac{1}{2x^4}\right)}{x^3}.$$

This term is certainly unbounded. We can be very specific. Let us define the sequence $x_n = \frac{1}{\sqrt{2}\,\sqrt[4]{\pi}\,\sqrt[4]{n}}$, $n \in \mathbf{N}_0$. Because $y = x$, we have also $y_n = \frac{1}{\sqrt{2}\,\sqrt[4]{\pi}\,\sqrt[4]{n}}$. These sequences converge both to 0. We calculate the image of this sequence.

$$h(x_n,y_n) = h\left(\frac{1}{\sqrt{2}\,\sqrt[4]{\pi}\,\sqrt[4]{n}}, \frac{1}{\sqrt{2}\,\sqrt[4]{\pi}\,\sqrt[4]{n}}\right) = -4\sqrt{2}\,\pi^{3/4}\,n^{3/4}.$$

Let us sketch this partial derivative with $y = \lambda\, x$ and $\lambda = 1$.

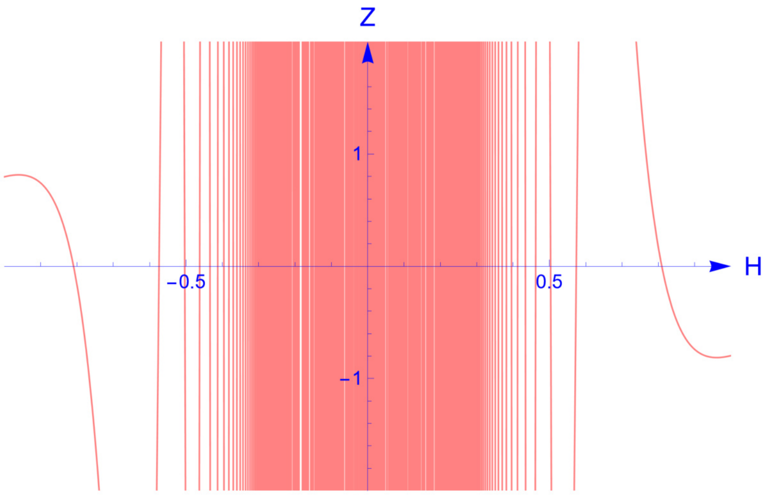

Figure 361. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with equation $y = x$. The predicted unboundedness is not explicit but suggested. We have drawn here the function $f(h,h)$.

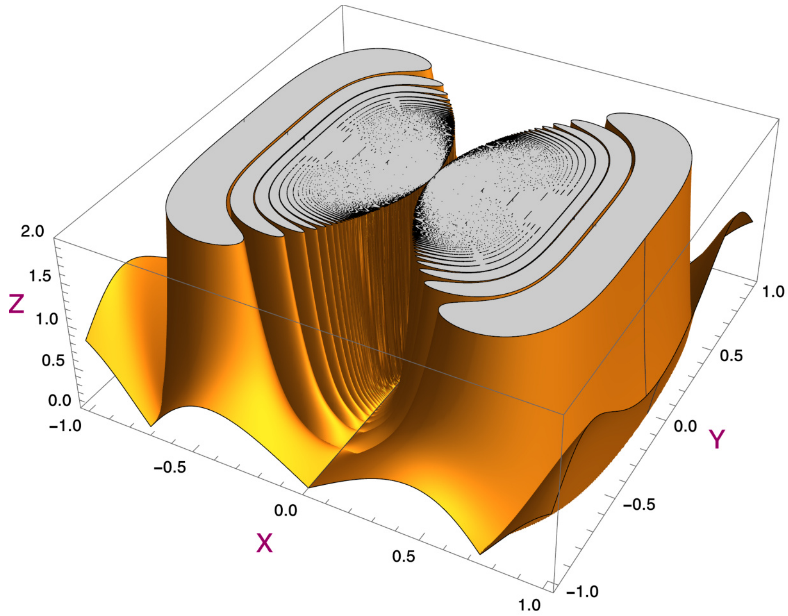

Figure 362. We see here the absolute value of the first partial derivative $\left|\frac{\partial f}{\partial x}\right|$. We can observe the unboundedness from this picture.

We do not draw the absolute value of the second partial derivative. This will be a similar graph by symmetry reasons.

44.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

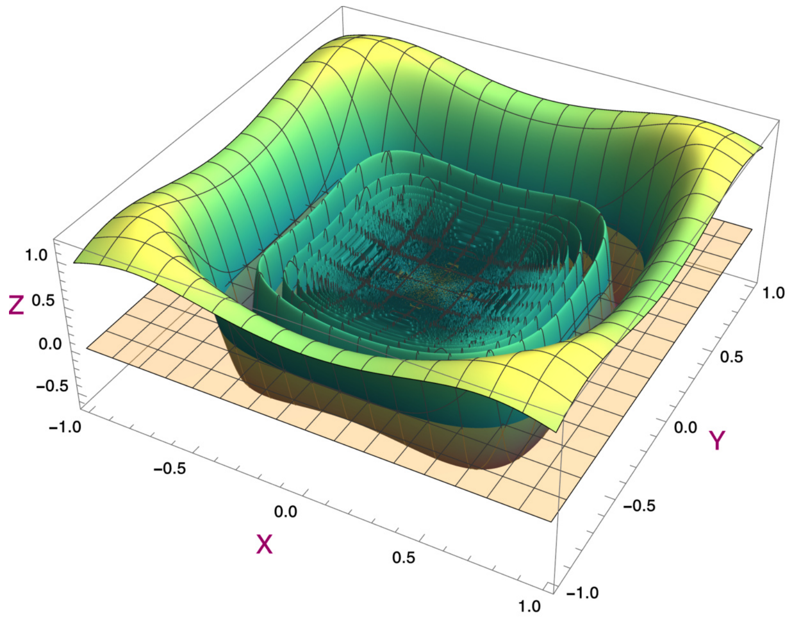

Figure 363. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function very nicely.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

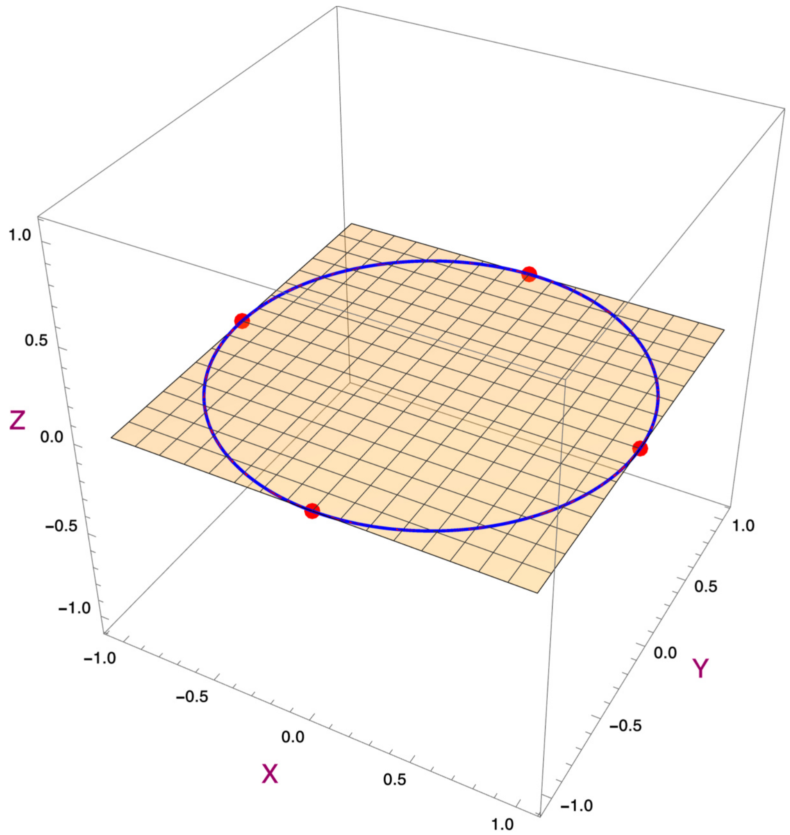

Figure 364. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \sqrt{h^2+k^2}\,\sin\left(\frac{1}{h^4+k^4}\right) & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left|\sqrt{h^2+k^2}\,\sin\left(\frac{1}{h^4+k^4}\right) - 0\right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\sqrt{h^2+k^2}\sin\left(\frac{1}{h^4+k^4}\right)\right| &\le \sqrt{h^2+k^2}\left|\sin\left(\frac{1}{h^4+k^4}\right)\right| \\ &\le \sqrt{h^2+k^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous.

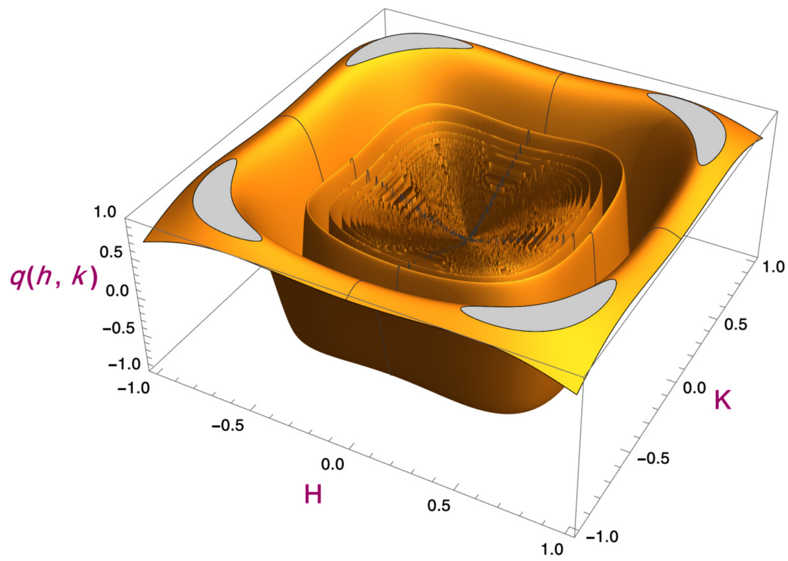

Figure 365. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

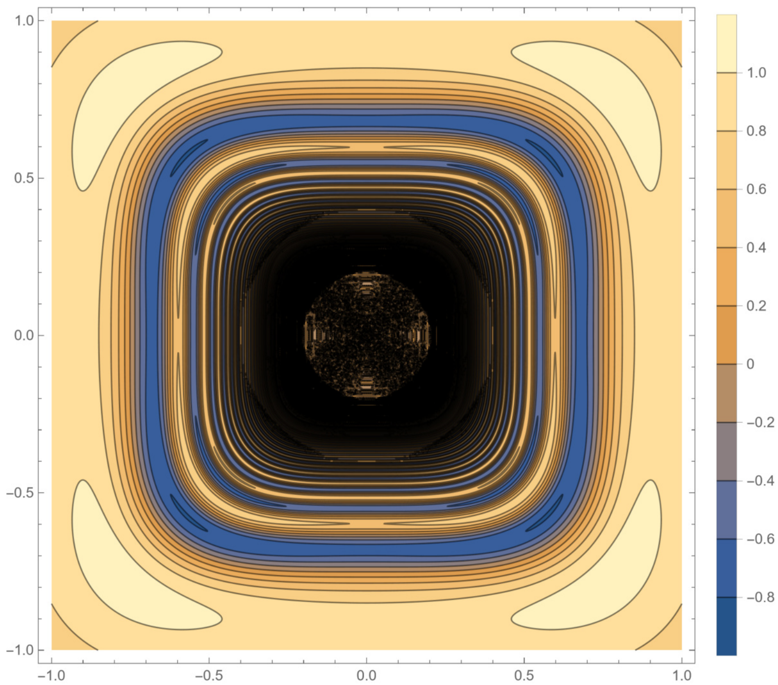

Figure 366. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

44.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\, u + \frac{\partial f}{\partial y}(0,0)\, v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the func-

tion $f(x, y)$ is locally Lipschitz continuous in $(0, 0)$. We cite here the criterion that we will use.

A function is differentiable in (a, b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.

2. The directional derivatives have the following form: $D_{(u,v)}(a, b) = \nabla f(a, b) \cdot (u, v) = \frac{\partial f}{\partial x}(a, b)\, u + \frac{\partial f}{\partial y}(a, b)\, v$.

3. The function is locally Lipschitz continuous in (a, b). This means that there exists at least one neighbourhood of (a, b) and a number $K > 0$ such that for all (x_1, y_1) and (x_2, y_2) in the neighbourhood, we have $|f(x_1, y_1) - f(x_2, y_2)| < K \left\|(x_1, y_1) - (x_2, y_2)\right\|$. Remark that the same K must be valid for all (x_1, y_1) and (x_2, y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be “locally flat" and thus “locally linear" which is partly expressed by this Lipschitz continuity condition.

The function cannot be Lipschitz continuous because the first partial derivative is unbounded. We have calculated that fact in section 44.4. So we cannot apply this particular criterion for differentiability.

44.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

The partial derivative to y is not bounded in any neighbourhood of $(0, 0)$. So this derivative is not continuous in $(0, 0)$. We conclude that this particular criterion cannot be used. Please consult section 44.4.

44.8 Overview

$$f(x,y) = \begin{cases} (x^2+y^2)\,\sin\left(\dfrac{1}{x^4+y^4}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

Exercise 45.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

45.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \left(x^2+y^2\right) \sin\left(\frac{1}{x^2+y^2}\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned} \left| \left(x^2+y^2\right) \sin\left(\frac{1}{x^2+y^2}\right) \right| &\leq \left(x^2+y^2\right) \left| \sin\left(\frac{1}{x^2+y^2}\right) \right| \\ &\leq x^2+y^2. \end{aligned}$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function is continuous.

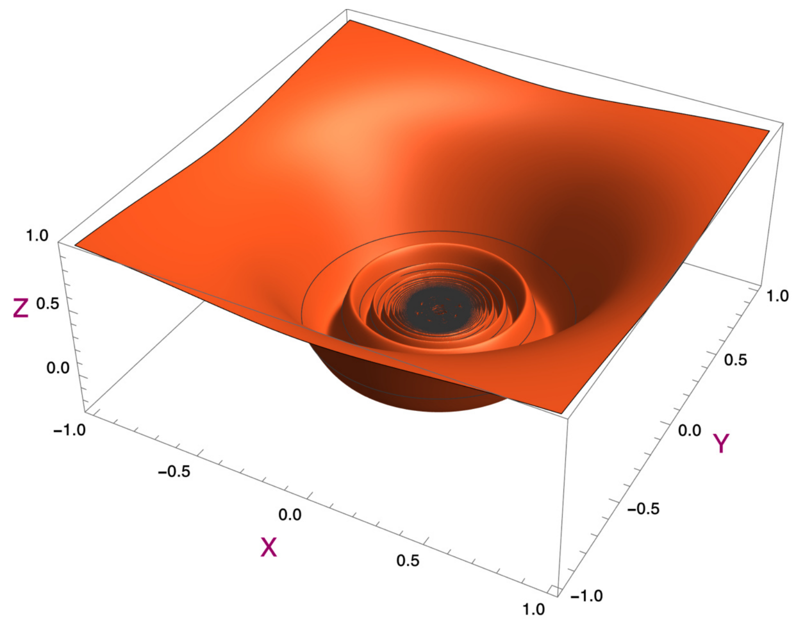

Figure 367. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

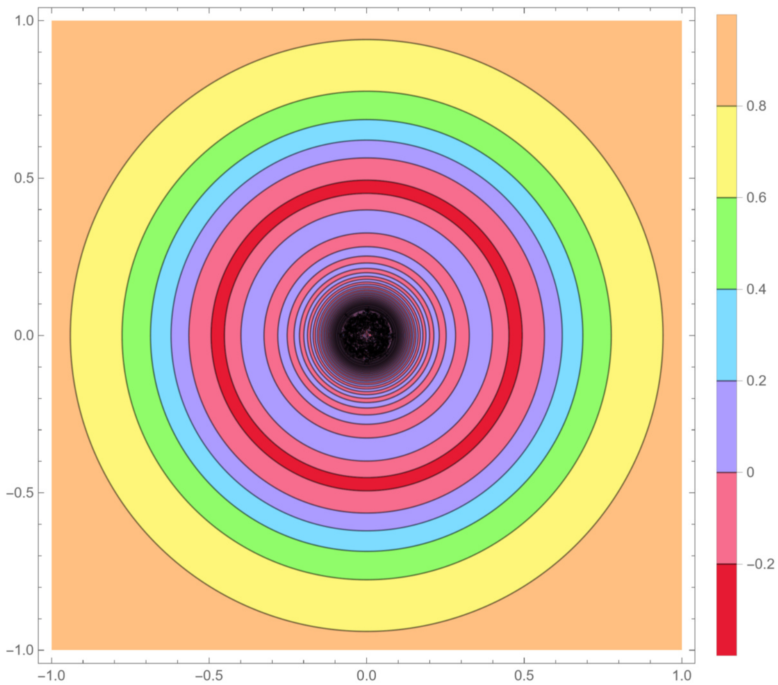

Figure 368. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

45.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f\big|_{y=0}(x,y) = \begin{cases} f(x,0) = x^2 \sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} h\sin\left(\frac{1}{h^2}\right)\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
The calculations are very similar here due to the symmetry of the function.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

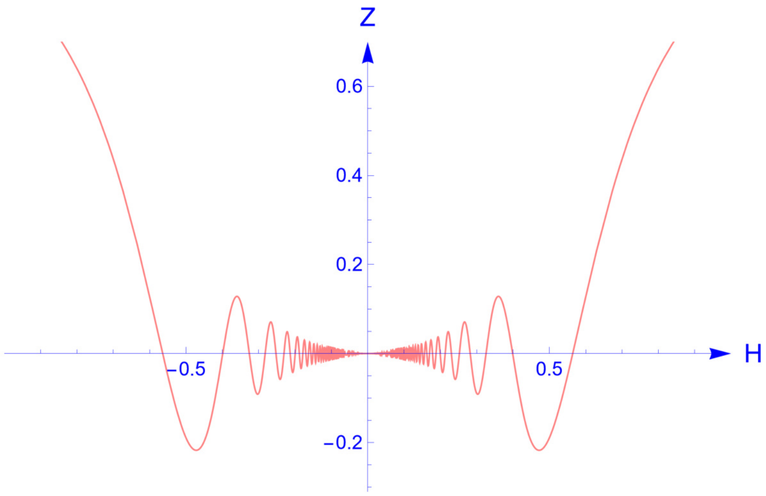

Figure 369. We see here a figure of the graph of the function restricted to the horizontal X-axis through $(0,0)$. We have plotted here the function $f(h,0)$.

45.3 Directional derivatives

Because of the symmetry caused by the function definition, we conclude that all directional derivatives exist and are 0.

45.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.
Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} 2x\sin\left(\frac{1}{x^2+y^2}\right) - \frac{2x\cos\left(\frac{1}{x^2+y^2}\right)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We see that the first term in the main definition is continuous and thus locally bounded. The second term gives rise to suspicion. Let us call the second term $h(x,y)$.

$$h(x,y) = -\frac{2x\cos\left(\frac{1}{x^2+y^2}\right)}{x^2+y^2}.$$

We substitute $y = x$. Then we have

$$h(x,x) = -\frac{\cos\left(\frac{1}{2x^2}\right)}{x}.$$

This term is unbounded. To be more specific, let us define the sequence $x_n = \frac{1}{2\sqrt{\pi}\sqrt{n}}$, $n \in \mathbf{N}_0$. Then because $y = x$, $y_n = \frac{1}{2\sqrt{\pi}\sqrt{n}}$, $n \in \mathbf{N}_0$. These sequences converge to 0. We calculate now the image.

$$h(x_n, y_n) = -2\sqrt{\pi}\sqrt{n}.$$

This illustrates the unboundedness of the partial derivative.

So we cannot use this particular criterion for continuity.

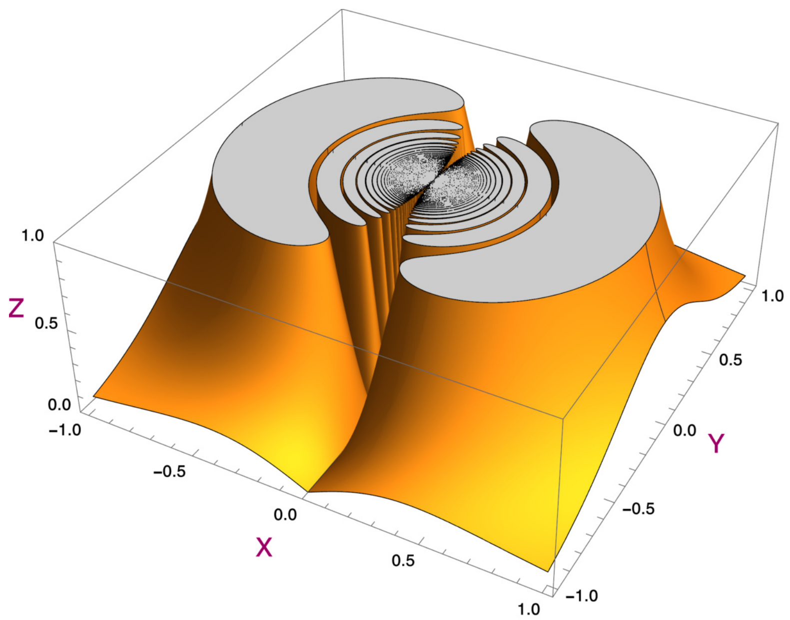

Figure 370. We see here the absolute value of the first partial derivative $\left|\frac{\partial f}{\partial x}\right|$. We can observe the unboundedness from this picture.

45.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

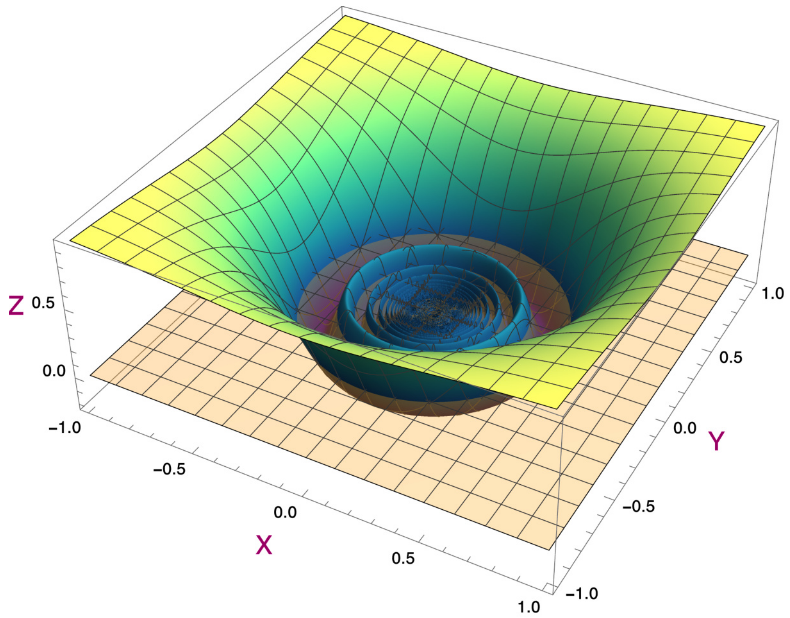

Figure 371. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function very nicely.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

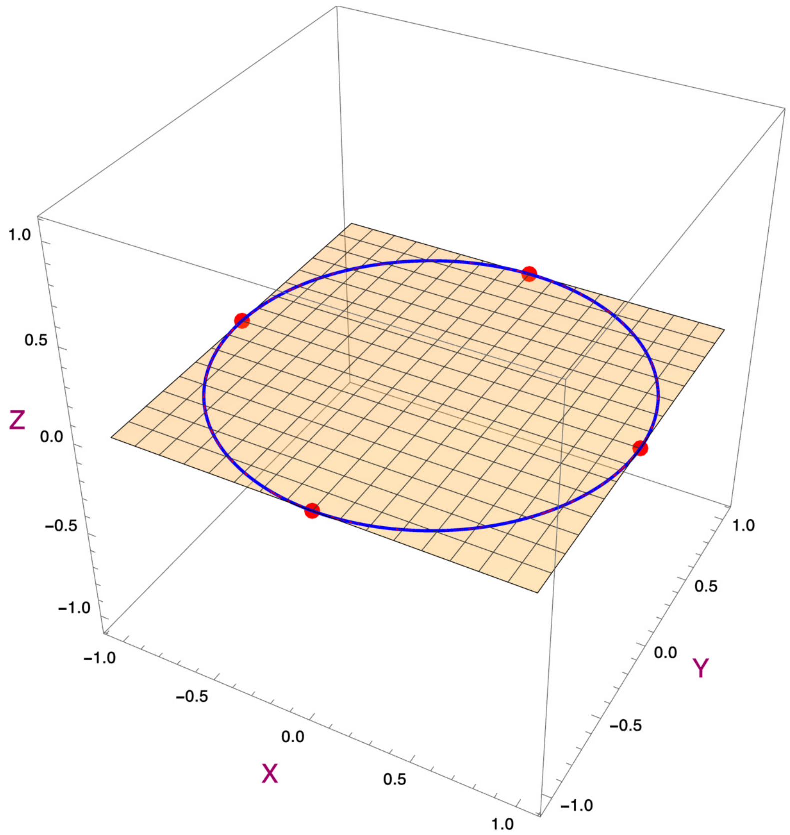

Figure 372. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h, k)$ in $(0, 0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \sqrt{h^2+k^2}\,\sin\left(\dfrac{1}{h^2+k^2}\right) & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left|\sqrt{h^2+k^2}\,\sin\left(\frac{1}{h^2+k^2}\right) - 0\right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\left|\sqrt{h^2+k^2}\,\sin\left(\frac{1}{h^2+k^2}\right)\right| \le \sqrt{h^2+k^2}\,\left|\sin\left(\frac{1}{h^2+k^2}\right)\right|$$
$$\le \sqrt{h^2+k^2}.$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous.

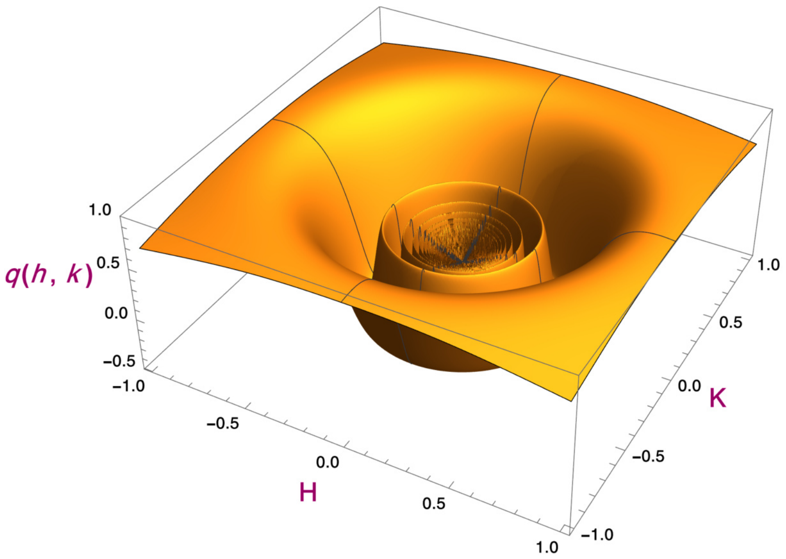

Figure 373. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

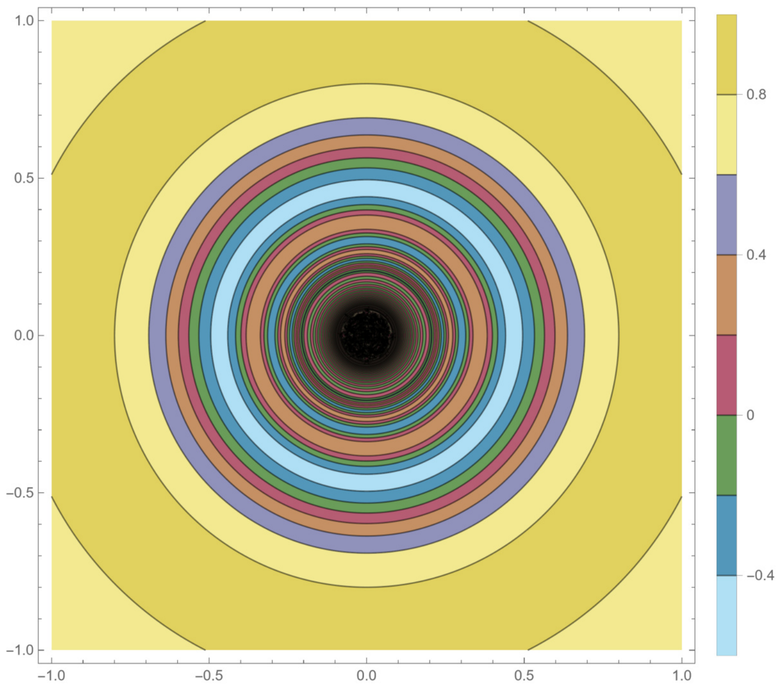

Figure 374. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

45.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\, u + \frac{\partial f}{\partial y}(0,0)\, v\,.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the func-

tion $f(x, y)$ is locally Lipschitz continuous in $(0, 0)$. We cite here the criterion that we will use.

A function is differentiable in (a, b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.

2. The directional derivatives have the following form: $D_{(u,v)}(a, b) = \nabla f(a, b) \cdot (u, v) = \frac{\partial f}{\partial x}(a, b)\, u + \frac{\partial f}{\partial y}(a, b)\, v$.

3. The function is locally Lipschitz continuous in (a, b). This means that there exists at least one neighbourhood of (a, b) and a number $K > 0$ such that for all (x_1, y_1) and (x_2, y_2) in the neighbourhood, we have $|f(x_1, y_1) - f(x_2, y_2)| < K \|(x_1, y_1) - (x_2, y_2)\|$. Remark that the same K must be valid for all (x_1, y_1) and (x_2, y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be "locally flat" and thus "locally linear" which is partly expressed by this Lipschitz continuity condition.

Lipschitz continuity is not possible because the first partial derivative is not bounded in any neighbourhood of $(0, 0)$. See section 45.4. So the function cannot be Lipschitz continuous.

45.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

We have seen that the first partial derivative is unbounded in any neighbourhood of $(0, 0)$. See section 45.4. Continuity is then impossible in $(0, 0)$.

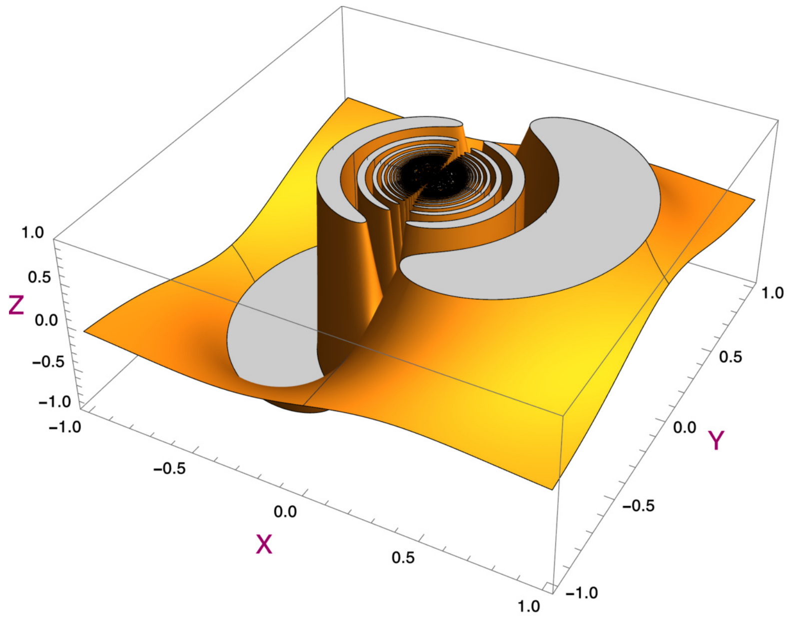

Figure 375. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial x}(x, y)$. This looks like an unbounded function.

We will not repeat those remarks for the second partial derivative to y. The computations are completely similar due to the rotational symmetries of the function.

45.8 Overview

$$f(x,y) = \begin{cases} (x^2+y^2)\,\sin\left(\dfrac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

Exercise 46.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} x\,y^2 \sin\left(\frac{1}{y}\right) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

46.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| x\,y^2 \sin\left(\frac{1}{y}\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|x\,y^2 \sin\left(\frac{1}{y}\right)\right| &\le |x|\,y^2 \left|\sin\left(\frac{1}{y}\right)\right| \\ &\le \sqrt{x^2+y^2}\,\sqrt{x^2+y^2}^{\,2} \\ &\le \sqrt{x^2+y^2}^{\,3}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon^{1/3}$. We can find a δ, so we conclude that the function is continuous.

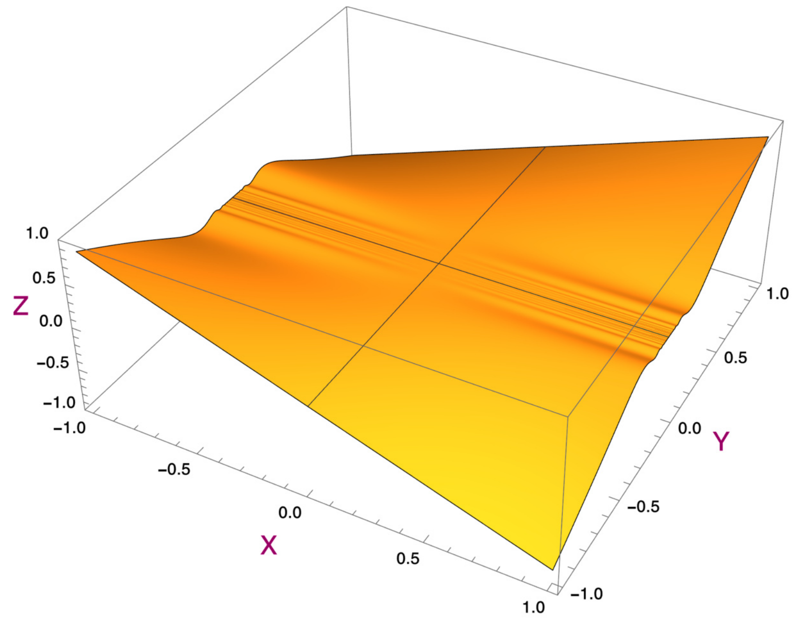

Figure 376. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

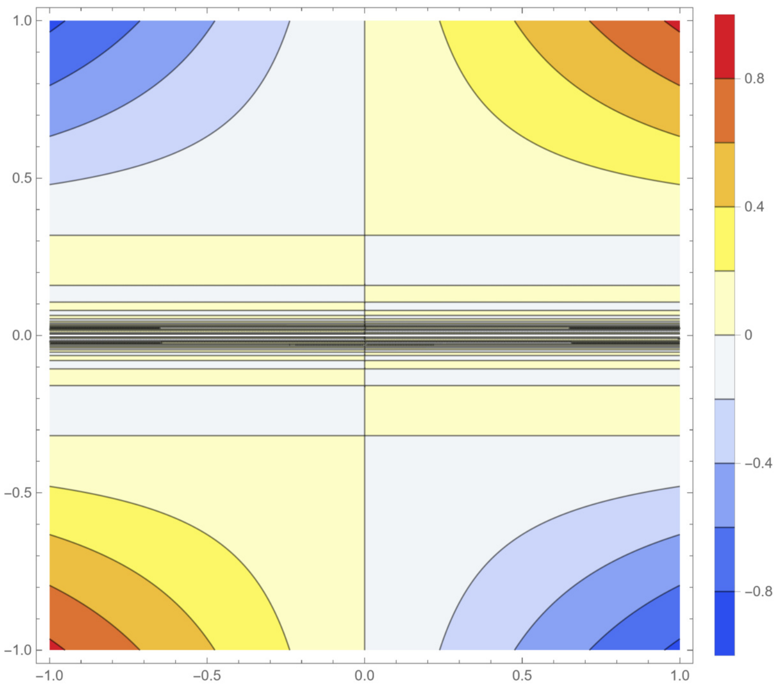

Figure 377. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

46.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

46.3 Directional derivatives

Let (u, v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2 + v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u, v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} h^2 u\, v^2 \sin\left(\frac{1}{h\,v}\right) \\ &= 0. \end{aligned}$$

We can use the squeeze theorem for this calculation.

So the directional derivatives do always exist.

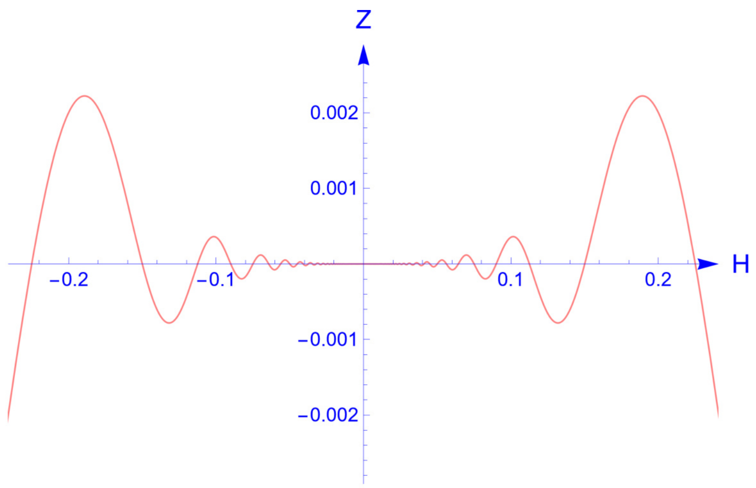

Figure 378. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

46.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.
Remember that we already calculated the partial derivatives in $(0,0)$.

Looking for a definition of the partial derivatives. We have to be able to define the partial derivatives in at least one neighbourhood around $(0,0)$. We have no problems with points that are in the interior of the definition domains of the classical functions. We have there the classical calculation rules for defining those functions and the partial derivatives always exist there and are even continuous.

But in our case, we have to investigate the points that are in exceptional subsets in the definition of the function.

In this case these are the points $(a,0)$. We do not have to investigate other points.

The partial derivative to x exists in the points $(a,0)$ because the function is then by definition identically 0. So the partial derivative to x is also identically 0.

We draw our attention now to the partial derivative to y. Let us look at a point $(a,0)$. We are going to investigate the function in $(a,0)$ in the Y-direction. This function is defined by

$$f(a,h) = \begin{cases} a\,h^2 \sin\left(\frac{1}{h}\right) & \text{if } h \neq 0, \\ 0 & \text{if } h = 0. \end{cases}$$

We calculate the limit

$$\frac{\partial f}{\partial y}(a,0) = \lim_{h\to 0} \frac{a\,h^2 \sin\left(\frac{1}{h}\right)}{h} = 0.$$

We see that this function is differentiable in $h = 0$. We conclude that the partial derivative $\frac{\partial f}{\partial y}(a,0)$ does exist and is equal to 0.

We consult a figure for this last observation.

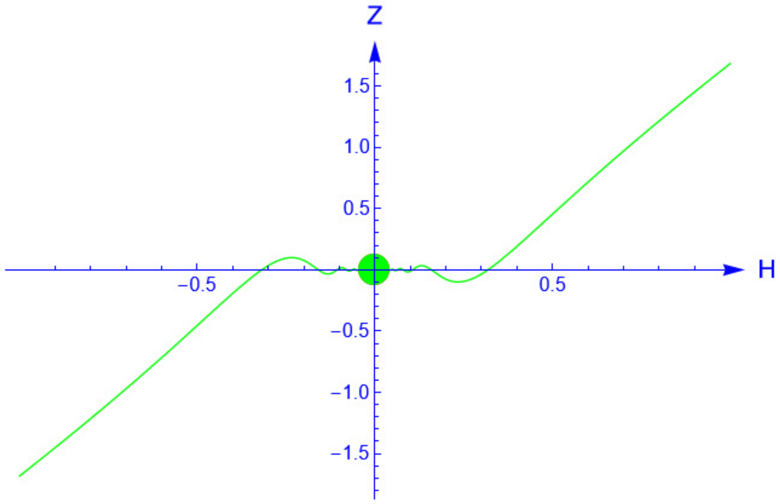

Figure 379. We see here a figure of the graph of the function restricted to the line through $(a,0)$ with direction $(0,1)$, this is the Y-direction. We see that this function is differentiable in $h = 0$. We have plotted here the function $f(a,h)$.

The partial derivative to x is:

$$\frac{\partial f}{\partial x} = \begin{cases} y^2 \sin\left(\frac{1}{y}\right) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

The partial derivative to y is together with the observations we made:

$$\frac{\partial f}{\partial y} = \begin{cases} x\left(2\,y \sin\left(\frac{1}{y}\right) - \cos\left(\frac{1}{y}\right)\right) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

$$\begin{aligned}
\left|\frac{\partial f}{\partial x}(x,y)\right| &\le \left|y^2 \sin\left(\frac{1}{y}\right)\right| \\
&\le y^2 \left|\sin\left(\frac{1}{y}\right)\right| \\
&\le y^2 \\
&\le \sqrt{x^2+y^2}^{\,2} \\
&\le 1.
\end{aligned}$$

We have chosen here the restriction to the neighbourhood defined by $\sqrt{x^2+y^2} < 1$ in order to get the last inequality.

Let us try to prove that $\left|\frac{\partial f}{\partial y}\right|$ is bounded.

$$\begin{aligned}
\left|\frac{\partial f}{\partial y}(x,y)\right| &\le \left|x\left(2\,y\,\sin\left(\frac{1}{y}\right) - \cos\left(\frac{1}{y}\right)\right)\right| \\
&\le |x|\left(2\,|y|\left|\sin\left(\frac{1}{y}\right)\right| + \left|\cos\left(\frac{1}{y}\right)\right|\right) \\
&\le |x|\,(2\,|y|+1) \\
&\le \sqrt{x^2+y^2}\left(2\sqrt{x^2+y^2}+1\right) \\
&\le 3.
\end{aligned}$$

We have chosen here the restriction to the neighbourhood defined by $\sqrt{x^2+y^2} < 1$ in order to get the last inequality.

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

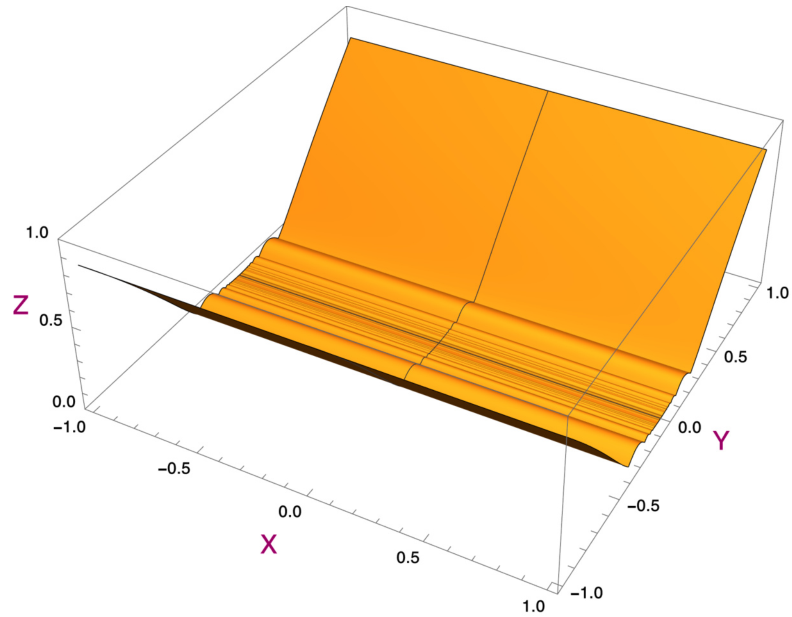

Figure 380. We see here the absolute value of the first partial derivative $\left|\dfrac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

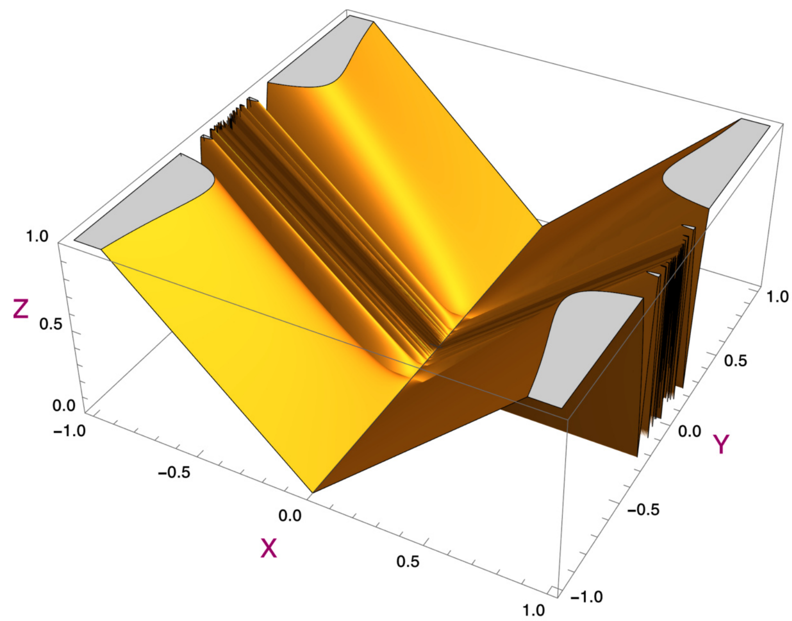

Figure 381. We see here the absolute value of the second partial derivative $\left|\dfrac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture.

46.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

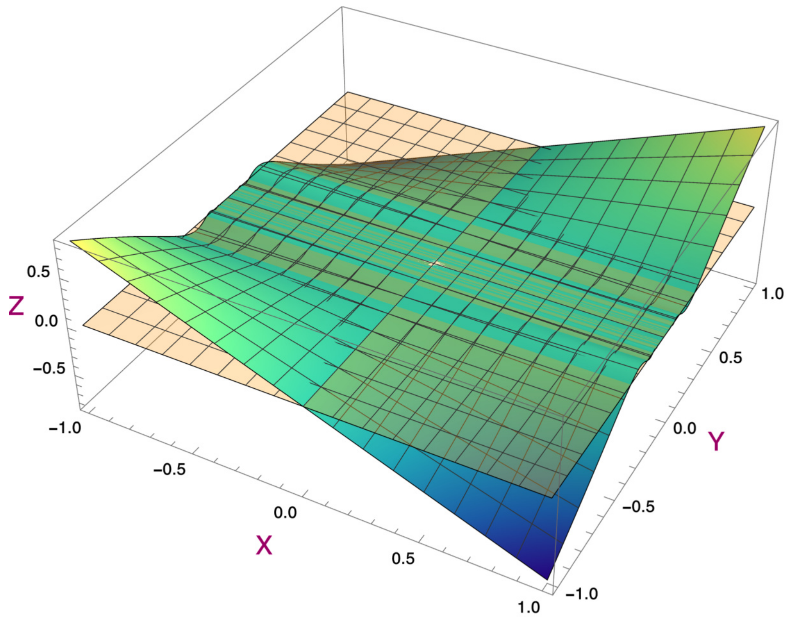

Figure 382. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function very nicely.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we

know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

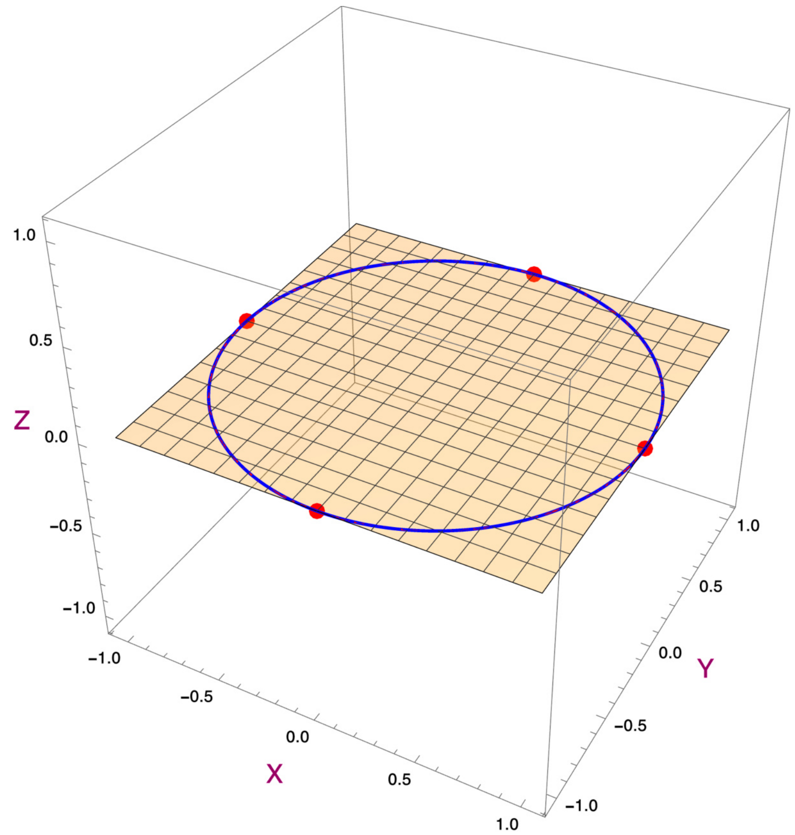

Figure 383. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h\,k^2 \sin\left(\dfrac{1}{k}\right)}{\sqrt{h^2+k^2}} & \text{if } (h,k) \neq (0,0) \text{ and } k \neq 0; \\ 0 & \text{if } (h,k) = (0,0) \text{ or } k = 0 \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left| \frac{h\,k^2 \sin\left(\frac{1}{k}\right)}{\sqrt{h^2+k^2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left|\frac{h\,k^2\,\sin\left(\frac{1}{k}\right)}{\sqrt{h^2+k^2}}\right| &\le \frac{|h|\,k^2\,\left|\sin\left(\frac{1}{k}\right)\right|}{\sqrt{h^2+k^2}} \\
&\le \frac{|h|\,k^2}{\sqrt{h^2+k^2}} \\
&\le \frac{\sqrt{h^2+k^2}\,\sqrt{h^2+k^2}^{\,2}}{\sqrt{h^2+k^2}} \\
&\le \sqrt{h^2+k^2}^{\,2}.
\end{aligned}$$

It is sufficient to take $\delta = \epsilon^{1/2}$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous.

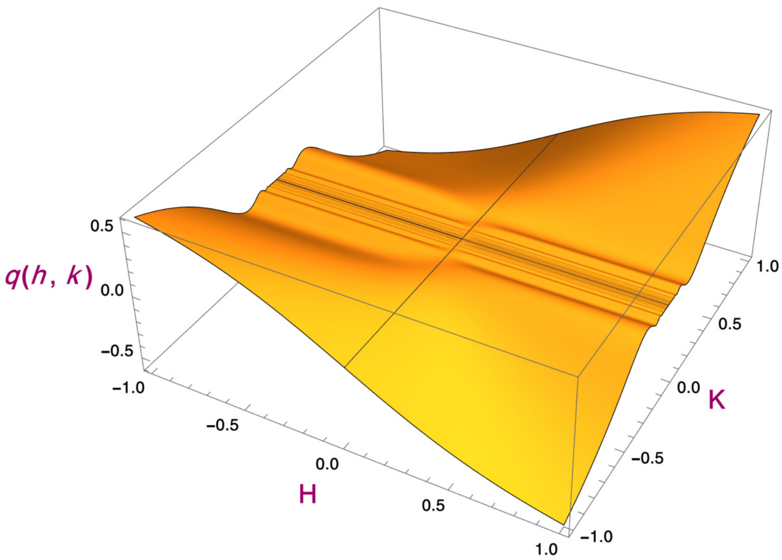

Figure 384. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

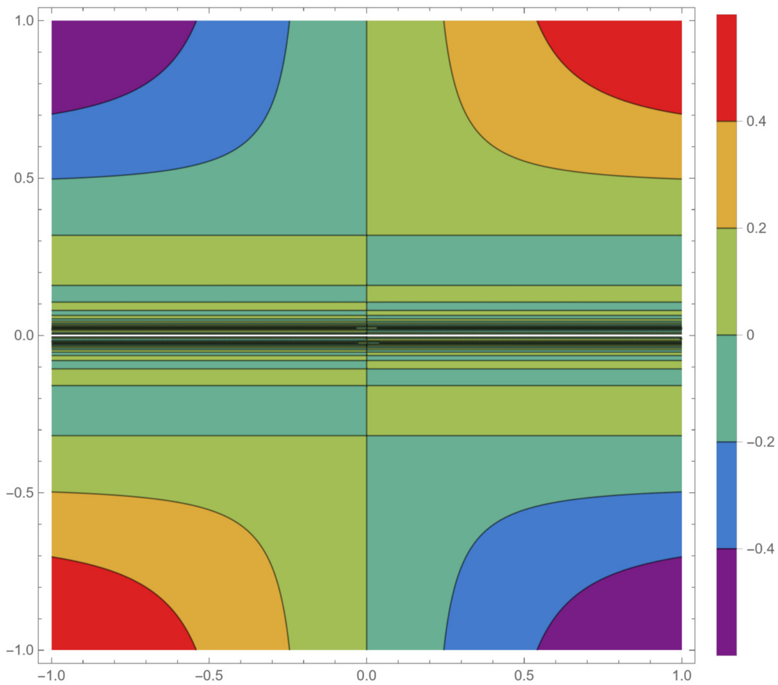

Figure 385. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

46.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u,v,D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\,u + \frac{\partial f}{\partial y}(0,0)\,v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the func-

tion $f(x, y)$ is locally Lipschitz continuous in $(0, 0)$. We cite here the criterion that we will use.

A function is differentiable in (a, b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.

2. The directional derivatives have the following form: $D_{(u,v)}(a, b) = \nabla f(a, b) \cdot (u, v) = \frac{\partial f}{\partial x}(a, b)\, u + \frac{\partial f}{\partial y}(a, b)\, v$.

3. The function is locally Lipschitz continuous in (a, b). This means that there exists at least one neighbourhood of (a, b) and a number $K > 0$ such that for all (x_1, y_1) and (x_2, y_2) in the neighbourhood, we have $|f(x_1, y_1) - f(x_2, y_2)| < K \left\|(x_1, y_1) - (x_2, y_2)\right\|$. Remark that the same K must be valid for all (x_1, y_1) and (x_2, y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be “locally flat" and thus “locally linear" which is partly expressed by this Lipschitz continuity condition.

Let us try to prove the Lipschitz condition. We are going to use a well known method.

$$
\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)| \\
&\qquad = |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\qquad \le |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)|.
\end{aligned}
$$

We focus now on the first term $|f(x_1, y_1) - f(x_1, y_2)|$. We fix now x_1 and look upon $f(x_1, y)$ as a function in one variable y. So it springs to mind that we can estimate this by using Lagrange’s intermediate value theorem. So we have

$$
|f(x_1, y_1) - f(x_1, y_2)| = \left|\frac{\partial f}{\partial y}(x_1, \xi)\right| \; |y_1 - y_2|
$$

where ξ is a number lying in the open interval (y_1, y_2) if e.g. $y_1 \leq y_2$. Remark that we already have proven that the partial derivatives are bounded in a neighbourhood, we can take that bound found for $\frac{\partial f}{\partial y}$, say M_2 and that neighbourhood in this case also. So $\left|\frac{\partial f}{\partial y}(x_1, \xi)\right| \leq M_2$. We work in a completely analogous way for the second term $|f(x_1, y_2) - f(x_2, y_2)|$. We have in that case $\left|\frac{\partial f}{\partial x}(\xi, y_2)\right| \leq M_1$ where ξ is now a number in the open interval (x_1, x_2) if e.g. $x_1 \leq x_2$.

We take up our inequality again

$$
\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)| \\
&\quad \leq |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\quad \leq |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)| \\
&\quad \leq M_2 \, |(y_1 - y_2)| + M_1 \, |(x_1 - x_2)| \\
&\quad \leq M_2 \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + M_1 \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&\quad \leq (M_1 + M_2) \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.
\end{aligned}
$$

So this function is locally Lipschitz continuous. We have an alternative proof for the differentiability.

46.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

We already now that the partials exist everywhere in at least one neighbourhood of $(0, 0)$. The question is now if they are continuous.

We have to distinguish three cases. The first case is where $y \neq 0$. The second case is the point $(0,0)$ itself. The third case is in the points $(a,0)$ with $a \neq 0$.

The **first** case is a region in which the partial derivative to x is composed of classical functions which are known to be infinitely differentiable.

The **second** case is the point $(0,0)$.

Discussion of the continuity of the first partial derivative in $(0,0)$.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove then that the inequality $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$ holds under certain conditions. The problem is now to find a $\delta > 0$ such that if the inequality $||(x,y) - (0,0)|| < \delta$ holds, then it follows that $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left|\frac{\partial f}{\partial x}\right| &\leq \left|y^2 \sin\left(\frac{1}{y}\right)\right| \\
&\leq y^2 \left|\sin\left(\frac{1}{y}\right)\right| \\
&\leq y^2 \\
&\leq \sqrt{x^2 + y^2}^2 .
\end{aligned}$$

It is sufficient to take $\delta = \epsilon^{1/2}$. We can find a δ, so we conclude that the function $\frac{\partial f}{\partial x}$ is continuous.

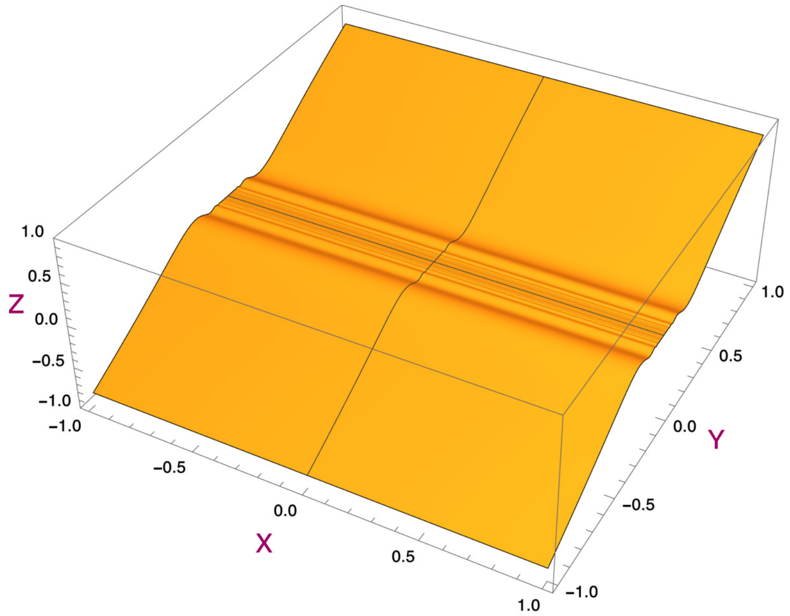

Figure 386. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial x}(x, y)$. This looks like a continuous function in $(0, 0)$.

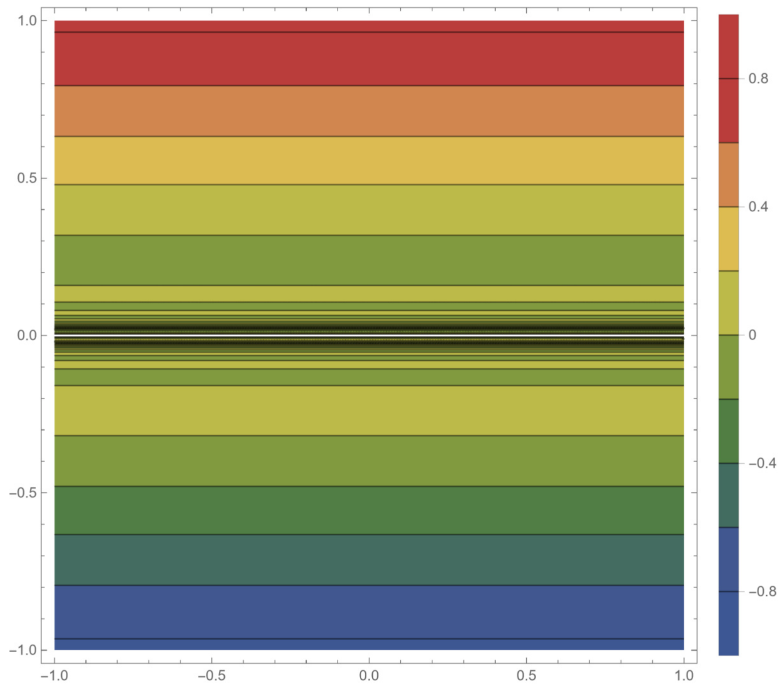

Figure 387. We see here a figure of the contour plot of the $\frac{\partial f}{\partial x}(x, y)$. Only level curves of level around 0 come close to $(0, 0)$. This looks like a continuous function in $(0, 0)$.

The **third** case is in the points $(a, 0)$ with $a \neq 0$. We remember that the partial derivative is

$$\frac{\partial f}{\partial x}(x, y) = \begin{cases} y^2 \sin\left(\frac{1}{y}\right) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

We have to prove continuity in $(a, 0)$. We start from the inequality $|\frac{\partial f}{\partial x}(x, y) - \frac{\partial f}{\partial x}(a, 0)| < \epsilon$ and look for a δ such that if $\sqrt{(x-a)^2 + y^2}^2 < \delta$ holds, the inequality $|\frac{\partial f}{\partial x}(x, y) - \frac{\partial f}{\partial x}(a, 0)| < \epsilon$ also holds.

$$\begin{aligned} \left| y^2 \sin\left(\frac{1}{y}\right) - 0 \right| &\leq \left| y^2 \right| \\ &\leq \sqrt{(x-a)^2 + y^2}^2. \end{aligned}$$

So we can take $\delta = \epsilon^{1/2}$ and the partial derivative $\frac{\partial f}{\partial x}$ is in $(a, 0)$ continuous.

Discussion of the continuity of the second partial derivative in $(0, 0)$.

Let us now investigate the continuity of the partial derivative to y.

We have to distinguish three cases. The first case is where $y \neq 0$. The second case is the point $(0, 0)$ itself. The third case is in the points $(a, 0)$ with $a \neq 0$.

We remember

$$\frac{\partial f}{\partial y} = \begin{cases} x\left(2\,y\,\sin\left(\frac{1}{y}\right) - \cos\left(\frac{1}{y}\right)\right) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

The **first** case is a region in which the partial derivative to x is composed of classical functions which are known to be infinitely differentiable.

The **second** case is the point $(0, 0)$.

Discussion of the continuity in $(0, 0)$.
We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that the inequality $|\frac{\partial f}{\partial y}(x, y) - \frac{\partial f}{\partial y}(0, 0)| < \epsilon$ holds subject to conditions. The problem is now to find a $\delta > 0$ such that if $||(x, y) - (0, 0)|| < \delta$ it follows that $|\frac{\partial f}{\partial y}(x, y) - \frac{\partial f}{\partial y}(0, 0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0, 0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left|\frac{\partial f}{\partial y}(x, y) - \frac{\partial f}{\partial y}(0, 0)\right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left|\frac{\partial f}{\partial y}(x,y)\right| &\leq \left|x\left(2\,y\,\sin\left(\frac{1}{y}\right)-\cos\left(\frac{1}{y}\right)\right)\right| \\
&\leq |x|\left(2\,|y|\,\left|\sin\left(\frac{1}{y}\right)\right|+\left|\cos\left(\frac{1}{y}\right)\right|\right) \\
&\leq |x|\,(2\,|y|+1) \\
&\leq \sqrt{x^2+y^2}\,(2\,\sqrt{x^2+y^2}+1) \\
&\leq 3\,\sqrt{x^2+y^2}.
\end{aligned}$$

We have chosen here the restriction to the neighbourhood defined by $\sqrt{x^2+y^2}<1$ in order to get the last inequality.

It is sufficient to take $\delta=\min\{1,\epsilon/3\}$. We can find a δ, so we conclude that the function $\frac{\partial f}{\partial y}$ is continuous.

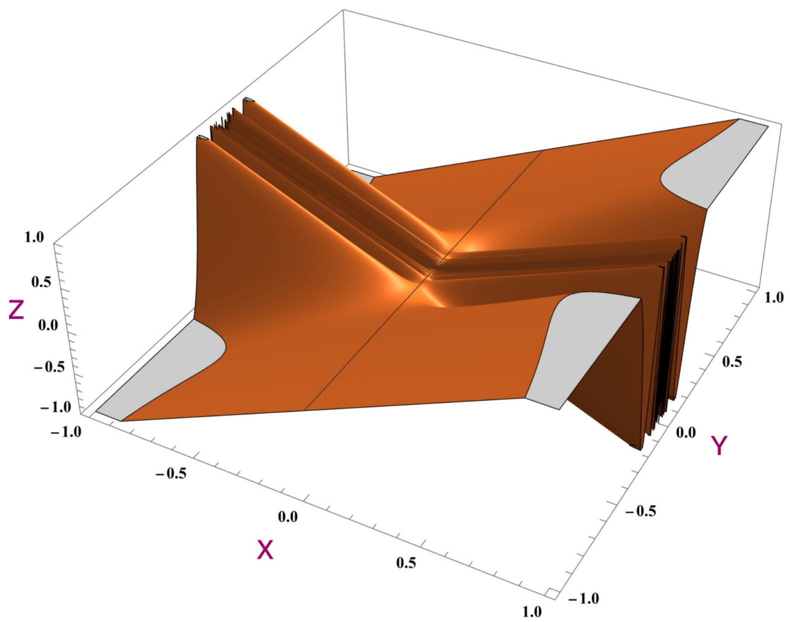

Figure 388. We see here a three dimensional figure of the graph of the second partial derivative $\frac{\partial f}{\partial y}(x,y)$. This looks like a continuous function in $(0,0)$.

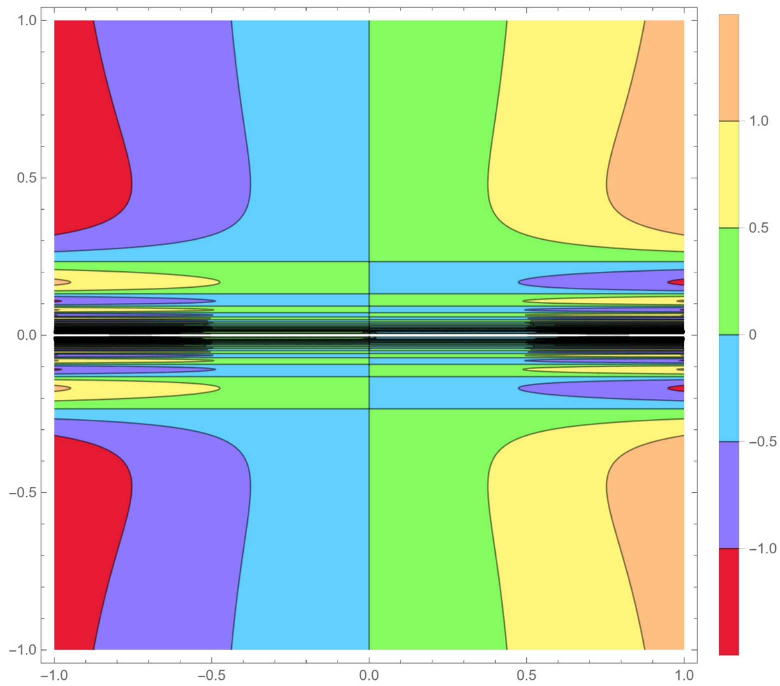

Figure 389. We see here a figure of the contour plot of the $\frac{\partial f}{\partial y}(x, y)$. Only level curves of level around 0 come close to $(0, 0)$. This looks like a continuous function in $(0, 0)$.

The **third** case is the investigation of the continuity in points $(a, 0)$ with $a \neq 0$.

Let us translate the function to the origin. We substitute $x = u + a$, $y = v$ and have then for the main part of the function definition

$$g(u, v) = 2\, v\, (a + u) \sin\left(\frac{1}{v}\right) - (a + u) \cos\left(\frac{1}{v}\right)$$

which we have to investigate in $(0, 0)$.

So we have the following function in the new coordinates.

$$g(u, v) = \begin{cases} 2\, v\, (a + u) \sin\left(\frac{1}{v}\right) - (a + u) \cos\left(\frac{1}{v}\right) & \text{if } v \neq 0, \\ 0 & \text{if } v = 0. \end{cases}$$

We see that the first term in the main part of the definition of $g(u, v)$ is surely continuous. We turn our attention to the second term. By expanding the term $a + u$ we see that the second term is

$$-a \cos\left(\frac{1}{v}\right) - u \cos\left(\frac{1}{v}\right).$$

The second term is surely continuous, so we are left with investigating the first term. After dropping the non zero coefficient, we are left with investigating the term

$$h(v) = \cos\left(\frac{1}{v}\right).$$

This term is a classical example of a non continuous function. We can be more explicit here. We define a sequence $v_n = \frac{1}{2\pi n}$, $n \in \mathbf{N}_0$, that converges to zero. We have that $\lim_{n\to\infty} h(v_n) = \lim_{n\to\infty} 1 = 1$. But we define another sequence $v_n = \frac{2}{\pi(4n+1)}$, $n \in \mathbf{N}_0$, that converges to zero. We have that $\lim_{n\to\infty} h(v_n) = \lim_{n\to\infty} 0 = 0$. This last term cannot be continuous.

We cannot apply this criterion for an alternative proof of the differentiability.

46.8 Overview

$$f(x,y) = \begin{cases} x y^2 \sin\left(\frac{1}{y}\right) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

46.9 One step further

We want to know if this function is further uneventful from the point of view of differentiability. Let us take a look at the second order partial derivative

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{x\left((2\,y^2-1)\,\sin\left(\frac{1}{y}\right)-2\,y\,\cos\left(\frac{1}{y}\right)\right)}{y^2}.$$

Let us take a look of a three dimensional plot of this second order partial derivative of the function.

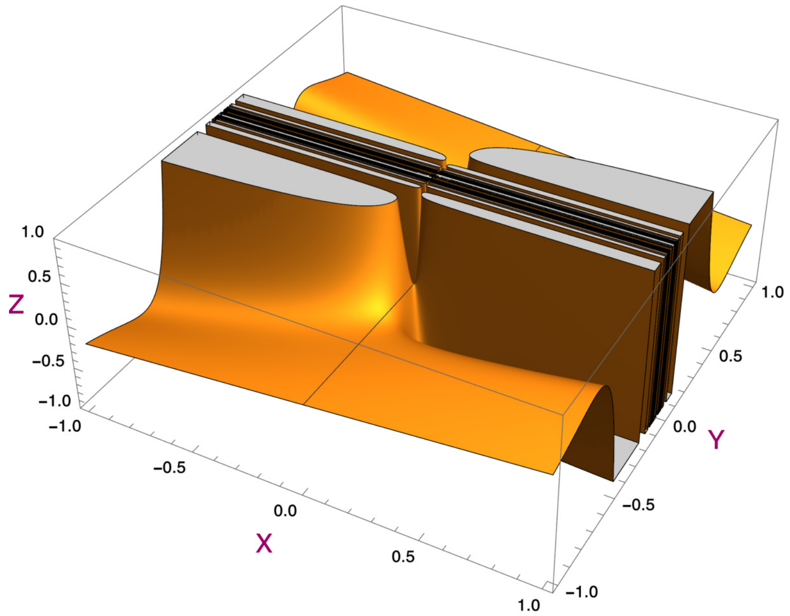

Figure 390. We see here a figure of the second order partial derivative $\frac{\partial^2 f}{\partial y^2}(x,y)$. It seems quite improbable that this second order partial derivative is continuous. We stop however our investigations here and leave this to the initiative of the interested reader.

Exercise 47.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y)=\begin{cases}\dfrac{|y|\,2^{-\frac{|y|}{x^2}}}{x^2} & \text{if } x\neq 0,\\ 0 & \text{if } x=0.\end{cases}$$

47.1 Continuity

We restrict the function to the continuous curves with equations $y = \lambda x^2$. We observe then that

$$f|_{y=\lambda x}(x,y)=\begin{cases}f(x,\lambda x)=\dfrac{2^{-\frac{|\lambda x^2|}{x^2}}\left|\lambda x^2\right|}{x^2}=2^{-|\lambda|}\,|\lambda| & \text{if } x\neq 0;\\ 0 & \text{if } x=0.\end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0)=0$. So this function $f(x,y)$ is not continuous.

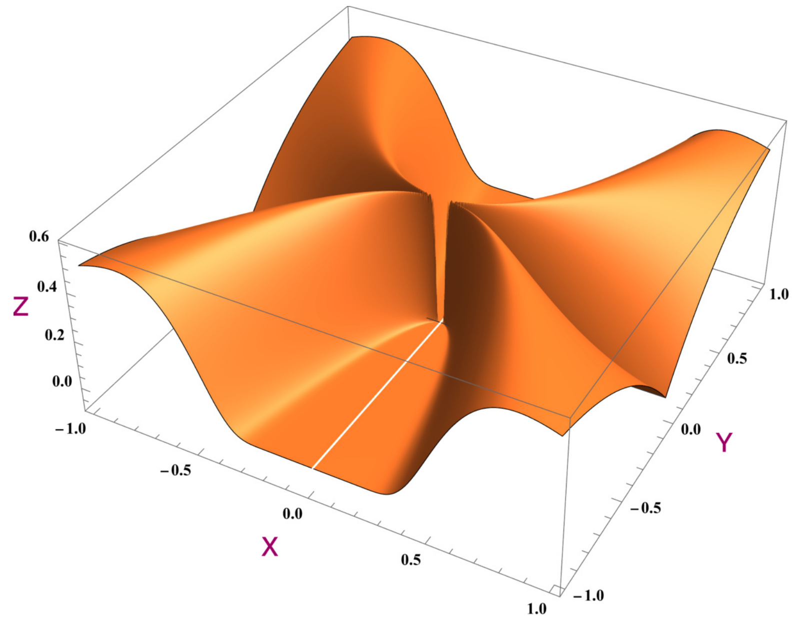

Figure 391. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

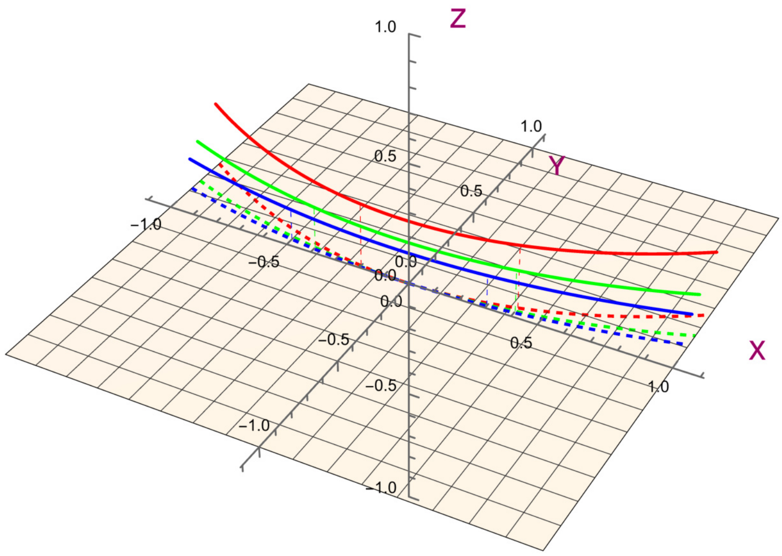

Figure 392. We have restricted the function here to $y = 1/3\,x^2$ and $y = 1/5\,x^2$ and $y = 1/7\,x^2$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

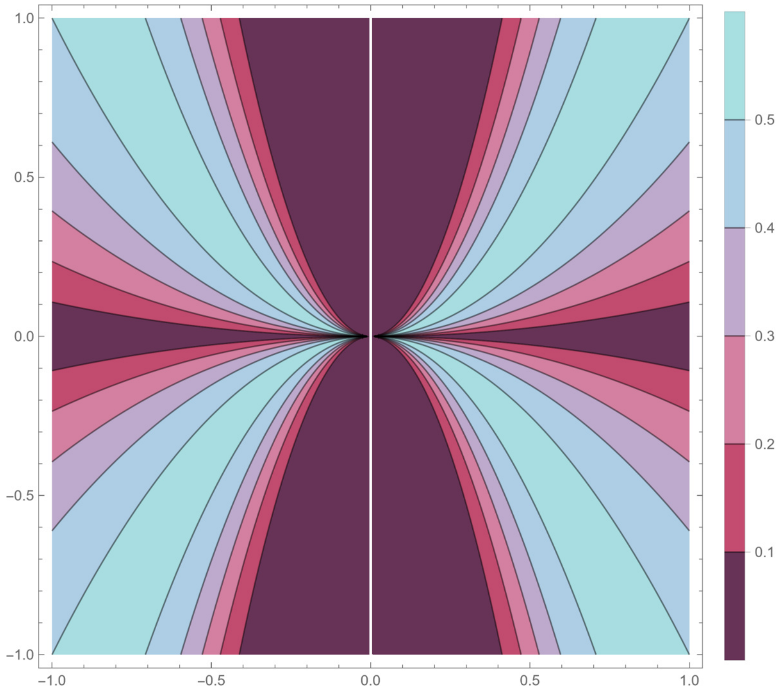

Figure 393. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

47.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

47.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{|hv|\, 2^{-\frac{|hv|}{h^2 u^2}}}{h^3 u^2}. \end{aligned}$$

We compute the left limit first. We will substitute $\alpha = -1/h$ and observing that $h \underset{<}{\to} 0$ causes $\alpha \to +\infty$,

$$\begin{aligned} \lim_{\substack{h\to 0 \\ <}} \frac{-h|v|2^{-\frac{-h|v|}{h^2 u^2}}}{h^3 u^2} &= \lim_{\substack{h\to 0 \\ <}} -\frac{|v|2^{\frac{|v|}{h u^2}}}{h^2 u^2} \\ &= \lim_{\alpha\to +\infty} -2^{\frac{-\alpha|v|}{u^2}} \frac{\alpha^2 |v|}{u^2} \\ &= 0. \end{aligned}$$

The last result is obtained by applying the theorem of de l'Hospital. We do not to cover the cases $uv = 0$ because we already did that in the section on partial derivatives.

We compute now the right limit. We will substitute $\alpha = 1/h$ and observing that $h \underset{>}{\to} 0$ causes $\alpha \to +\infty$,

$$\lim_{\substack{h \to 0 \\ >}} \frac{h|v|2^{-\frac{h|v|}{h^2 u^2}}}{h^3 u^2} = \lim_{\substack{h \to 0 \\ >}} \frac{|v|2^{-\frac{|v|}{h u^2}}}{h^2 u^2}$$

$$= \lim_{\alpha \to \infty} 2^{\frac{-\alpha|v|}{u^2}} \frac{\alpha^2 |v|}{u^2}$$

$$= 0.$$

The last result is again obtained by applying the theorem of de l'Hospital. We do not to cover the case $u\,v = 0$ because we already did that in the section on partial derivatives.

So the directional derivatives do always exist.

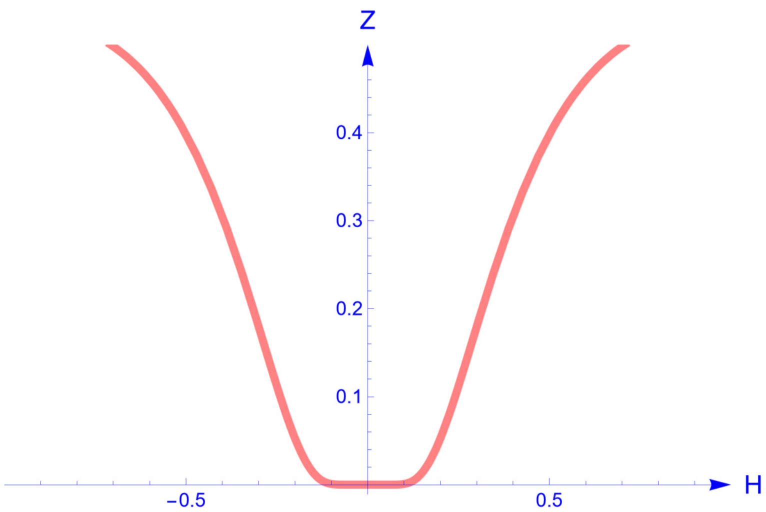

Figure 394. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

47.4 Alternative proof of continuity (optional)

The function is not continuous. So this is irrelevant.

47.5 Differentiability

We have that the function is not continuous and therefore not differentiable.

47.6 Alternative proof of differentiability (optional)

This section is **irrelevant** for this exercise, because the function is not differentiable.

47.7 Continuity of the partial derivatives

This section is **irrelevant** for this exercise, because the function is not differentiable.

47.8 Overview

$$f(x,y) = \begin{cases} \dfrac{|y|\,2^{-\frac{|y|}{x^2}}}{x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

continuous	no
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

Exercise 48.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \frac{x\,y}{|x|} + x\,\sin\left(\frac{1}{y}\right) & \text{if } x\,y \neq 0, \\ 0 & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

48.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{x\,y}{|x|} + x\,\sin\left(\frac{1}{y}\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{x\,y}{|x|}+x\,\sin\left(\frac{1}{y}\right)\right| &\le \frac{|x|\,|y|}{|x|}+|x|\,\left|\sin\left(\frac{1}{y}\right)\right|\\ &\le |y|+|x|\\ &\le \sqrt{x^2+y^2}+\sqrt{x^2+y^2}\\ &\le 2\sqrt{x^2+y^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon/2$. We can find a δ, so we conclude that the function is continuous.

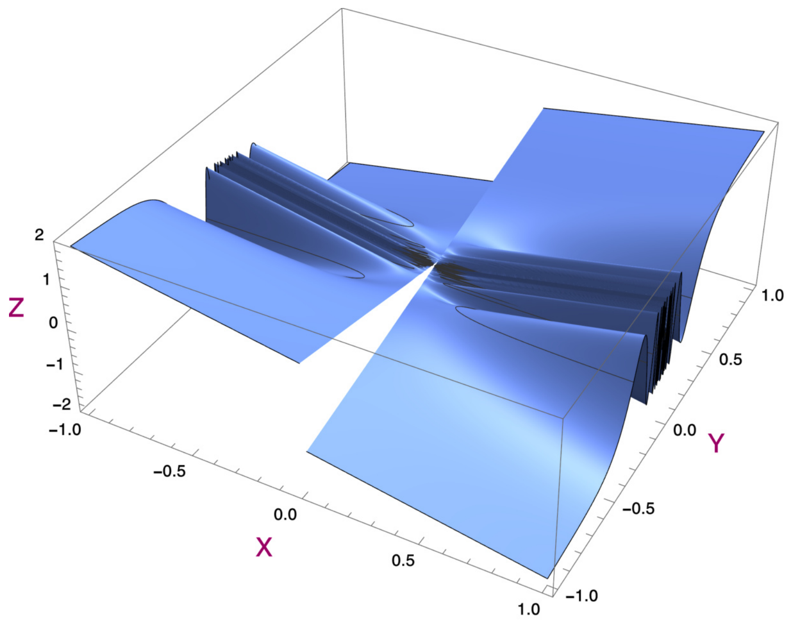

Figure 395. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

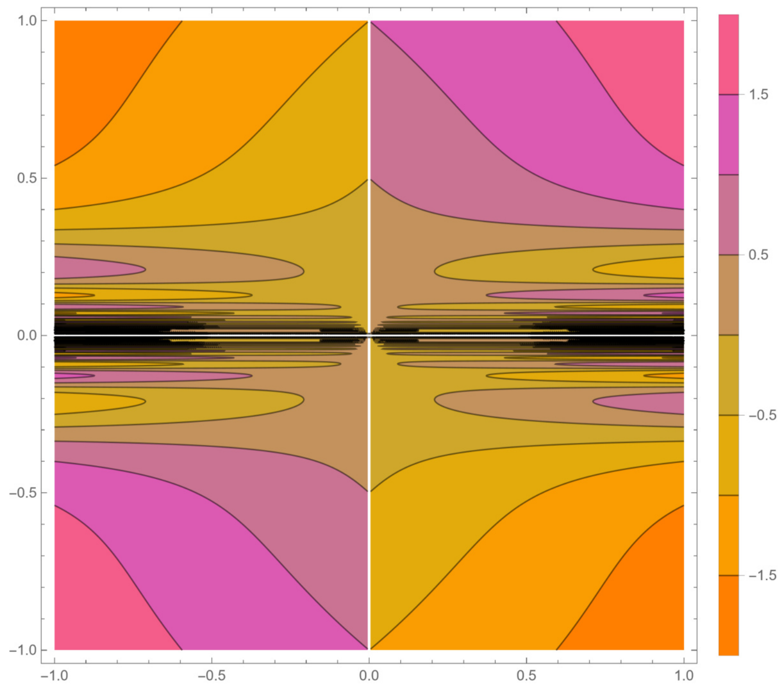

Figure 396. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

48.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

48.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\ &= \lim_{h\to 0} u\left(\frac{hv}{|hu|} + \sin\left(\frac{1}{hv}\right)\right) \\ &= \lim_{h\to 0} u\left(\frac{\operatorname{sgn}(h)\, v}{|u|} + \sin\left(\frac{1}{hv}\right)\right). \end{aligned}$$

This limit does not exist if $u\,v \neq 0$. We have already calculated these exceptional cases. Let us try to be more specific about the non existence of the limit. Let us define first $g(h) = u\left(\frac{\operatorname{sgn}(h)\, v}{|u|} + \sin\left(\frac{1}{hv}\right)\right)$. Let us define the sequence $h_n = \frac{2}{\pi\,(2n+1)v}$, $n \in \mathbf{N}$. This sequence converges to 0. We evaluate this sequence in the function $g(h)$ and find $g(h_n) = u\left(\frac{1}{(2n+1)\left|\frac{u}{2nv+v}\right|} + \cos(\pi n)\right)$ or $g(h_n) = u\left(\frac{1}{|u/v|} + \cos(\pi n)\right)$. But this sequence does not have a limit at all. If the function $g(h)$ is continuous, then there is a limit.

So the directional derivatives do not always exist.

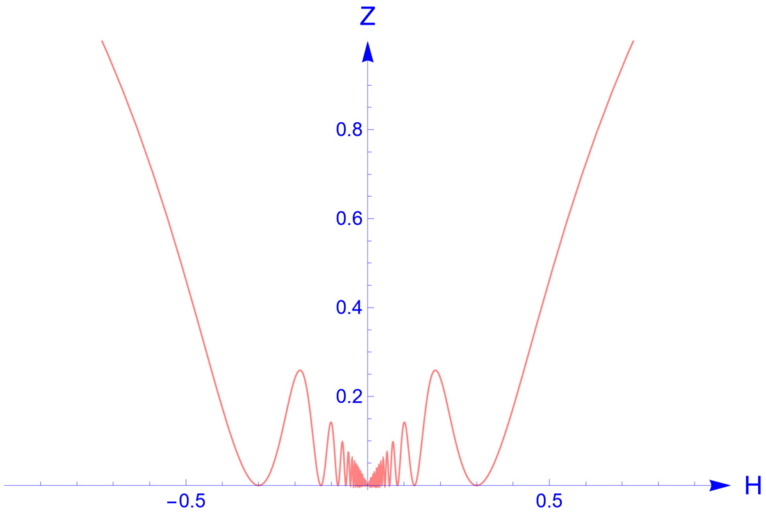

Figure 397. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$. This function is not differentiable.

48.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Looking for a definition of the partial derivatives.

We have to be able to define the partial derivatives in at least one neighbourhood around $(0,0)$. We have no problems with points that are in the interior of the definition domains of the classical functions. We have there the classical calculation rules for defining those functions and the partial derivatives always exist there and are even continuous.

But in our case, we have to investigate the points that are in exceptional subsets in the definition of the function.

In this case these are the points $(a, 0)$ and $(0, b)$.

Let us look at a point $(a, 0)$ with $a \neq 0$. We are going to investigate the function in $(a, 0)$ in the Y-direction. This function is defined by

$$f(a,h) = \begin{cases} a\left(\frac{h}{|a|} + \sin\left(\frac{1}{h}\right)\right) & \text{if } h \neq 0, \\ 0 & \text{if } h = 0. \end{cases}$$

We see that this function is not continuous in $h = 0$ if $a \neq 0$, so the derivative does not exist. The conclusion is that the partial derivative $\frac{\partial f}{\partial y}(a, 0)$ does not exist for all a with $a \neq 0$.

We consult a figure for this observation.

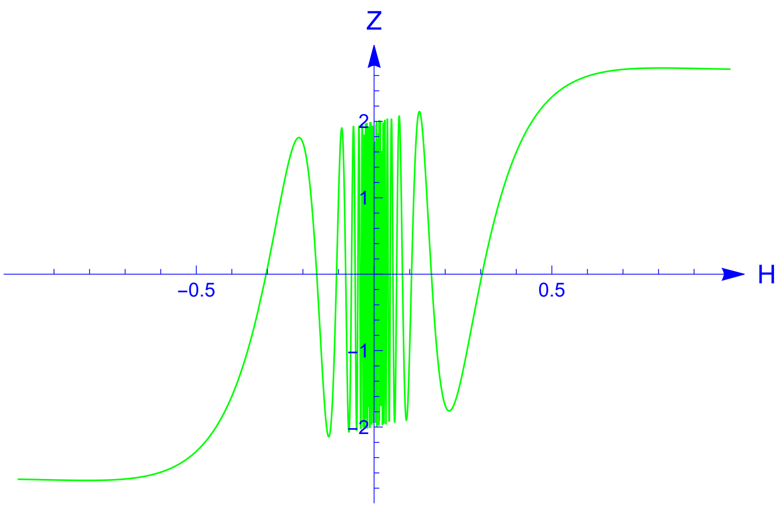

Figure 398. We see here a figure of the graph of the function restricted to the line through $(a, 0)$ with direction $(0, 1)$, this is the Y-direction. We have drawn this figure with the value $a = 2$. This is a figure of the function with function definition $f(a, h)$.

Some of the partial derivatives do not exist in any neighbourhood of $(0, 0)$. So the partial derivatives cannot be defined in any neighbourhood of $(0, 0)$. The conclusion is that an alternative proof following the lines described at the start of this section cannot be given. Other alternative proofs can of course exist.

48.5 Differentiability

Not all directional derivatives exist, so this function is not differentiable.

48.6 Alternative proof of differentiability (optional)

The function is not differentiable, so this is irrelevant.

48.7 Continuity of the partial derivatives

The function is not differentiable, so this section is irrelevant.

48.8 Overview

$$f(x,y) = \begin{cases} \dfrac{x\,y}{|x|} + x\,\sin\left(\dfrac{1}{y}\right) & \text{if } x\,y \neq 0, \\ 0 & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 49.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} x^3 \sin\left(\frac{1}{x^2}\right) + y^3 \sin\left(\frac{1}{y^2}\right) & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0. \end{cases}$$

49.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| x^3 \sin\left(\frac{1}{x^2}\right) + y^3 \sin\left(\frac{1}{y^2}\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|x^3 \sin\left(\frac{1}{x^2}\right) + y^3 \sin\left(\frac{1}{y^2}\right)\right| &\le |x|^3 \left|\sin\left(\frac{1}{x^2}\right)\right| + |y|^3 \left|\sin\left(\frac{1}{y^2}\right)\right| \\ &\le |x|^3 + |y|^3 \\ &\le \sqrt{x^2+y^2}^3 + \sqrt{x^2+y^2}^3 \\ &\le 2\sqrt{x^2+y^2}^3 .\end{aligned}$$

It is sufficient to take $\delta = (\epsilon/2)^{1/3}$. We can find a δ, so we conclude that the function is continuous.

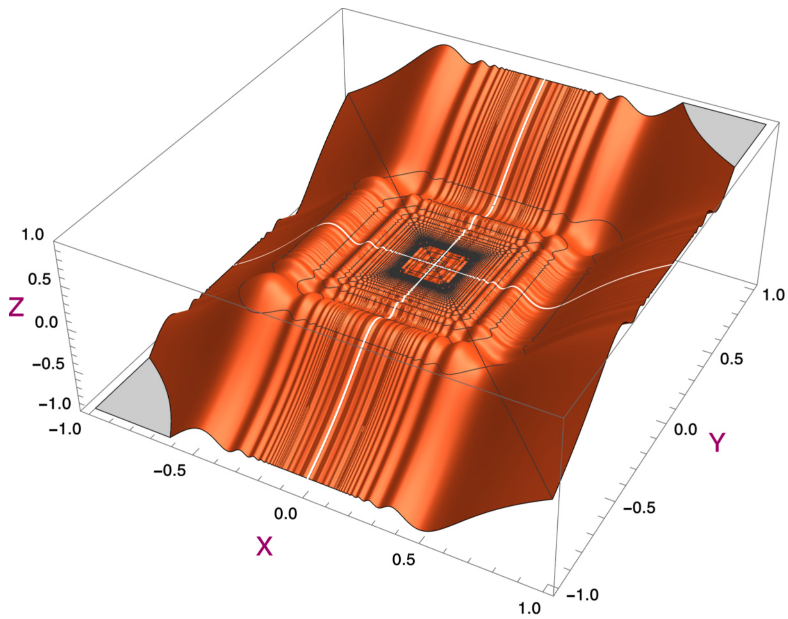

Figure 399. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

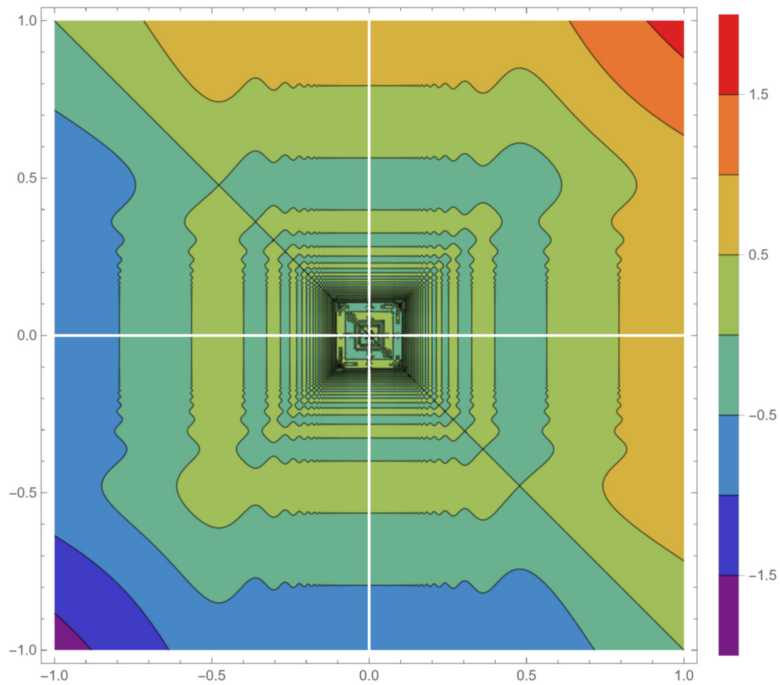

Figure 400. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

49.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

We can use the squeeze theorem for this limit.
So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

49.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\ &= \lim_{h\to 0} h^2 \left(u^3 \sin\left(\frac{1}{h^2 u^2}\right) + v^3 \sin\left(\frac{1}{h^2 v^2}\right)\right) \\ &= 0. \end{aligned}$$

We can use the squeeze theorem for this. Remark that this calculation is only valid if $u\,v \neq 0$. We did the other cases before.

So the directional derivatives do always exist.

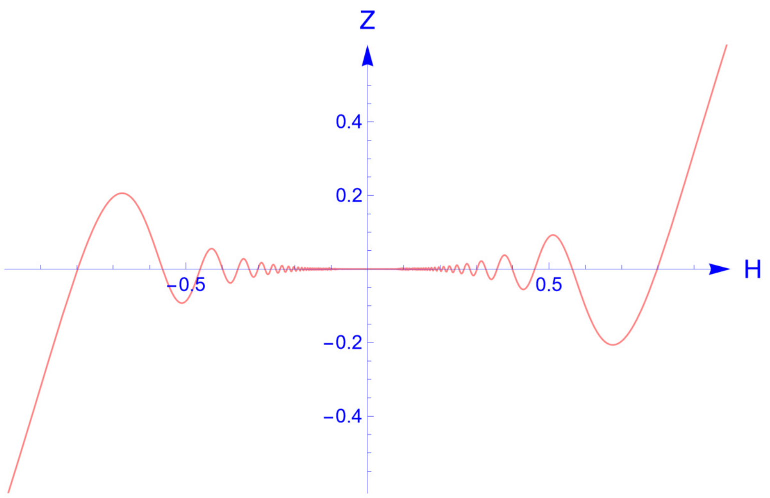

Figure 401. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$.

49.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

We remark that some of the partial derivatives do not exist in any neighbourhood of $(0,0)$. We have e.g. in $(a,0)$ with $a \neq 0$ in the Y-direction the following function.

$$f(a,h) = \begin{cases} a^3 \sin\left(\frac{1}{a^2}\right) + h^3 \sin\left(\frac{1}{h^2}\right) & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

This function is almost never continuous except in those cases where $a^3 \sin\left(\frac{1}{a^2}\right) = 0$. This only happens in a countable number of cases.
We see that this limit does not exist if $a^3 \sin\left(\frac{1}{a^2}\right) \neq 0$.

So an alternative proof following this criterion is not possible.

Consult the following figure.

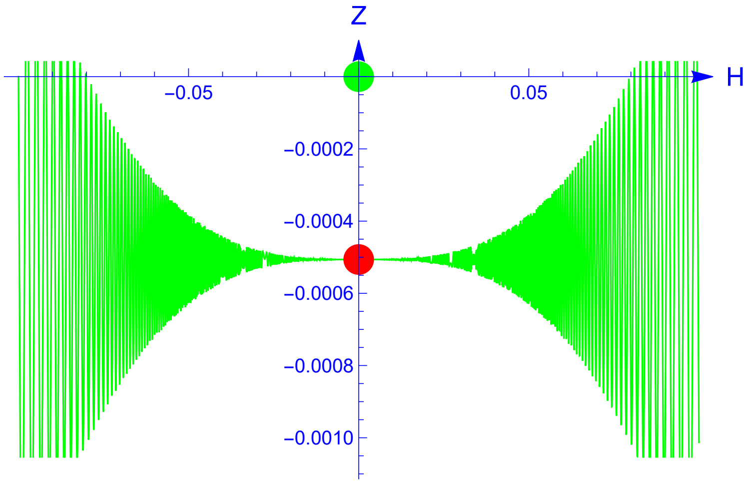

Figure 402. We see here a figure of the graph of the function $f(a, h)$. We have drawn the function here for the value $a = 1/10$ which is exemplary for the values of a close to 0. Remark that in $h = 0$ the function is 0 causing a discontinuous point in $h = 0$. This does not look like a continuous function. We have plotted here the function $f(a, h)$.

49.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

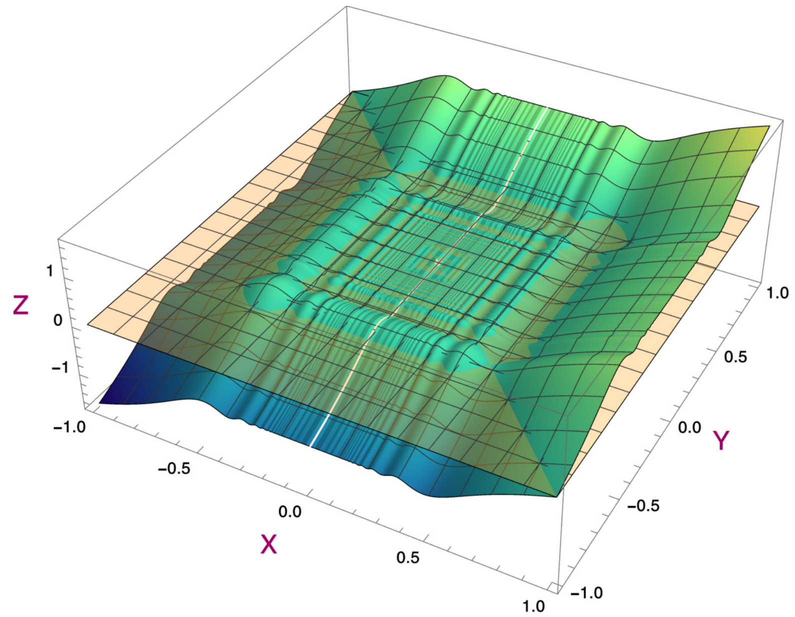

Figure 403. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function very nicely.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

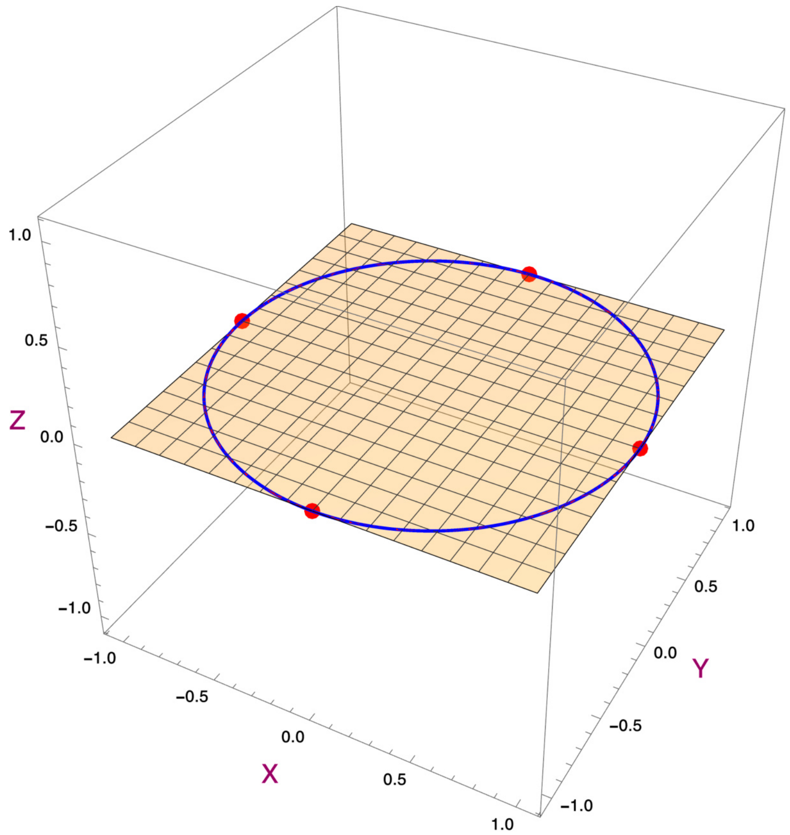

Figure 404. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0, 0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0, 0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0, 0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h, k)$ in $(0, 0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\, h + \frac{\partial f}{\partial y}(0,0)\, k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h^3 \sin\left(\dfrac{1}{h^2}\right) + k^3 \sin\left(\dfrac{1}{k^2}\right)}{\sqrt{h^2+k^2}} & \text{if } (h,k) \neq (0,0) \text{ and } h \neq 0 \text{ and } k \neq 0; \\ 0 & \text{if } (h,k) = (0,0) \text{ or } h = 0 \text{ or } k = 0 \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left| \frac{h^3 \sin\left(\frac{1}{h^2}\right) + k^3 \sin\left(\frac{1}{k^2}\right)}{\sqrt{h^2+k^2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{h^3 \sin\left(\frac{1}{h^2}\right) + k^3 \sin\left(\frac{1}{k^2}\right)}{\sqrt{h^2+k^2}} \right| &\leq \frac{|h|^3 \left|\sin\left(\frac{1}{h^2}\right)\right| + |k|^3 \left|\sin\left(\frac{1}{k^2}\right)\right|}{\sqrt{h^2+k^2}} \\
&\leq \frac{|h|^3 + |k|^3}{\sqrt{h^2+k^2}} \\
&\leq \frac{\sqrt{h^2+k^2}^3 + \sqrt{h^2+k^2}^3}{\sqrt{h^2+k^2}} \\
&\leq 2\,\frac{\sqrt{h^2+k^2}^3}{\sqrt{h^2+k^2}} \\
&\leq 2\sqrt{h^2+k^2}^2.
\end{aligned}$$

It is sufficient to take $\delta = (\epsilon/2)^{1/2}$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous.

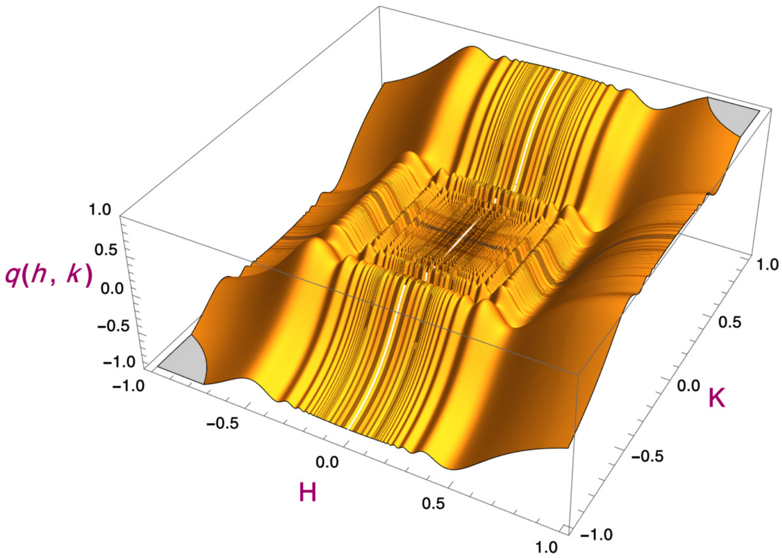

Figure 405. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

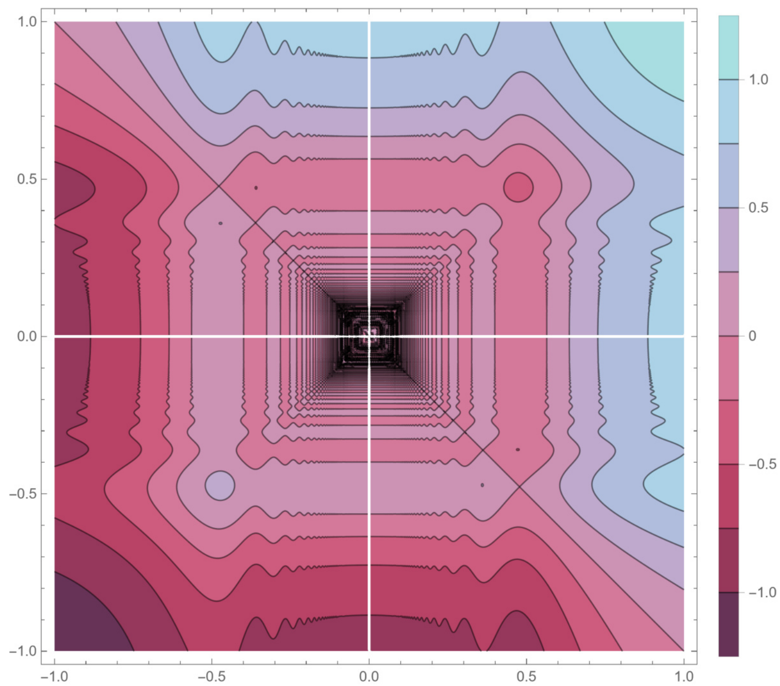

Figure 406. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

49.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\,u + \frac{\partial f}{\partial y}(0,0)\,v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the function $f(x,y)$ is locally Lipschitz continuous in $(0,0)$. We cite here the criterion that we will use.

A function is differentiable in (a,b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.
2. The directional derivatives have the following form: $D_{(u,v)}(a,b) = \nabla f(a,b)\cdot(u,v) = \frac{\partial f}{\partial x}(a,b)\,u + \frac{\partial f}{\partial y}(a,b)\,v$.
3. The function is locally Lipschitz continuous in (a,b). This means that there exists at least one neighbourhood of (a,b) and a number $K > 0$ such that for all (x_1,y_1) and (x_2,y_2) in the neighbourhood, we have $|f(x_1,y_1) - f(x_2,y_2)| < K\,||(x_1,y_1) - (x_2,y_2)||$. Remark that the same K must be valid for all (x_1,y_1) and (x_2,y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be “locally flat" and thus “locally linear" which is partly expressed by this Lipschitz continuity condition.

We have seen in section 49.4 that that in every neighbourhood of $(0,0)$, there are points where the function is not continuous. So the function cannot be Lipschitz continuous.

49.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

We have seen in section 49.4 that not all partial derivatives exist in any neighbourhood of $(0, 0)$. So we cannot use this criterion.

49.8 Overview

$$f(x, y) = \begin{cases} x^3 \sin\left(\frac{1}{x^2}\right) + y^3 \sin\left(\frac{1}{y^2}\right) & \text{if } x\,y \neq 0, \\ 0 & \text{if } x\,y = 0. \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

Exercise 50.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0, 0)$.

$$f(x, y) = \begin{cases} \dfrac{y\sqrt{x^2 + y^2}}{|y|} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

50.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x, y) - f(0, 0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x, y) - (0, 0)|| < \delta$ it follows that $|f(x, y) - f(0, 0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0, 0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{y\sqrt{x^2 + y^2}}{|y|} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\left|\frac{y\sqrt{x^2+y^2}}{|y|}\right| \leq \frac{|y|\sqrt{x^2+y^2}}{|y|}$$
$$\leq \sqrt{x^2+y^2}.$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function is continuous.

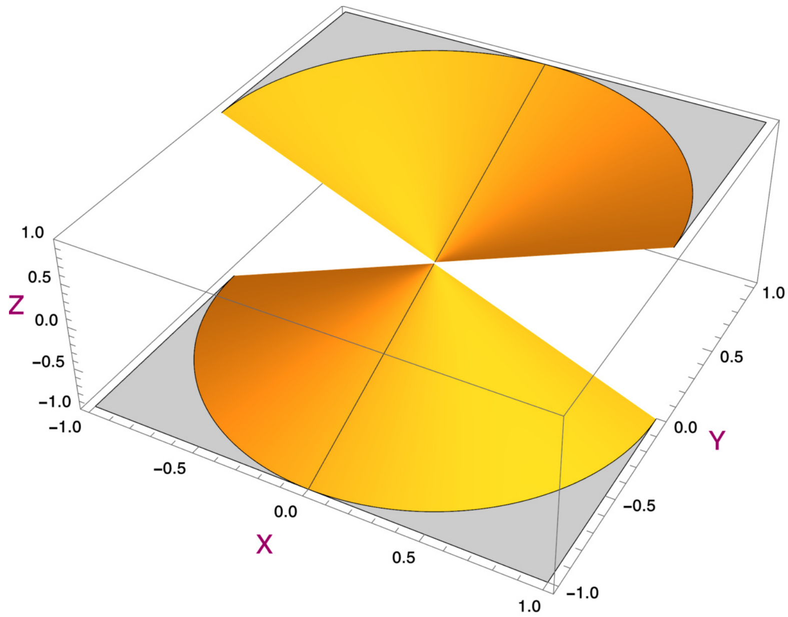

Figure 407. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

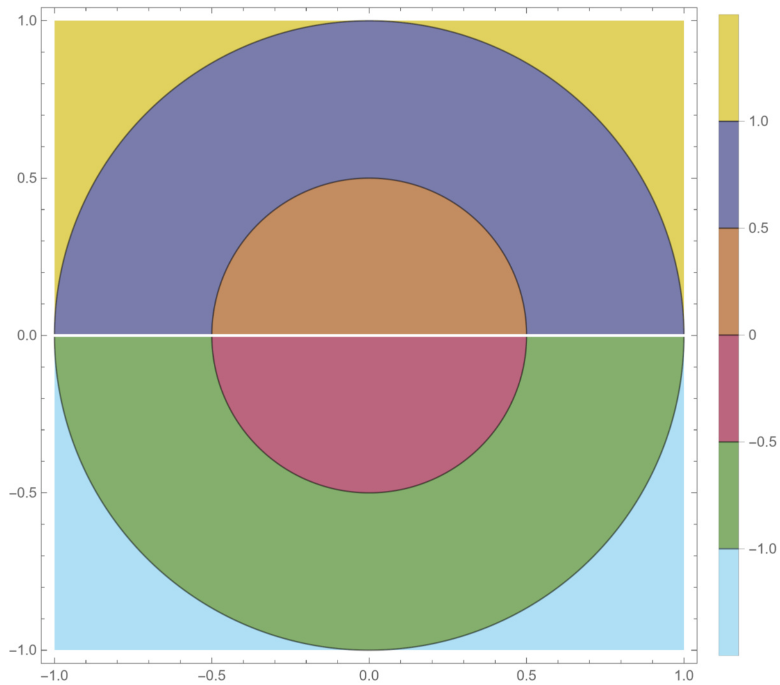

Figure 408. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

50.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f\,\big|_{x=0}(x,y) = \begin{cases} f(0,y) = \dfrac{y\sqrt{y^2}}{|y|} & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0}\frac{\sqrt{h^2}}{|h|}\\ &= 1.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 1.$$

50.3 Directional derivatives

Let (u, v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2 + v^2} = 1$. Then the directional derivative in $(0, 0)$ in the direction (u, v), notation $D_{(u,v)}(0, 0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0 + hu, 0 + hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1, 0)$ and $(0, 1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0 + hu, 0 + hv) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{v\sqrt{h^2\,(u^2 + v^2)}}{|hv|} \\ &= \lim_{h\to 0} \frac{v|h|\sqrt{(u^2 + v^2)}}{|hv|} \\ &= \frac{v\sqrt{(u^2 + v^2)}}{|v|}. \end{aligned}$$

These calculations are done with $v \neq 0$. This is a case that we have covered before.

So the directional derivatives do always exist.

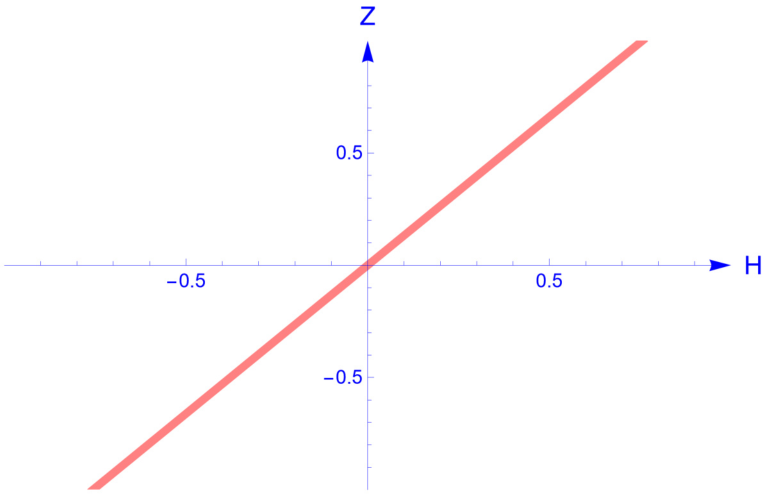

Figure 409. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$. This function is differentiable.

50.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

The partial derivative does not exist in points $(a,0)$ with $a \neq 0$ in the Y-direction. The function is in those points not even continuous as can be seen on the figure.

So the partial derivatives do not all exist in any neighbourhood of $(0,0)$. We cannot use this alternative criterion.

50.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks of the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

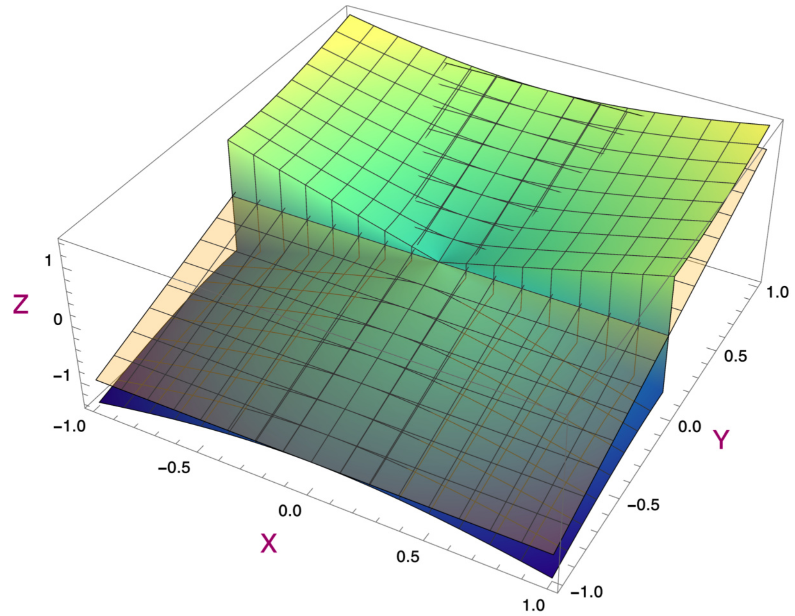

Figure 410. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x, y)$. We see here that the candidate tangent plane fits the function very nicely.

We make our second observation. We can look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the vectors $(u, v, D_{(u,v)}(0, 0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

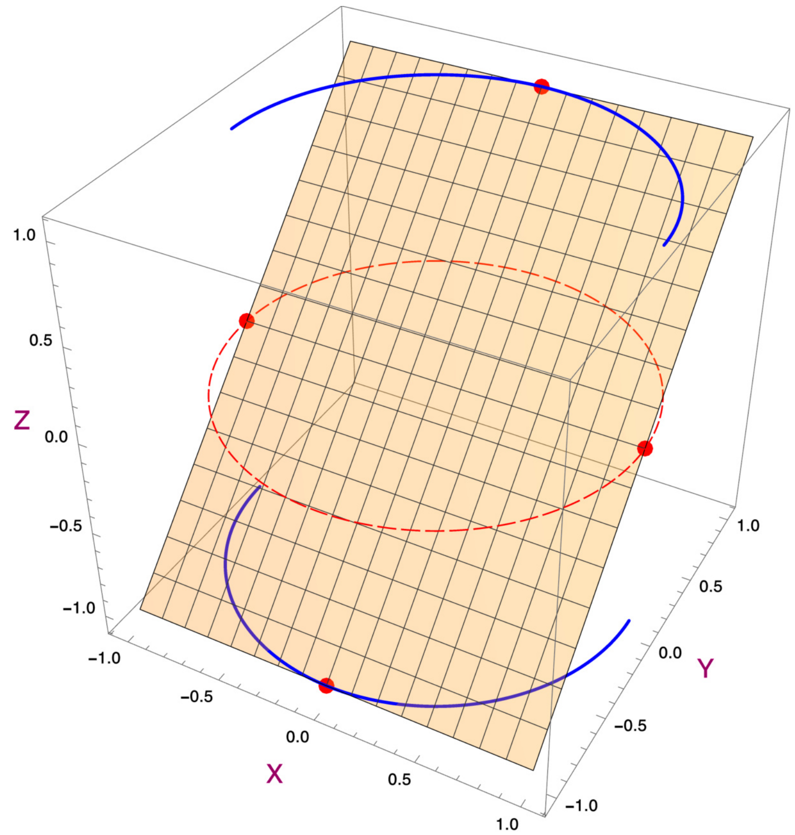

Figure 411. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ do not sweep out a nice ellipse lying in one plane! This is very bad news for the differentiability.

Discussion of the continuity of the quotient.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(x, y)$ and not the differential quotient.

If the

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x, y)$ is by definition differentiable in $(0, 0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} k\left(\frac{1}{|k|} - \frac{1}{\sqrt{h^2+k^2}}\right) & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0). \end{cases}$$

is continuous in $(0, 0)$.

We restrict the function $q(h, k)$ to the continuous curves with equations $k = \lambda\, h$. We observe then that

$$q\big|_{k=\lambda h}(h,k) = \begin{cases} q(h,\lambda\, h) = h\lambda\left(\frac{1}{|h\lambda|} - \frac{1}{\sqrt{h^2(\lambda^2+1)}}\right) & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

We see that these restricted functions have different limits if $\lambda \neq 0$. But if $q(h, k)$ is continuous, all these limit values should be $q(0, 0) = 0$. So this function $q(h, k)$ is not continuous in $(0, 0)$. The function $f(x, y)$ is not differentiable in $(0, 0)$.

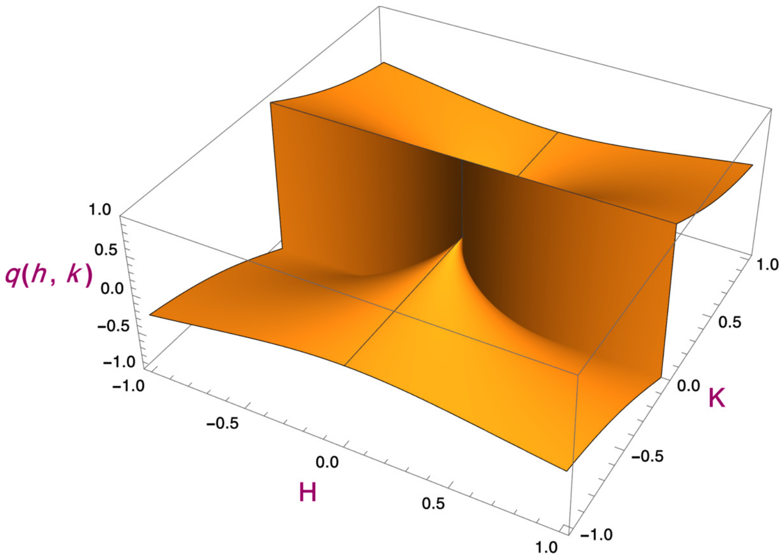

Figure 412. We see here a three dimensional figure of the graph of the function $q(h,k)$. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

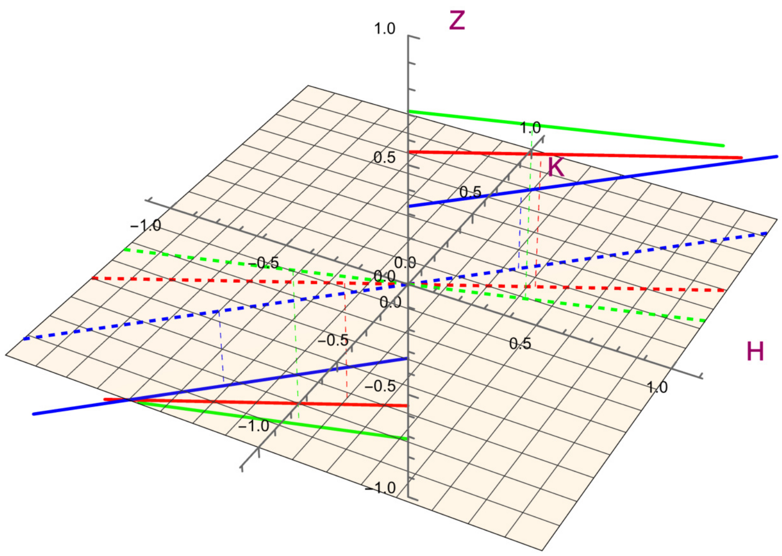

Figure 413. We have restricted the function $q(h,k)$ here to $k = 1/2\,h$ and $k = 3/10\,h$ and $k = 9/10\,h$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

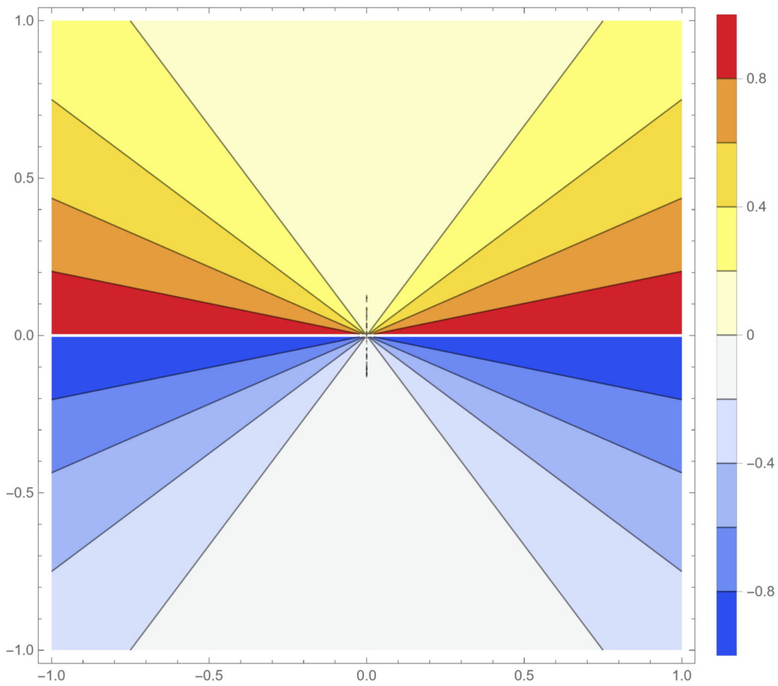

Figure 414. We see here a figure of the contour plot of the function $q(h,k)$. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

50.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\,u + \frac{\partial f}{\partial y}(0,0)\,v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the function $f(x,y)$ is locally Lipschitz continuous in $(0,0)$. We cite here the criterion that we will use.

A function is differentiable in (a,b) if and only if it satisfies the three following conditions

1. All the directional derivatives of the function exist.
2. The directional derivatives have the following form: $D_{(u,v)}(a,b) = \nabla f(a,b) \cdot (u,v) = \frac{\partial f}{\partial x}(a,b)\,u + \frac{\partial f}{\partial y}(a,b)\,v$.
3. The function is locally Lipschitz continuous in (a,b). This means that there exists at least one neighbourhood of (a,b) and a number $K > 0$ such that for all (x_1,y_1) and (x_2,y_2) in the neighbourhood, we have $|f(x_1,y_1) - f(x_2,y_2)| < K\,||(x_1,y_1) - (x_2,y_2)||$. Remark that the same K must be valid for all (x_1,y_1) and (x_2,y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is a indeed very strong condition but this is not unexpected because the differentiable function f must be "locally flat" and thus "locally linear" which is partly expressed by this Lipschitz continuity condition.

This function is not locally Lipschitz continuous because it is not locally continuous. We have no alternative proof for the differentiability following these lines.

50.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

The function is not locally continuous. Please consult section 50.4.

50.8 Overview

$$f(x,y) = \begin{cases} \dfrac{y\sqrt{x^2+y^2}}{|y|} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

50.9 One step further

We have used in the calculations for differentiability that we had some magical curves $k = \lambda h$ which behaved very strangely when mapped by $q(h,k)$ to the Z-direction. We want to see what is going on with these curves. Let us define the 3-dimensional curve in parametric form that projects in the (h,k)-plane to our curve $k = \lambda h$ where we use $\lambda = 1$: $(x(t), y(t), z(t)) = (t, \lambda t, f(t, \lambda t)) = \left(t, t, \sqrt{2}\, t\right)$.

This curve lies completely in the surface defined by the function. It is clear that the tangent vector lies in the tangent plane if the function is differentiable. Now we have a candidate tangent plane, we draw that and draw also the curve defined above which is in this case a line. The tangent vector in $t = 0$ coincides with this line. But the tangent vector should be in the candidate tangent plane if it is a tangent plane. This is not the case.

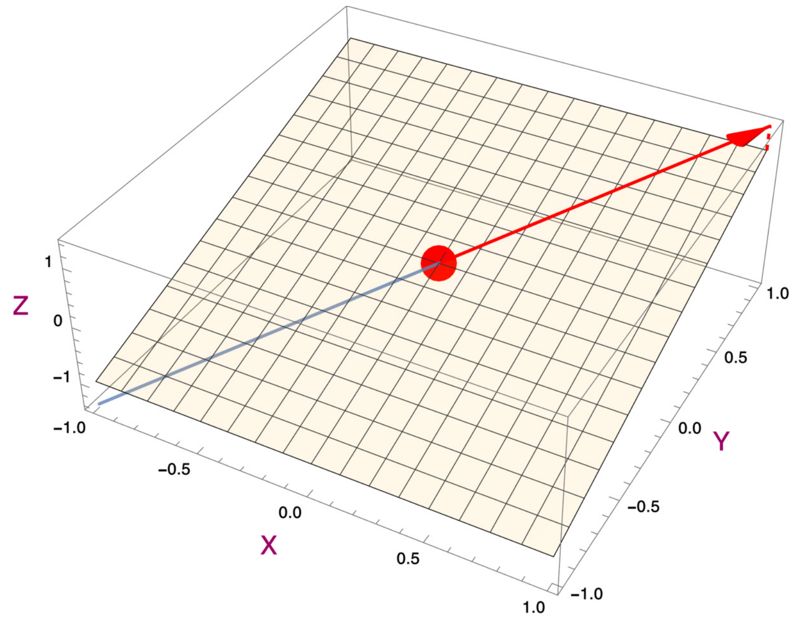

Figure 415. The tangent vector is not in the candidate tangent plane. The curve intersects the candidate tangent plane transversally and not tangentially. We have used $\lambda = 1$ in this figure.

Exercise 51.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{x^2 y \sqrt{|y|}}{x^4 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

51.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{x^2 y \sqrt{|y|}}{x^4 + y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$
\begin{aligned}
\left| \frac{x^2 |y| \sqrt{|y|}}{x^4 + y^2} \right| &\le \frac{x^2 |y| \sqrt{|y|}}{x^4 + y^2} \\
&\le \frac{\sqrt{|y|}}{2} \\
&\le \frac{\sqrt{\sqrt{x^2 + y^2}}}{2} \\
&\le \frac{\sqrt{x^2 + y^2}^{1/2}}{2}.
\end{aligned}
$$

We know that $0 \le (a-b)^2$ so that $2ab \le a^2 + b^2$. So $ab/(a^2+b^2) \le 1/2$. We have applied that with $a = x^2$ and $b = |y|$.

It is sufficient to take $\delta = (2\epsilon)^2$. We can find a δ, so we conclude that the function is continuous.

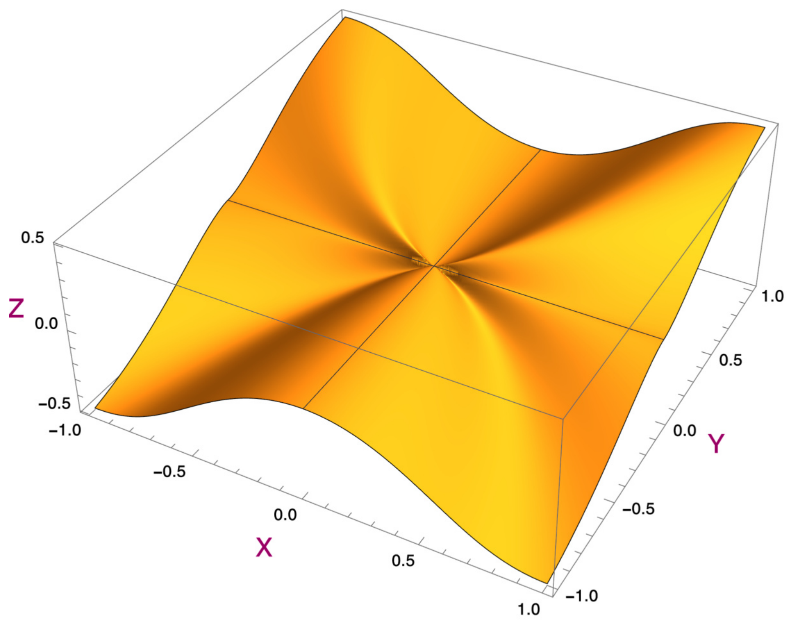

Figure 416. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

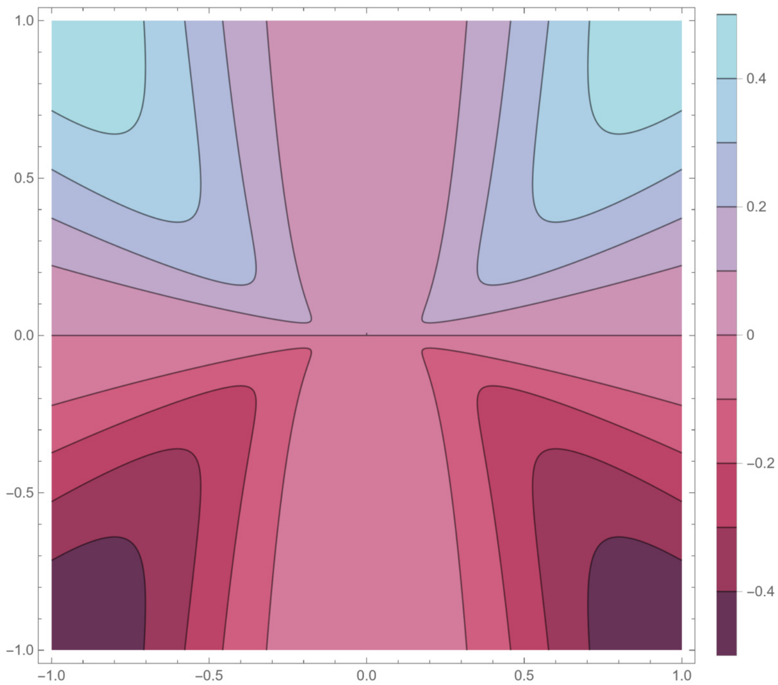

Figure 417. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

51.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y)=0 & \text{if } y\neq 0;\\ 0 & \text{if } y=0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0)=0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0)=0.$$

51.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{u^2\, v\, \sqrt{|h\,v|}}{h^2\, u^4 + v^2} \\ &= 0. \end{aligned}$$

We see that $v = 0$ must be excluded from this calculation but we discussed that direction before.

So the directional derivatives do always exist.

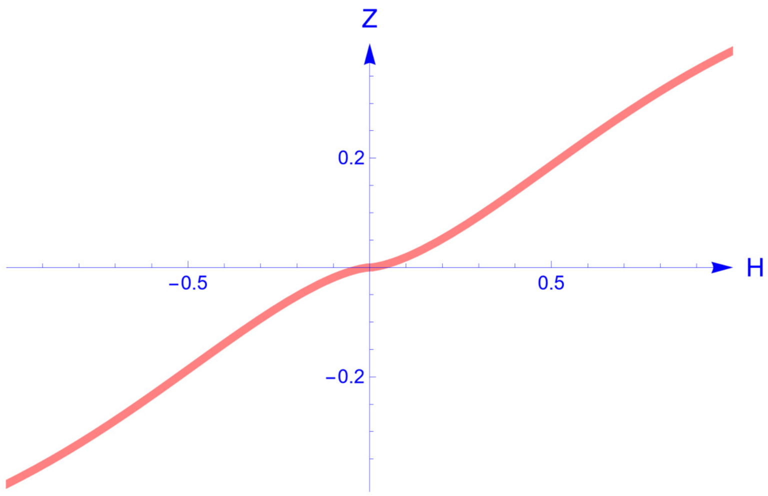

Figure 418. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

51.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Let us investigate the partial derivative to y.
If $y > 0$, this is equal to

$$\frac{\partial f}{\partial y}(x,y) = \frac{x^2\sqrt{y}\,(3\,x^4 - y^2)}{2\,(x^4+y^2)^2}.$$

By restricting to $y = x^2$ with $y > 0$, we have then

$$\frac{\partial f}{\partial y}(x,x^2) = \frac{1}{4\sqrt{x^2}}.$$

This function is not bounded in any neighbourhood of $(0,0)$. So this criterion cannot be used for an alternative proof of continuity.

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

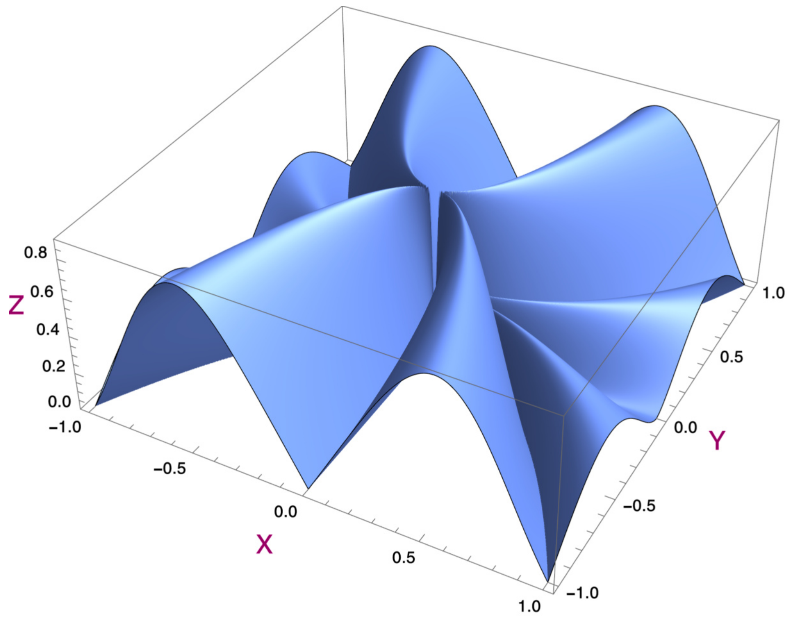

Figure 419. We see here the absolute value of the first partial derivative $\left|\dfrac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

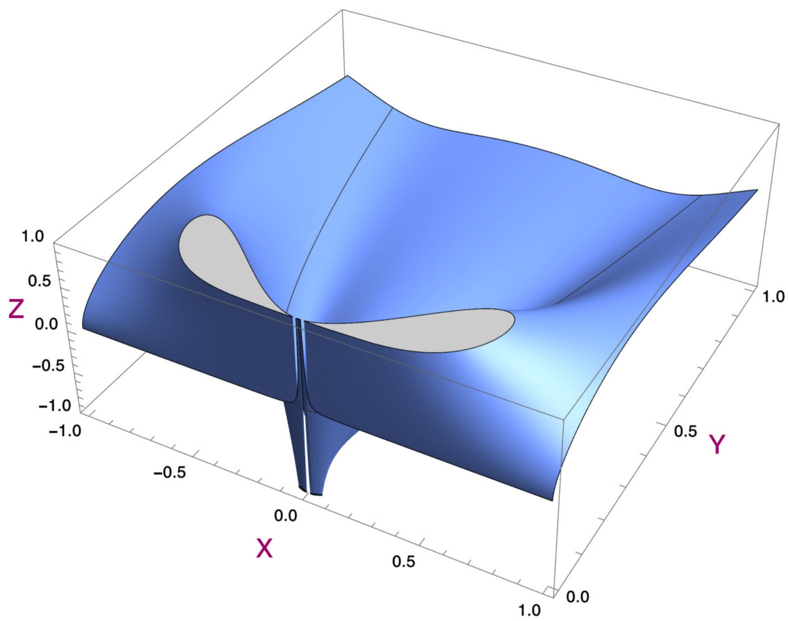

Figure 420. We see here the absolute value of the second partial derivative $\left|\dfrac{\partial f}{\partial y}\right|$. We can observe the unboundedness from this picture.

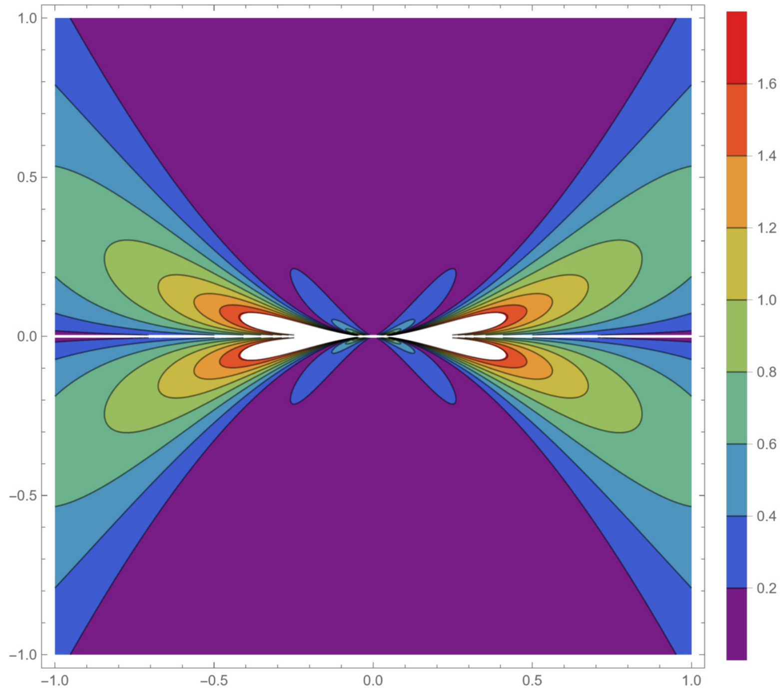

Figure 421. We see here a contour plot of the absolute value of the second partial derivative $\left|\dfrac{\partial f}{\partial y}\right|$. We see a picture that indicates that unboundedness is possible.

51.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

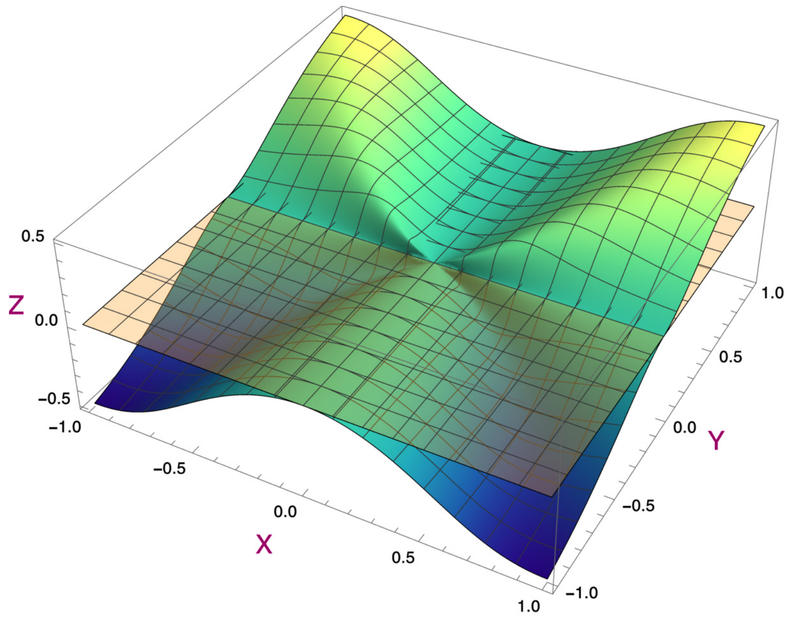

Figure 422. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function quite nicely. One cannot be sure. The calculations will decide if the fit is strong enough.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

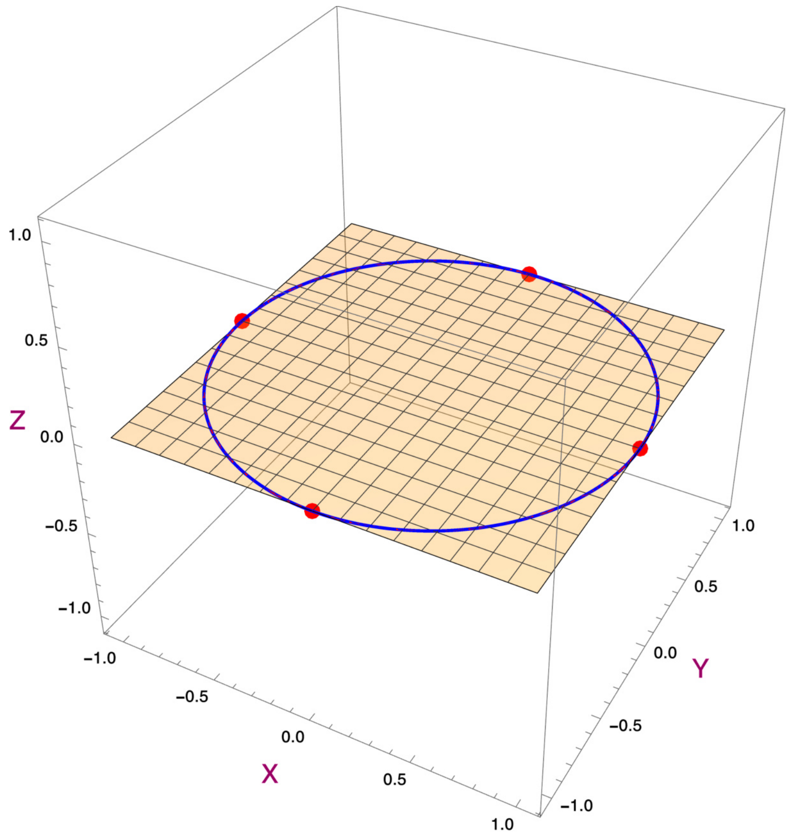

Figure 423. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h, k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h^2\,k\sqrt{|k|}}{\sqrt{h^2+k^2}\left(h^4+k^2\right)} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We restrict the function $q(h,k)$ to the continuous curves with equations $k = \lambda\,h^2$. We observe then that

$$q\,|_{k=\lambda h^2}(h,k) = \begin{cases} q(h,\lambda\,h^2) \\ \quad = \dfrac{\lambda\sqrt{|h^2\lambda|}}{\left(\lambda^2+1\right)\sqrt{h^4\lambda^2+h^2}} & \text{if } h \neq 0; \\ \quad = \dfrac{\lambda|h|\sqrt{|\lambda|}}{\left(\lambda^2+1\right)|h|\sqrt{h^2\lambda^2+1}} & \\ 0 & \text{if } h = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $q(h,k)$ is continuous, all these limit values should be $q(0,0) = 0$. So this function $q(h,k)$ is not continuous in $(0,0)$. The function $f(x,y)$ is not differentiable in $(0,0)$.

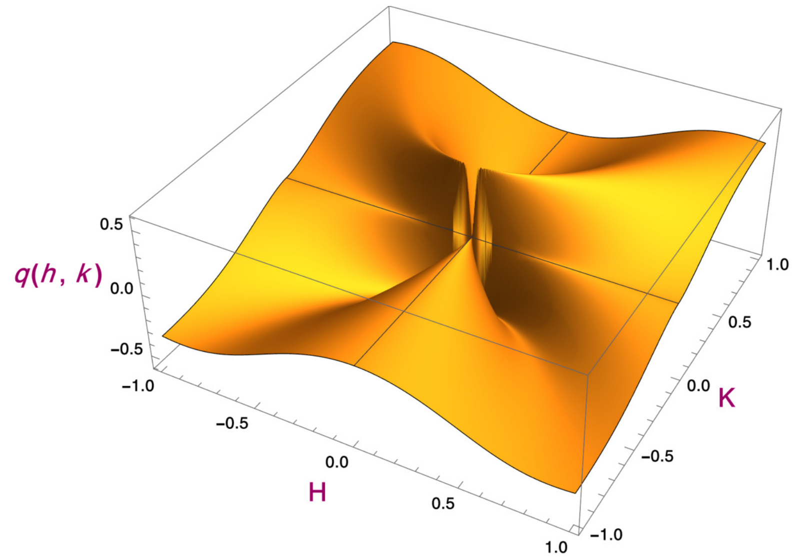

Figure 424. We see here a three dimensional figure of the graph of the function $q(h,k)$. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

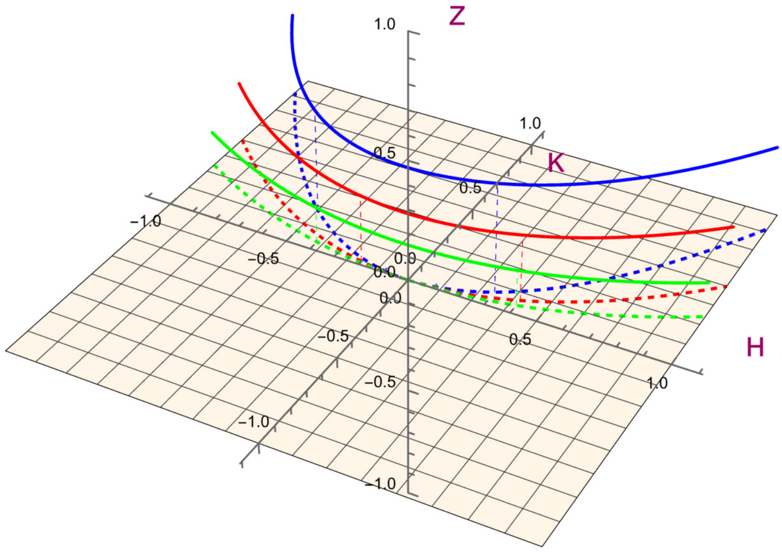

Figure 425. We have restricted the function $q(h,k)$ here to $k = 1/2\,h^2$ and $k = 3/10\,h^2$ and $k = 9/10\,h^2$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

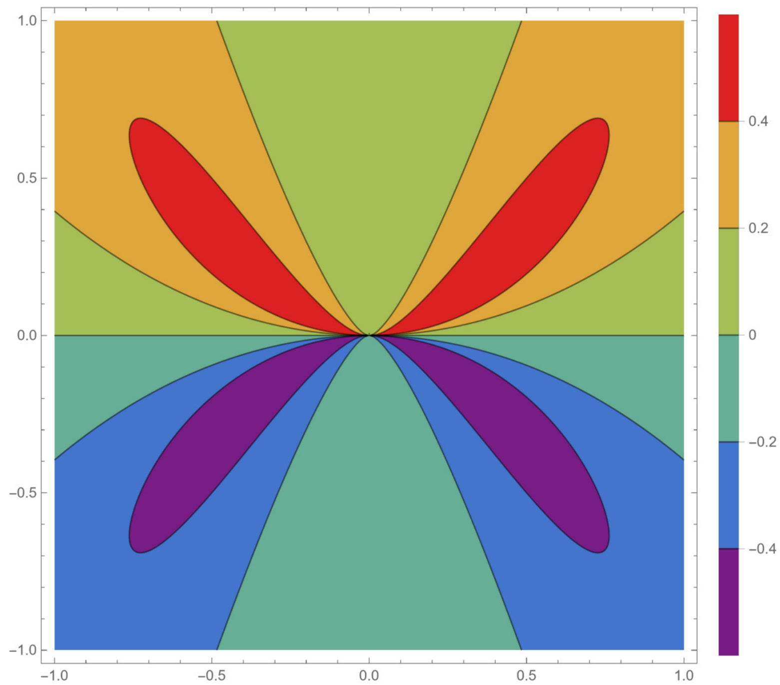

Figure 426. We see here a figure of the contour plot of the function $q(h,k)$. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

51.6 Alternative proof of differentiability (optional)

The function is not differentiable. So this section is irrelevant.

51.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability. The function is not differentiable. So this section is irrelevant.

51.8 Overview

$$f(x,y) = \begin{cases} \dfrac{x^2\, y\sqrt{|y|}}{x^4 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

51.9 One step further

We wonder what those magical curves $h = \lambda h^2$ could mean for the geometry of the graph of our function.
We calculate the curve

$$(x(t), y(t), z(t)) = (t, \lambda t^2, f(t, \lambda t^2))$$

which is entirely on the surface defined by the graph of our function.

By the composition law of differentiable functions, also called the chain rule, we have that this curve must be differentiable if f is differentiable.
We calculate the curve.

$$(x(t), y(t), z(t)) = \left(t, t^2, \frac{|t|}{2}\right).$$

We calculate the tangent vector in $t = 0$ for $t \leq 0$. It is $(1, 0, -1/2)$. We calculate the tangent vector in $t = 0$ for $t \geq 0$. It is $(1, 0, 1/2)$. So there is no tangent vector at all. This curve is not differentiable. We illustrate this situation.

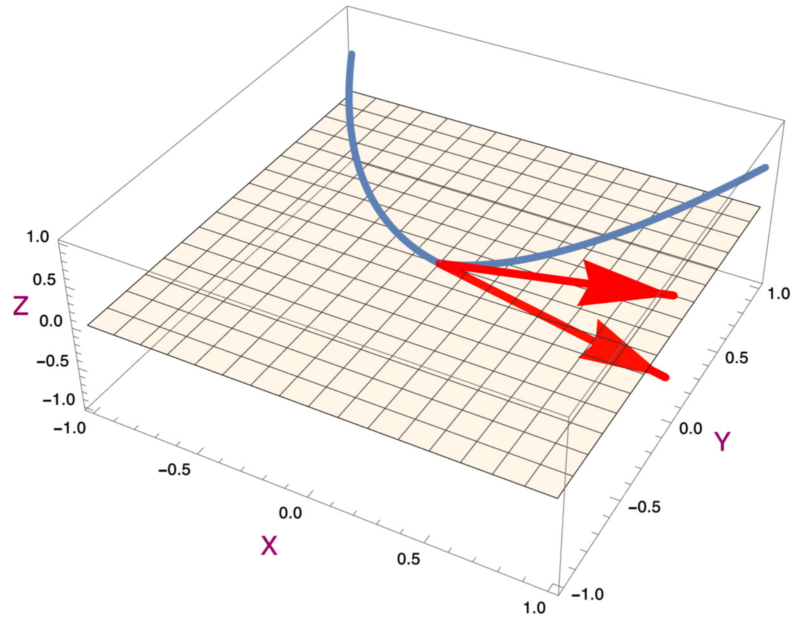

Figure 427. We see here a figure of the curve that is on our surface. We see the tangent vectors in $t = 0$. They are clearly different what cannot be possible if f is differentiable.

Exercise 52.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \sqrt[3]{x}\,\sqrt[3]{y}.$$

52.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left|\sqrt[3]{x}\,\sqrt[3]{y} - 0\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left|\sqrt[3]{x}\,\sqrt[3]{y}\right| &\leq \sqrt[3]{|x|}\,\sqrt[3]{|y|} \\
&\leq \sqrt[3]{\sqrt{x^2+y^2}}\,\sqrt[3]{\sqrt{x^2+y^2}} \\
&\leq \sqrt{x^2+y^2}^{\,2/3}.
\end{aligned}$$

It is sufficient to take $\delta = \epsilon^{3/2}$. We can find a δ, so we conclude that the function is continuous.

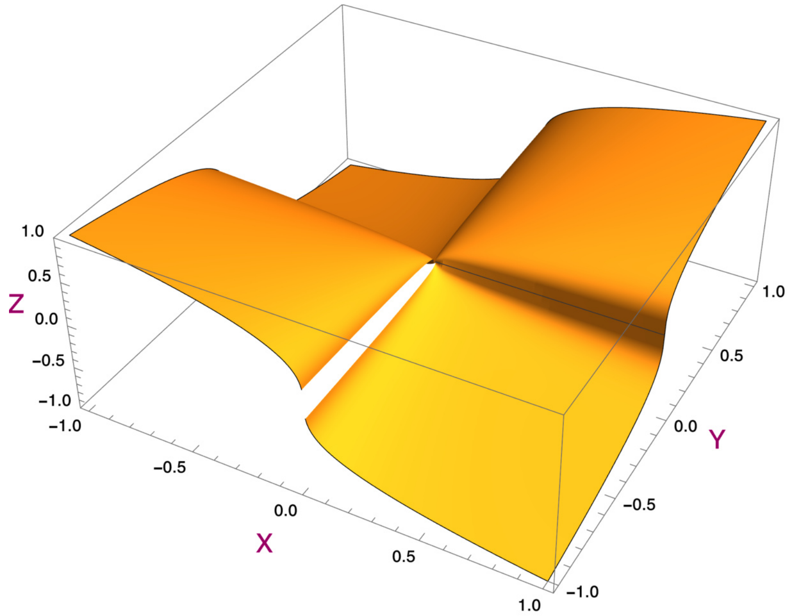

Figure 428. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

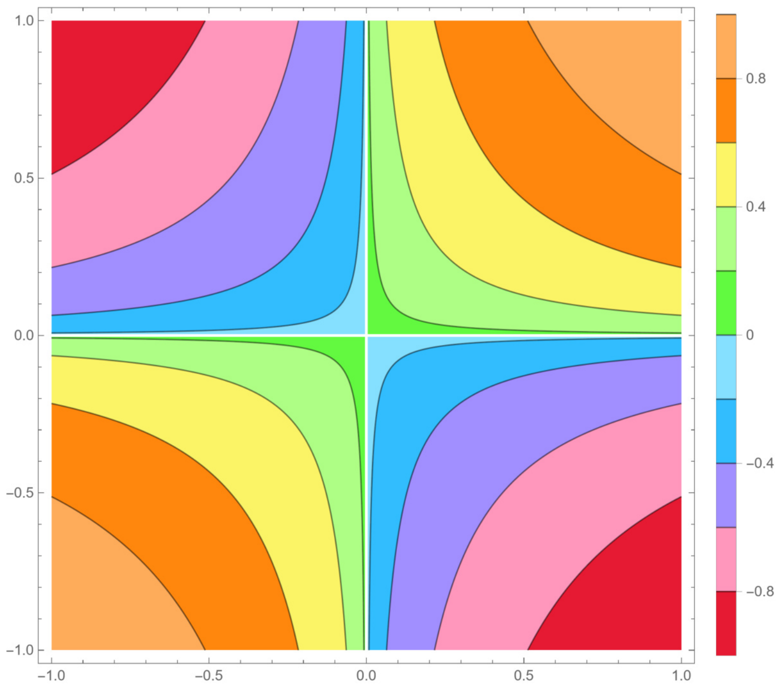

Figure 429. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

52.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y)=0 & \text{if } y\neq 0;\\ 0 & \text{if } y=0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0)=0 \quad\text{and}\quad \frac{\partial f}{\partial y}(0,0)=0.$$

52.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h \to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h \to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{\sqrt[3]{hu}\,\sqrt[3]{hv}}{h} \\ &= \lim_{h \to 0} \frac{h^{2/3}(\sqrt[3]{u}\,\sqrt[3]{v})}{h} \\ &= \lim_{h \to 0} \frac{\sqrt[3]{u}\,\sqrt[3]{v}}{h^{1/3}}. \end{aligned}$$

We see that the limit does not exist if $uv \neq 0$ but we dealt with these exceptional cases before.

So the directional derivatives do not always exist.

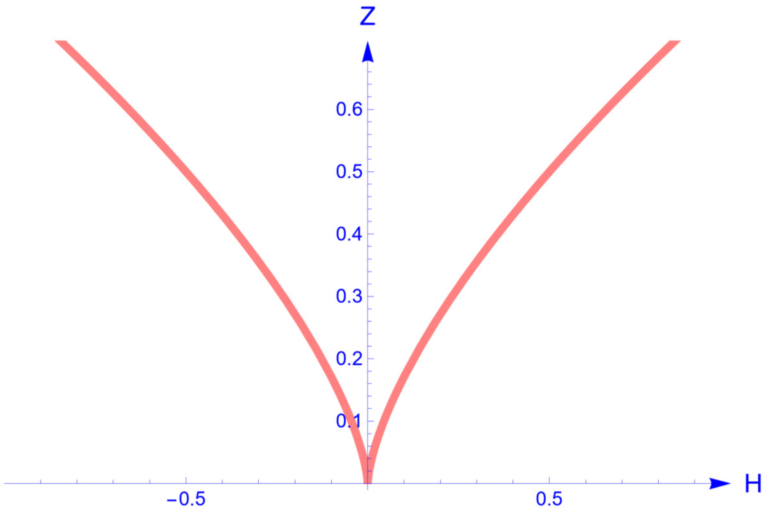

Figure 430. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$. This function is not differentiable in $h = 0$.

52.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

We doubt that all partial derivatives exist in a neighbourhood of $(0,0)$. Let us consider points $(a,0)$ with $a \neq 0$ and look upon the behaviour of the function in the Y-direction.

$$f(a,h) = \sqrt[3]{a}\,\sqrt[3]{h}.$$

The partial derivative to y in $(a,0)$ is then

$$\lim_{h\to 0} \frac{f(a,h) - f(a,0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{a}}{h^{2/3}}.$$

This limit is not finite if $a \neq 0$. This function is not differentiable in $h = 0$. Let us take a look at the behaviour in $a = 1/2$.

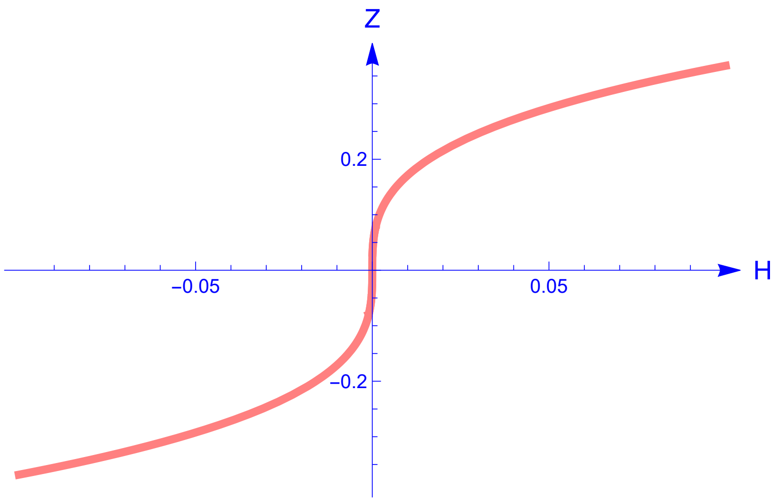

Figure 431. We see here a three dimensional figure of the graph of the function $f(a, h)$. We have drawn the function here for the value $a = 1/2$ which is exemplary for the values of a close to 0. This does not look like a differentiable function. The vertical tangent behaviour in $h = 0$ is problematic.

So an alternative proof using this criterion is not possible.

52.5 Differentiability

At least one of the directional derivatives does not exist. It follows then that the function is not differentiable.

52.6 Alternative proof of differentiability (optional)

Irrelevant because the function is not differentiable.

52.7 Continuity of the partial derivatives

Irrelevant because the function is not differentiable.

52.8 Overview

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 53.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \sin\left(\dfrac{x^2+y^2}{x^4+y^4}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

53.1 Continuity

We restrict the function to the continuous curves with equations $y = \lambda x$. We observe then that

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \sin\left(\dfrac{\lambda^2+1}{\left(\lambda^4+1\right)x^2}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have no limits. But if $f(x,y)$ is continuous, all the limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

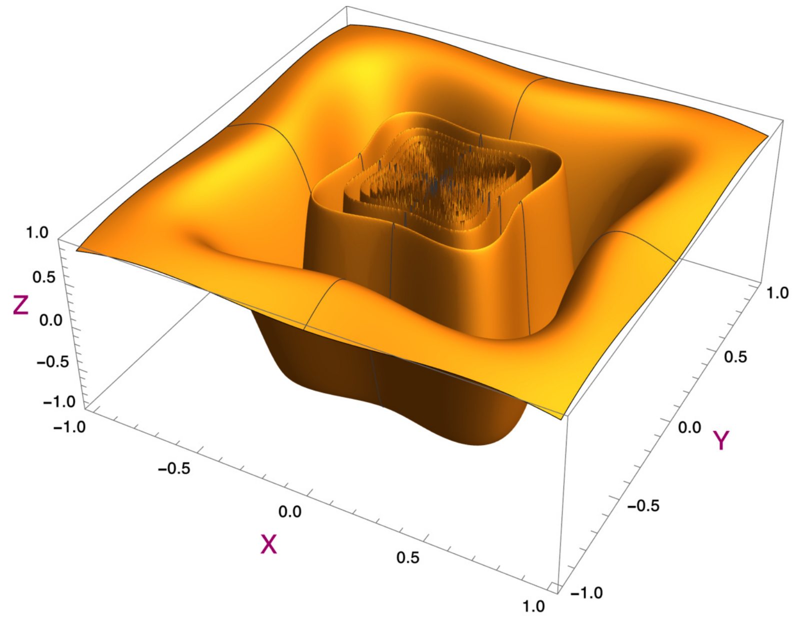

Figure 432. We see here a three dimensional figure of the graph of the function. This does not seem to be a graph of a continuous function.

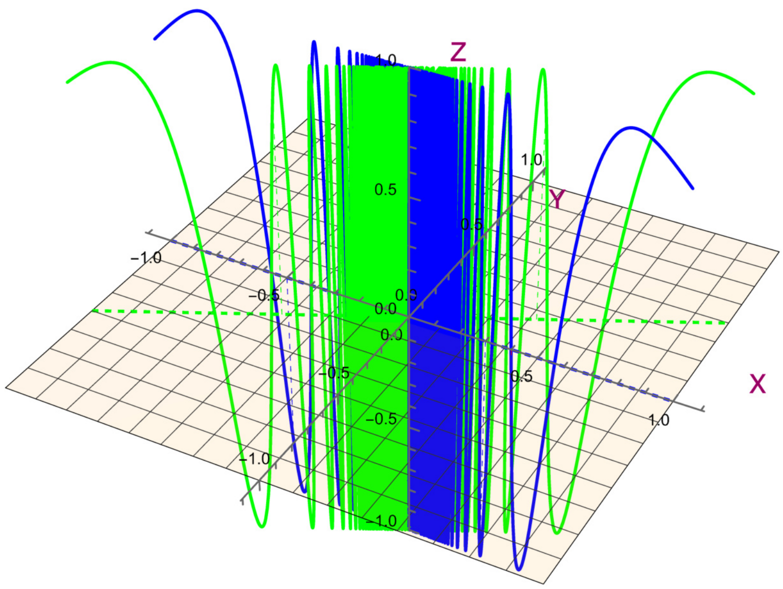

Figure 433. We have restricted the function here to $y = 0$ and $y = 1/2\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have no limits in 0. This is a standard example of the theory in one variable.

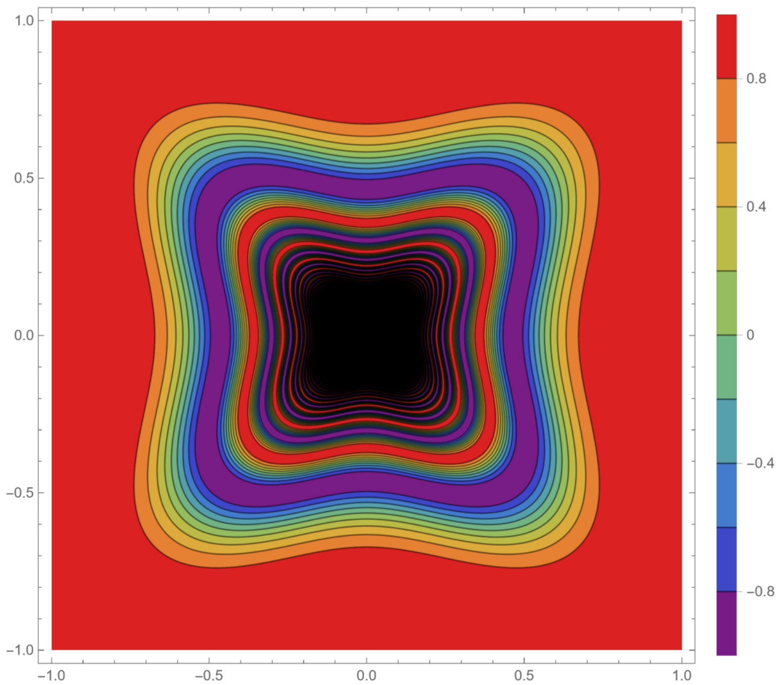

Figure 434. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

53.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = \sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

This function is not continuous and thus not differentiable. So the partial derivative $\frac{\partial f}{\partial x}(0,0)$ does not exist.

By symmetry reasons we have that the partial derivative to y does not exist.

We conclude that the partial derivative to x does not exist. and that the partial derivative to y does not exist.

53.3 Directional derivatives

Let (u, v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2 + v^2} = 1$. Then the directional derivative in $(0, 0)$ in the direction (u, v), notation $D_{(u,v)}(0, 0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0 + hu, 0 + hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1, 0)$ and $(0, 1)$.

We see that the function

$$f(0 + hu, 0 + hv) = \sin\left(\frac{u^2 + v^2}{h^2\left(u^4 + v^4\right)}\right)$$

is not continuous. So it cannot be differentiable.

So the directional derivatives do not always exist.

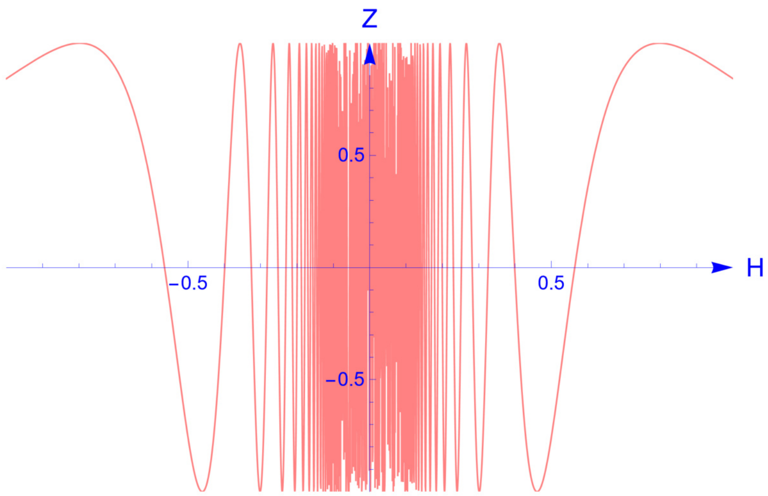

Figure 435. We see here a figure of the graph of the function restricted to the line through $(0, 0)$ with direction $(u, v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$. This function has no derivative in 0.

53.4 Alternative proof of continuity (optional)

Irrelevant. The function is not continuous.

53.5 Differentiability

The function is not continuous. It follows that the function is not differentiable.

53.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

53.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

53.8 Overview

$$f(x,y) = \begin{cases} \sin\left(\dfrac{x^2+y^2}{x^4+y^4}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	no
partial derivatives exist	no
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 54.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \sqrt{\sin^2(x) + \sin^2(y)}.$$

54.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \sqrt{\sin^2(x) + \sin^2(y)} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\left| \sqrt{\sin^2(x) + \sin^2(y)} \right| \leq \sqrt{x^2 + y^2}.$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function is continuous.

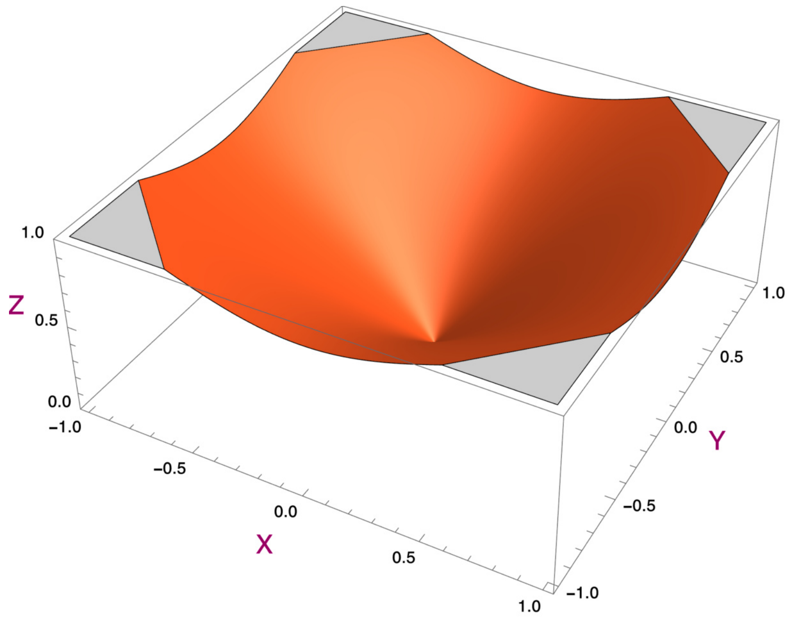

Figure 436. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

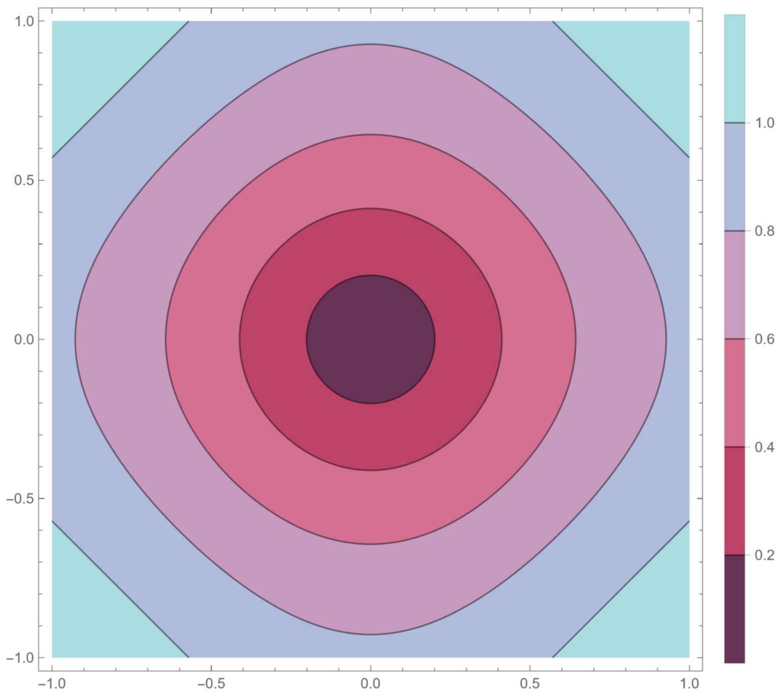

Figure 437. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

54.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = \sqrt{\sin^2(x)} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{\sin^2(h)}}{h}\\
&= \lim_{h\to 0}\frac{|\sin(h)|}{h}\frac{|h|}{|h|}\\
&= \lim_{h\to 0}\frac{|\sin(h)|}{|h|}\frac{|h|}{h}\\
&= \lim_{h\to 0}\operatorname{sgn}(h).
\end{aligned}$$

This limit does not exist.

So the partial derivative to x does not exist.

By the symmetry in the function definition, one can assume that it is the same case for y.

We conclude that the partial derivative to x does not exist and that the partial derivative to y does not exist.

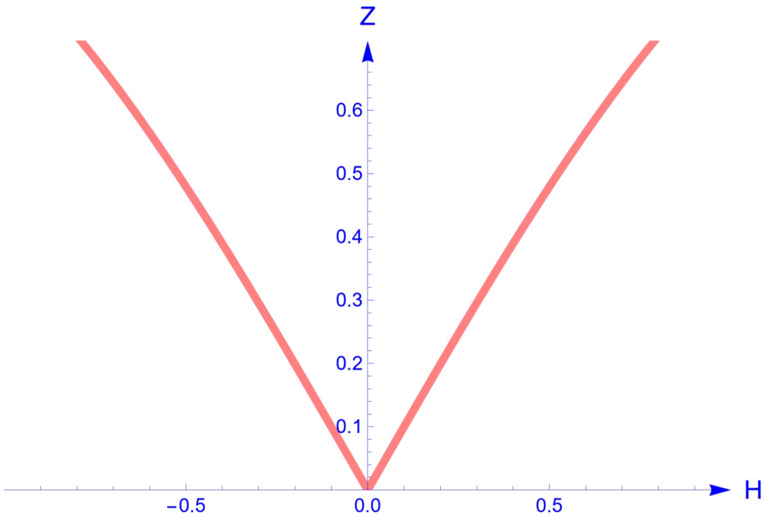

Figure 438. We see here a figure of the graph of the function restricted to the horizontal X-axis through $(0,0)$. We see that the derivative does not exist. We have plotted here the function $f(h,0)$.

54.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+h\,u,0+h\,v)-f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\lim_{h\to 0} \frac{f(0+h\,u,0+h\,v)-f(0,0)}{h} = \lim_{h\to 0} \frac{\sqrt{\sin^2(h\,u)+\sin^2(h\,v)}}{h}.$$

$$\begin{aligned}
D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu,0+hv)-f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{h^2u^2\sin^2(hu)/(hu)^2+h^2v^2\sin^2(hv)/(hv)^2}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{h^2(u^2\sin^2(hu)/(hu)^2+v^2\sin^2(hv)/(hv)^2)}}{h} \\
&= \lim_{h\to 0} \frac{|h|\sqrt{(u^2+v^2)}}{h}.
\end{aligned}$$

This limit does not exist.

So the directional derivatives do not always exist.

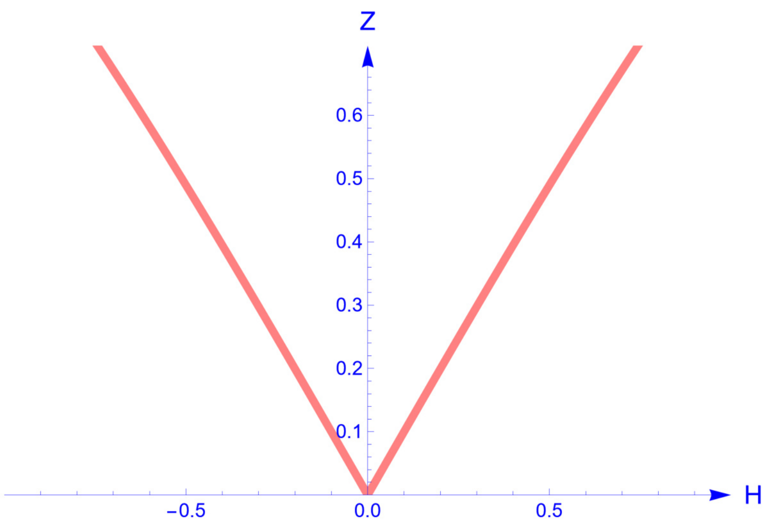

Figure 439. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$.

54.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0, 0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Irrelevant. The partials do not exist in any neighbourhood of $(0, 0)$.

54.5 Differentiability

The directional derivatives of the function do not exist. The function cannot be differentiable.

54.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

54.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

54.8 Overview

continuous	yes
partial derivatives exist	no
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 55.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{|x|+|y|-|x+y|}{\left(x^2+y^2\right)^{1/5}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

55.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{|x|+|y|-|x+y|}{(x^2+y^2)^{1/5}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left|\frac{|x|+|y|-|x+y|}{(x^2+y^2)^{1/5}}\right| &\le \frac{|x|+|y|+|x+y|}{(x^2+y^2)^{1/5}} \\
&\le \frac{|x|+|y|+|x|+|y|}{(x^2+y^2)^{1/5}} \\
&\le \frac{2|x|+2|y|}{(x^2+y^2)^{1/5}} \\
&\le \frac{2\sqrt{x^2+y^2}+2\sqrt{x^2+y^2}}{(x^2+y^2)^{1/5}} \\
&\le \frac{4\sqrt{x^2+y^2}}{(x^2+y^2)^{1/5}} \\
&\le 4\sqrt{x^2+y^2}^{4/5}.
\end{aligned}$$

It is sufficient to take $\delta = (\epsilon/4)^{5/4}$. We can find a δ, so we conclude that the function is continuous.

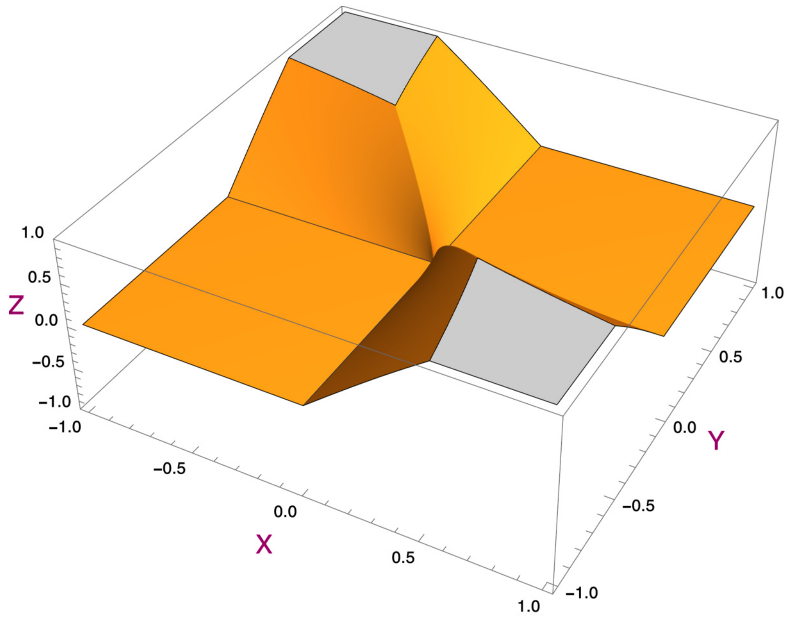

Figure 440. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

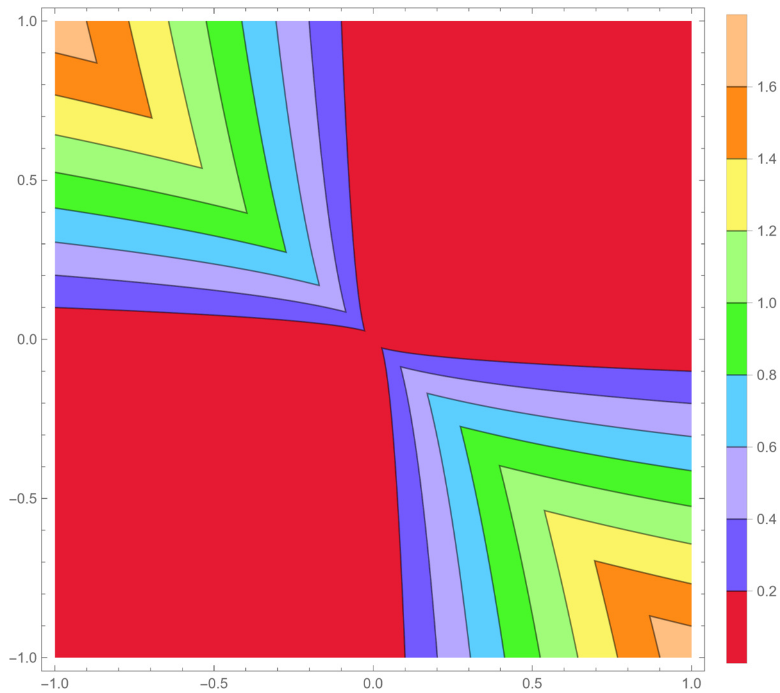

Figure 441. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

55.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

55.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+h\,u, 0+h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{|h|\,(-|(u+v)| + |u| + |v|)}{h\,h^{2/5}\,\sqrt[5]{(u^2+v^2)}} \\ &= \lim_{h\to 0} \frac{\operatorname{sgn}(h)\,(-|(u+v)| + |u| + |v|)}{h^{2/5}\,\sqrt[5]{(u^2+v^2)}}. \end{aligned}$$

This limit does not exist if $-|(u+v)| + |u| + |v| \neq 0$. It is equal to 0 if $-|(u+v)| + |u| + |v| = 0$.

So the directional derivatives do not always exist.

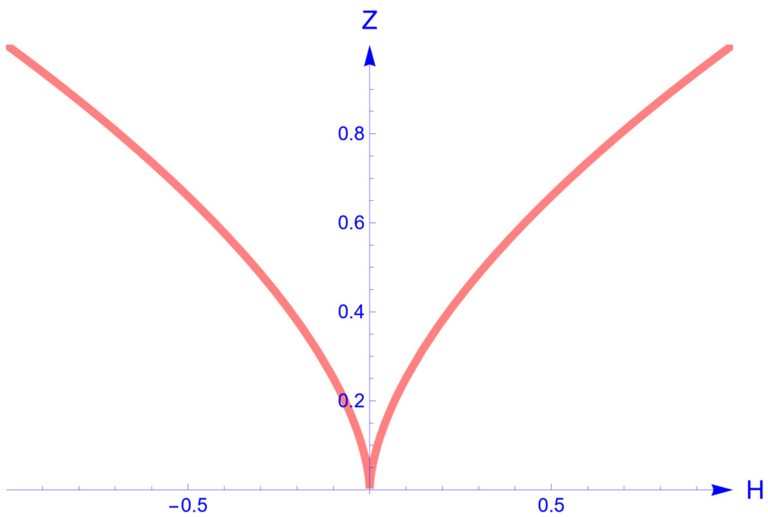

Figure 442. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(-\sqrt{3}/2, 1/2\right)$. We have plotted here the function $f(h\,u, h\,v)$.

55.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

We have doubts about the existence of the partial derivatives in a neighbourhood of $(0,0)$. Let us take a look at the point $(a,0)$ in the Y-direction. The function $f(a,h)$ with $a > 0$ is there the following.

$$f(a,h) = \begin{cases} \dfrac{-|a+h|+|a|+|h|}{\sqrt[5]{a^2+h^2}} = \dfrac{-2h}{\sqrt[5]{a^2+h^2}} & \text{if } h < 0 \text{ and } |h| < a; \\ 0 & \text{if } h \geq 0. \end{cases}$$

This function has no partial derivative to y in $(a,0)$. We conclude that an alternative proof following this criterion is not possible.

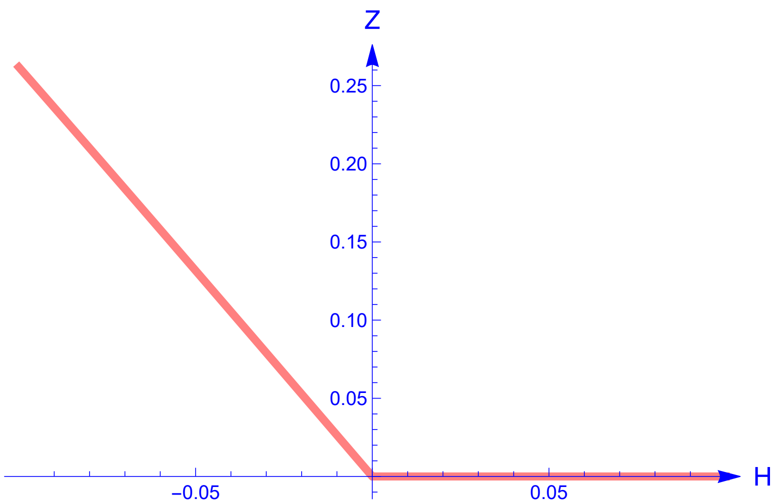

Figure 443. We see here a figure of the graph of the function $f(a,h)$. We have drawn the function here for the value $a = 1/2$ which is representative for the values of $a > 0$. This does not look like a differentiable function.

55.5 Differentiability

The function is not differentiable. At least one of the directional derivatives does not exist.

55.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

55.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

55.8 Overview

$$f(x,y) = \begin{cases} \dfrac{|x| + |y| - |x+y|}{\left(x^2+y^2\right)^{1/5}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 56.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y)=\begin{cases}\dfrac{|x|+|y|-|x+y|}{\left(x^2+y^2\right)^{1/2}} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

56.1 Continuity

We restrict the function to the continuous curves with equations $y=\lambda x$. We observe then that

$$f|_{y=\lambda x}(x,y)=\begin{cases}\dfrac{\partial f}{\partial x}(x,\lambda x) & \\ \quad=\dfrac{|\lambda x|-|\lambda x+x|+|x|}{\sqrt{\lambda^2x^2+x^2}} & \text{if } x\neq 0;\\ \quad=\dfrac{|x|\left(|\lambda|-|\lambda+1|+1\right)}{|x|\sqrt{\lambda^2+1}} & \\ 0 & \text{if } x=0.\end{cases}$$

We see that these restricted functions have different limits if $|\lambda|-|\lambda+1|+1\neq 0$. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0)=0$. So this function $f(x,y)$ is not continuous.

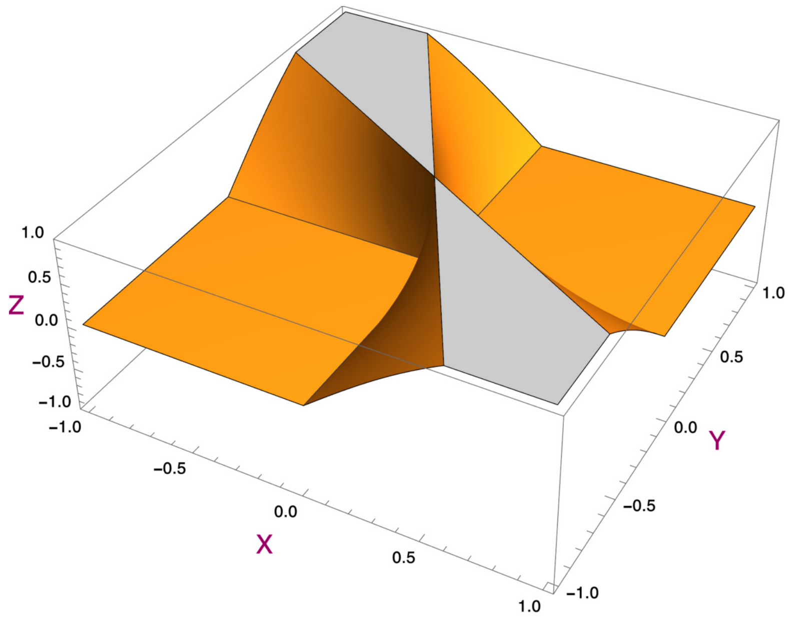

Figure 444. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

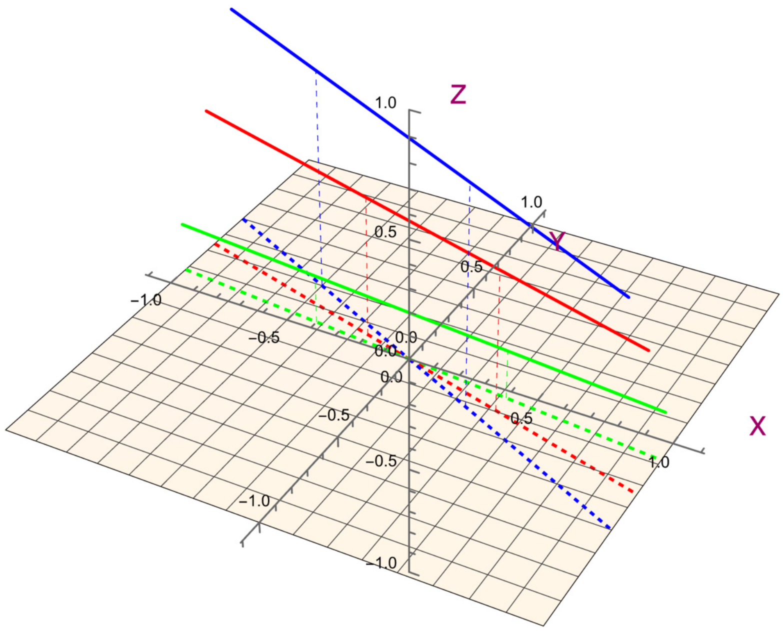

Figure 445. We have restricted the function here to $y = -3/10\,x$ and $y = -1/10\,x$ and $y = -1/2\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

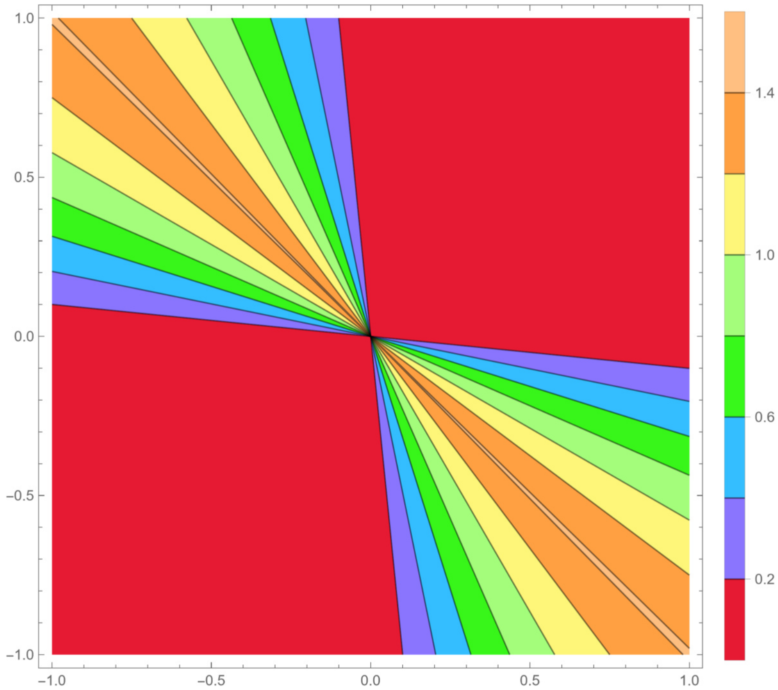

Figure 446. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

56.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.
By symmetry reasons, the partial derivative to y does exist.

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

56.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0}\frac{f(0+hu,0+hv)-f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$
\begin{aligned}
D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{-|h(u+v)| + |hu| + |hv|}{h\sqrt{h^2\ (u^2+v^2)}} \\
&= \lim_{h\to 0} \frac{|h|(-|u+v| + |u| + |v|)}{h|h|\sqrt{u^2+v^2}} \\
&= \lim_{h\to 0} \frac{-|u+v| + |u| + |v|}{h\sqrt{u^2+v^2}}.
\end{aligned}
$$

This limit is 0 if $-|u+v| + |u| + |v| = 0$. But if $-|u+v| + |u| + |v| \neq 0$, then the function $f(hu, hv)$ is not continuous, and thus evidently not differentiable. We consult the figure.

So the directional derivatives do not always exist.

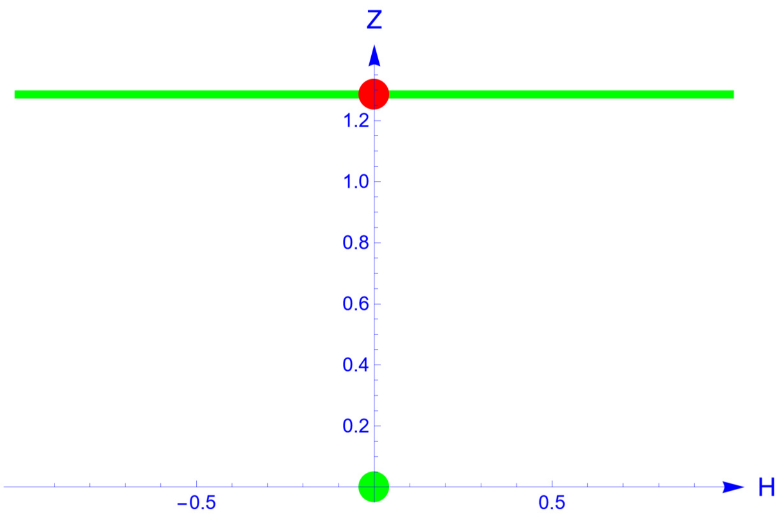

Figure 447. We see here a figure of the graph of the function restricted to the line through $(u, v) = (0, 0)$ with direction $(\cos\left(\frac{7\pi}{9}\right), \sin\left(\frac{7\pi}{9}\right))$. We have plotted here the function $f(hu, hv)$. This function is not continuous in this case.

56.4 Alternative proof of continuity (optional)

Irrelevant. The function is not continuous.

56.5 Differentiability

The function is not continuous and thus not differentiable.

56.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

56.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

56.8 Overview

$$f(x,y) = \begin{cases} \dfrac{|x| + |y| - |x+y|}{\left(x^2+y^2\right)^{1/2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	no
partial derivatives exist	yes
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 57.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y)=\begin{cases}\sin\left(\dfrac{1}{x^2+y^2}\right) & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

57.1 Continuity

We restrict the function to the continuous curves with equations $y = \lambda x$???. We observe then that

$$f|_{y=\lambda x}(x,y)=\begin{cases}f(x,\lambda x)=\sin\left(\dfrac{1}{\lambda^2x^2+x^2}\right) & \text{if } x\neq 0;\\ 0 & \text{if } x=0.\end{cases}$$

We see that these restricted functions are not continuous in 0. They have no limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0)=0$. So this function $f(x,y)$ is not continuous.

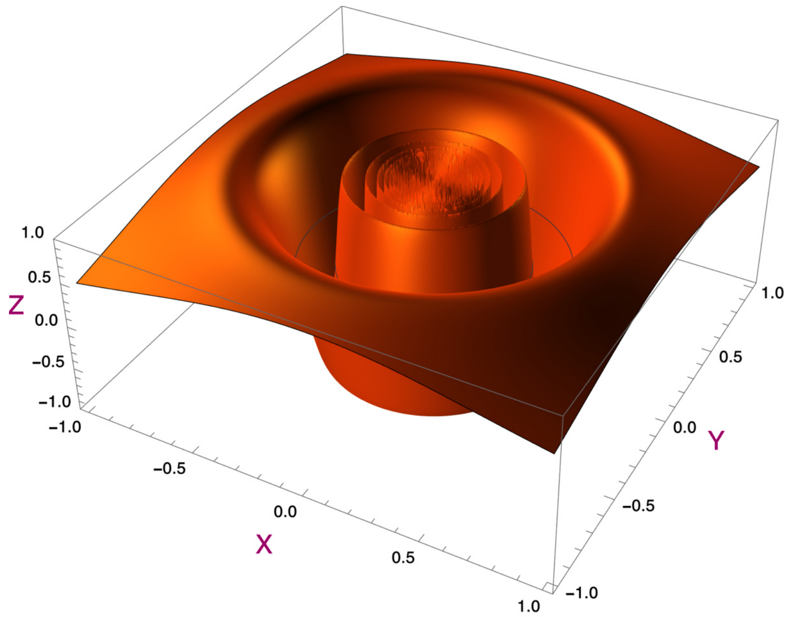

Figure 448. We see here a three dimensional figure of the graph of the function. This does not seem to be a graph of a continuous function.

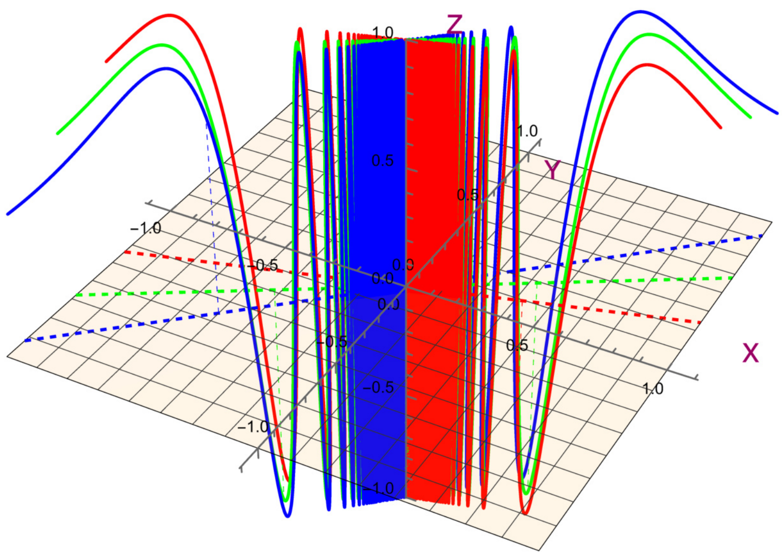

Figure 449. We have restricted the function here to $y = 3/10\,x$ and $y = 3/5\,x$ and $y = 9/10\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have no limits in 0.

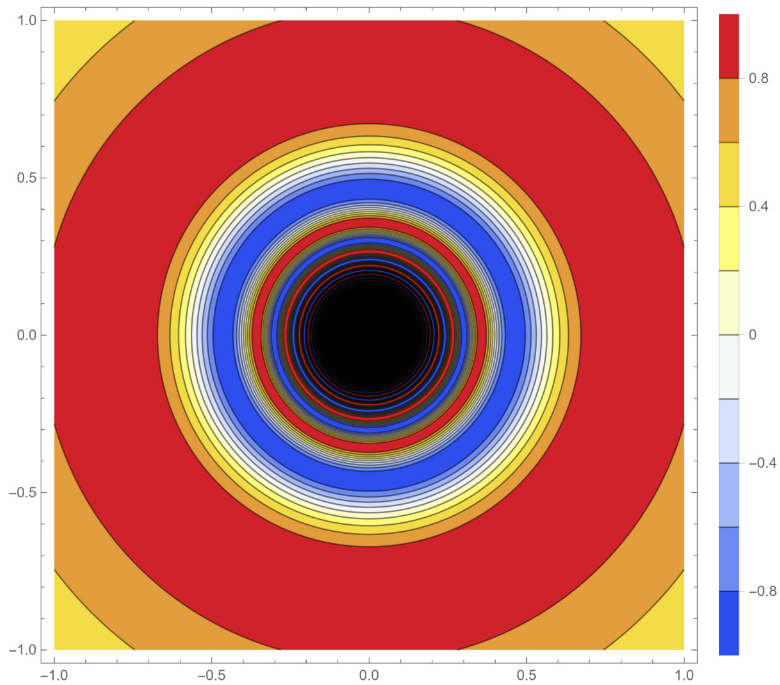

Figure 450. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

57.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = \sin\left(\dfrac{1}{x^2}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

This function is not continuous, so the partial derivative to x does not exist.

By rotational symmetry, the partial derivative to y does not exist.
We conclude that the partial derivative to x does not exist and that the partial derivative to y does not exist.

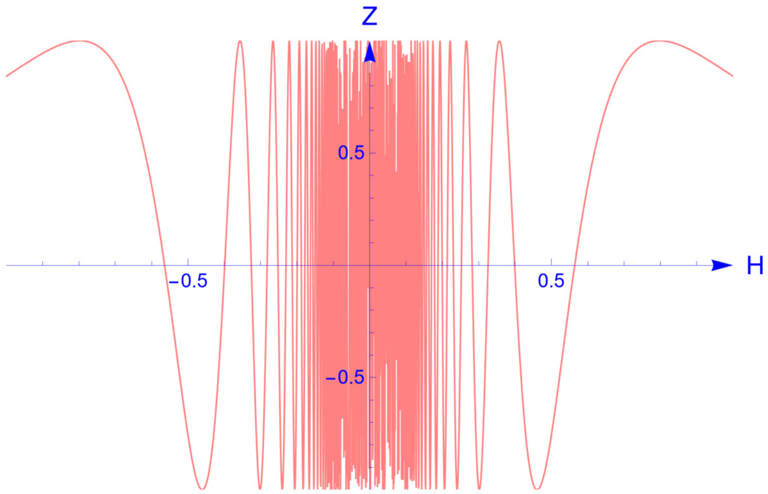

Figure 451. We see here a figure of the graph of the function restricted to the horizontal X-axis through $(0,0)$. This is a classical example of discontinuity and the function is of course also not differentiable.

57.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0)=\lim_{h\to 0}\frac{f(0+hu,0+hv)-f(0,0)}{h}.$$

By rotational symmetry, the directional derivatives do not exist.

57.4 Alternative proof of continuity (optional)

Irrelevant. The function is not continuous.

57.5 Differentiability

The function is not continuous. Thus it is not differentiable.

57.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

57.8 Overview

$$f(x,y)=\begin{cases}\sin\left(\dfrac{1}{x^2+y^2}\right) & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

continuous	no
partial derivatives exist	no
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 58.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0, 0)$.

$$f(x, y) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } y = 0 \text{ and } x \neq 0, \\ y \sin\left(\frac{1}{y}\right) & \text{if } x = 0 \text{ and } y \neq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

58.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x, y) - f(0, 0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x, y) - (0, 0)|| < \delta$ it follows that $|f(x, y) - f(0, 0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0, 0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| f(x, y) - 0 \right| < \epsilon.$$

We will be first investigate the function on the X-axis.

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$
\begin{aligned}
|f(x,0) - 0| &\le |x| \left|\sin\left(\frac{1}{x}\right)\right| \\
&\le |x| \\
&\le \sqrt{x^2 + y^2}.
\end{aligned}
$$

It is sufficient to take $\delta = \epsilon$.

We will now investigate the function on the Y-axis. This can be proven in a similar way and we can take the same δ.

We will now investigate the function elsewhere. The function is there equal to 0, so we can take every δ.

We conclude that δ can be chosen as $\delta = \epsilon$. The function is continuous.

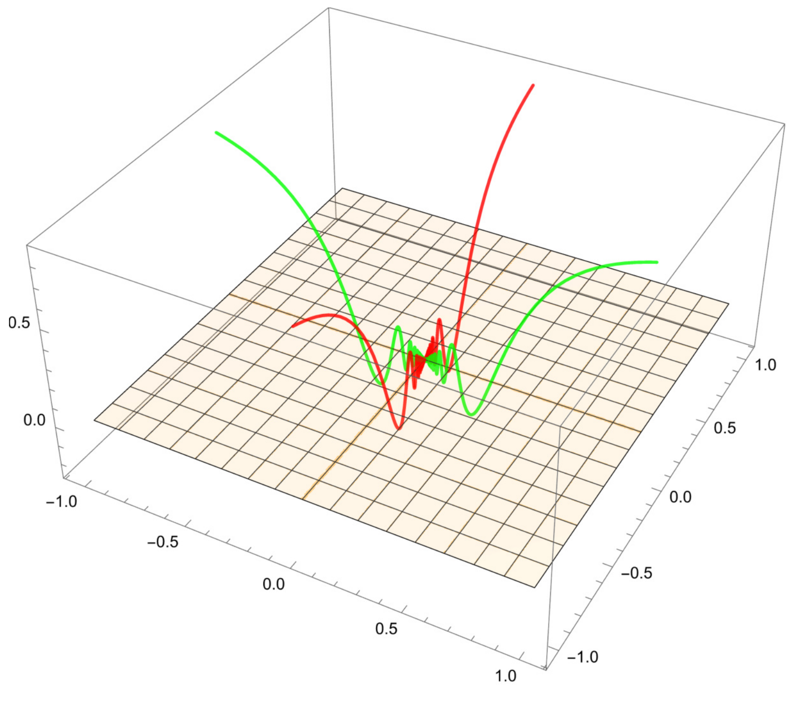

Figure 452. We see here a three dimensional figure of the graph of the function. For a good understanding: the graph consists of the green function above the X-axis and the red function above the Y-axis. Above all other points we have the yellow opaque plane. This looks like a continuous function.

A contour plot cannot be drawn. The function behaves too weird for that.

58.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).
We observe then that

$$f\big|_{y=0}(x,y) = \begin{cases} f(x,0) = x\,\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \sin\left(\frac{1}{h}\right). \end{aligned}$$

This limit does not exist. This is a standard example from the theory in one variable.

So the partial derivative to x does not exist.

The partial derivative to y does not exist by symmetry considerations.

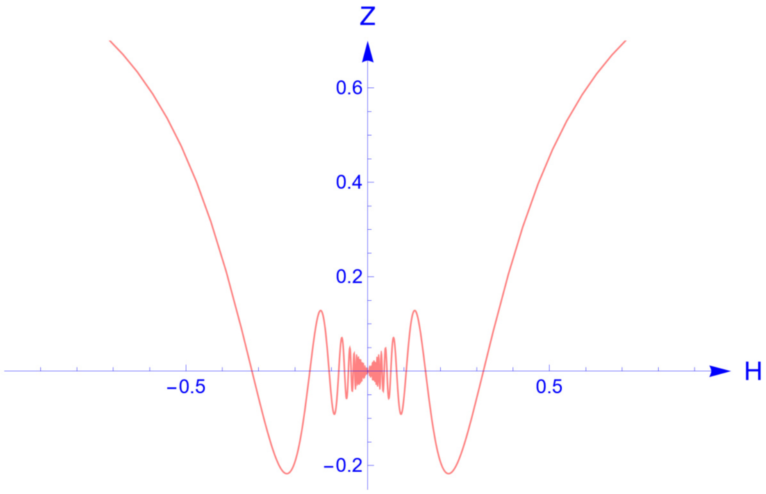

Figure 453. We see here a figure of the graph of the function restricted to the horizontal X-axis through $(0,0)$. This function is not differentiable. This is a standard example from the theory in one variable. We have plotted here the function $f(h,0)$.

58.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0)=\lim_{h\to 0}\frac{f(0+h\,u,0+h\,v)-f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit if $u\,v\neq 0$:

$$\lim_{h\to 0}\frac{f(0+h\,u,0+h\,v)-f(0,0)}{h}=0.$$

So the directional derivatives do not always exist because the partial derivatives do not exist.

58.4 Alternative proof of continuity (optional)

The partial derivatives do not exist. So a proof following this criterion cannot be given.

58.5 Differentiability

The function has no partial derivatives. So the function is not differentiable.

58.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

58.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

58.8 Overview

$$f(x,y) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } y = 0 \text{ and } x \neq 0, \\ y \sin\left(\frac{1}{y}\right) & \text{if } x = 0 \text{ and } y \neq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

continuous	yes
partial derivatives exist	no
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 59.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \min\{x,y\} = \frac{x+y}{2} - \frac{|x-y|}{2}.$$

59.1 Continuity

We can immediately conclude that this function is not continuous. All the composing functions are continuous. We are nevertheless going to reason by the definition.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \min\{x,y\} = \frac{x+y}{2} - \frac{|x-y|}{2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{x+y}{2}-\frac{|x-y|}{2}\right| &\leq \frac{|x|+|y|}{2}+\frac{|x|+|y|}{2}\\ &\leq \frac{\sqrt{x^2+y^2}+\sqrt{x^2+y^2}}{2}+\frac{\sqrt{x^2+y^2}+\sqrt{x^2+y^2}}{2}\\ &\leq 2\sqrt{x^2+y^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon/2$. We can find a δ, so we conclude that the function is continuous.

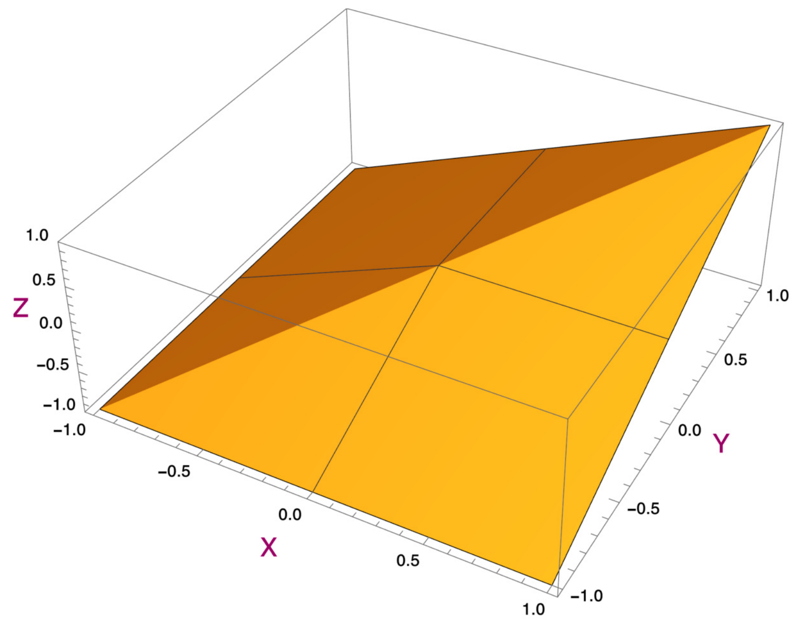

Figure 454. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

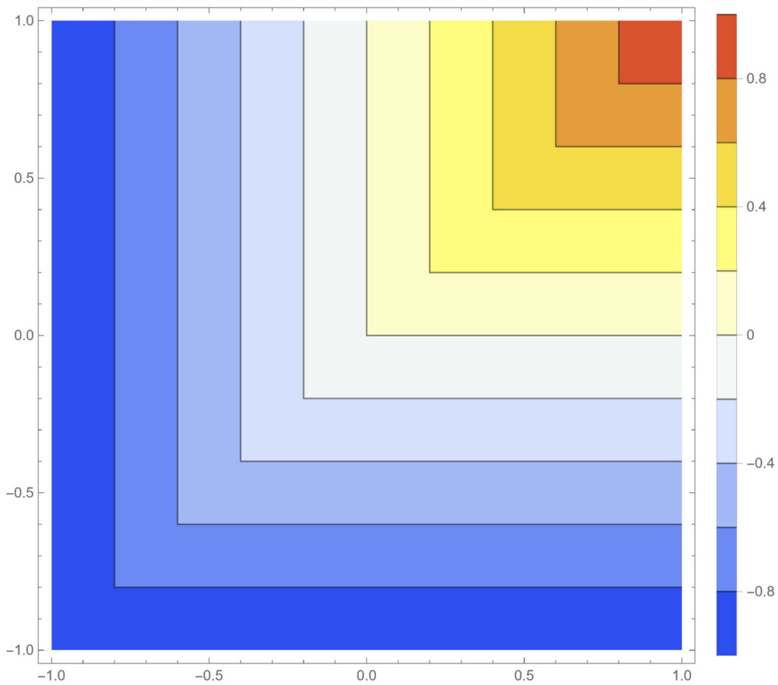

Figure 455. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

59.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \frac{x}{2} - \frac{|x|}{2}.$$

So

$$
\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{h}{2\,h} - \frac{|h|}{2\,h} \\
&= \lim_{h\to 0} \frac{1}{2} - \frac{\operatorname{sgn}(h)}{2}.
\end{aligned}
$$

So the partial derivative to x does not exist.

The partial derivative can be calculated by symmetric reasons in a completely similar way.

So the partial derivative to y does not exist.

We conclude that the partial derivative to x does not exist and that the partial derivative to y does not exist.

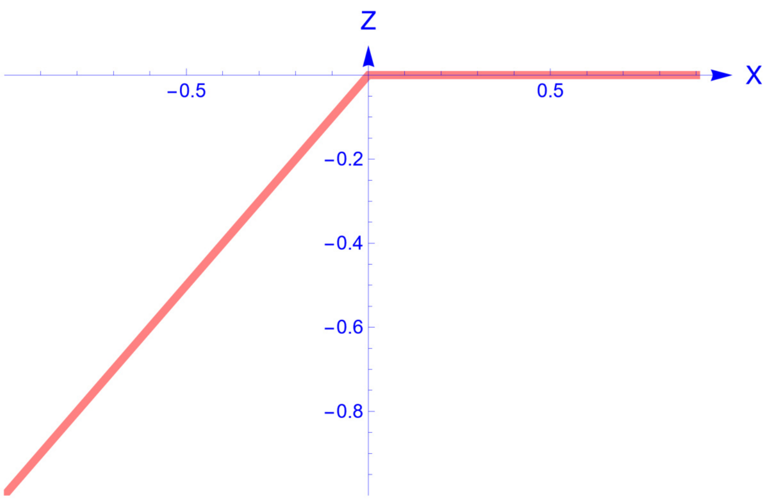

Figure 456. We see here a figure of the graph of the function restricted to the horizontal X-axis through $(0,0)$. The function is not differentiable in 0.

59.3 Directional derivatives

Let (u, v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2 + v^2} = 1$. Then the directional derivative in $(0, 0)$ in the direction (u, v), notation $D_{(u,v)}(0, 0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h \to 0} \frac{f(0 + hu, 0 + hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1, 0)$ and $(0, 1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h \to 0} \frac{f(0 + hu, 0 + hv) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{h(u + v) - |h(u - v)|}{2h} \\ &= \lim_{h \to 0} \frac{(u + v) - \operatorname{sgn}(h)|(u - v)|}{2}. \end{aligned}$$

This limit does not exist unless $u - v = 0$. So in almost all cases, the partial derivative does not exist.

So the directional derivatives do not always exist.

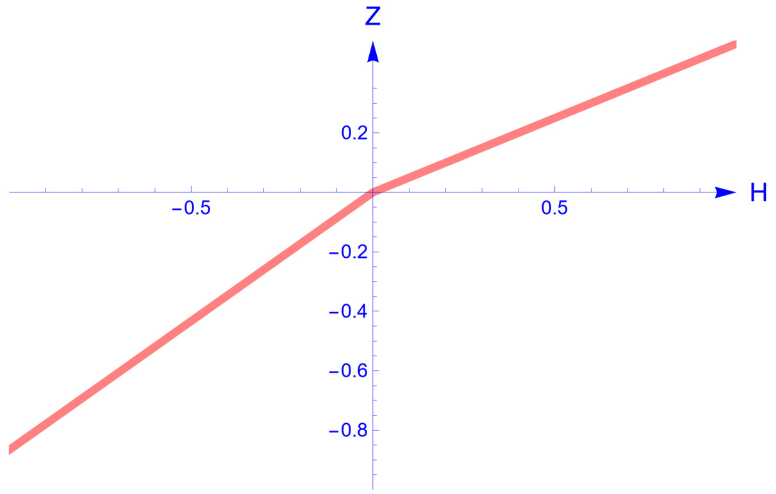

Figure 457. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $\left(\sqrt{3}/2, 1/2\right)$. We see that this function is not differentiable in $h = 0$. We have plotted here the function $f(h, 0)$.

59.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

The partial derivatives do not all exist in any neighbourhood of $(0,0)$. So this criterion cannot be used.

59.5 Differentiability

At least one of the directional derivatives does not exist, thus the function is not differentiable. So it is futile to continue.

59.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

59.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

59.8 Overview

$$f(x, y) = \min\{x, y\} = \frac{x + y}{2} - \frac{|x - y|}{2}.$$

continuous	yes
partial derivatives exist	no
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 60.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0, 0)$.

$$f(x, y) = |x| + |y|.$$

60.1 Continuity

This function is continuous because it is composed of continuous functions. We give nevertheless a classical proof. We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x, y) - f(0, 0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x, y) - (0, 0)|| < \delta$ it follows that $|f(x, y) - f(0, 0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x, y) - (0, 0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\big| |x| + |y| - 0 \big| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned} \big| |x| + |y| \big| &\leq \sqrt{x^2 + y^2} + \sqrt{x^2 + y^2} \\ &\leq 2\sqrt{x^2 + y^2}. \end{aligned}$$

It is sufficient to take $\delta = \epsilon/2$. We can find a δ, so we conclude that the function is continuous.

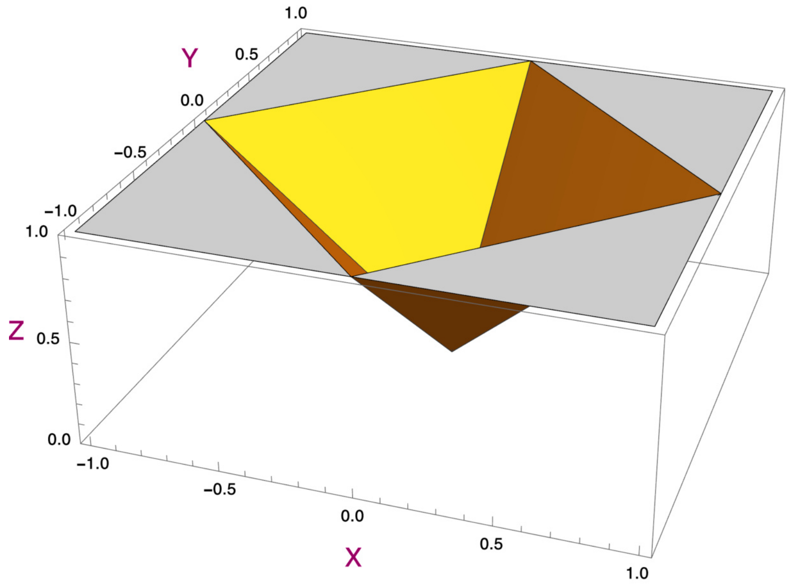

Figure 458. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

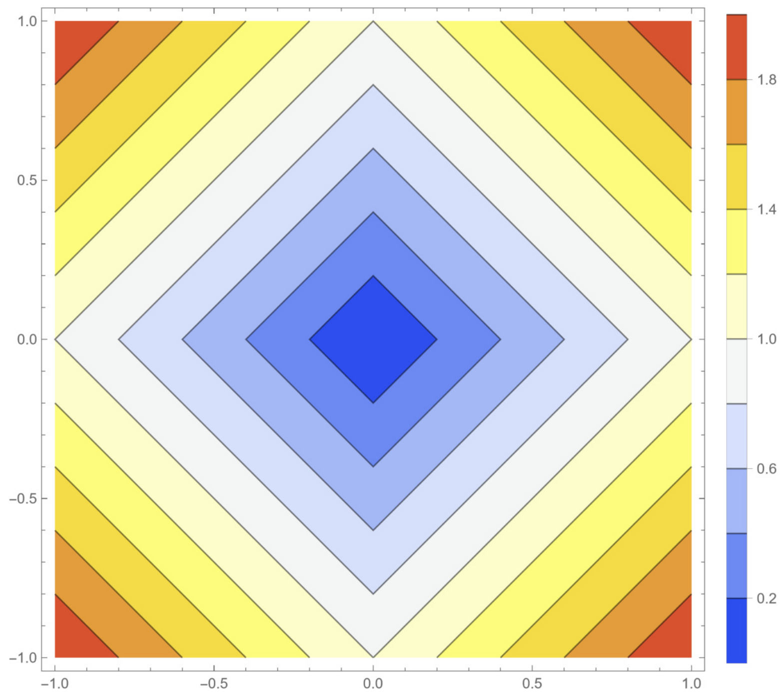

Figure 459. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

60.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = |x| & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0}\frac{|h|}{h}\\ &= \lim_{h\to 0}\operatorname{sgn}(h).\end{aligned}$$

So the partial derivative to x does not exist.

The partial derivative to y does not exist by symmetry reasons.

We conclude that the partial derivative to x does not exist and that the partial derivative to y does not exist.

60.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0}\frac{f(0+hu,0+hv)-f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned}D_{(u,v)}(0,0) &= \lim_{h\to 0}\frac{f(0+hu,0+hv)-f(0,0)}{h}\\ &= \lim_{h\to 0}\frac{|hu|+|hv|}{h}\\ &= \lim_{h\to 0}\frac{|h|(|u|+|v|)}{h}\\ &= \lim_{h\to 0}\operatorname{sgn}(h)\,(|u|+|v|).\end{aligned}$$

This limit does not exist unless $|u| + |v| = 0$ and that is not possible for unit vectors.

So the directional derivatives do not always exist.

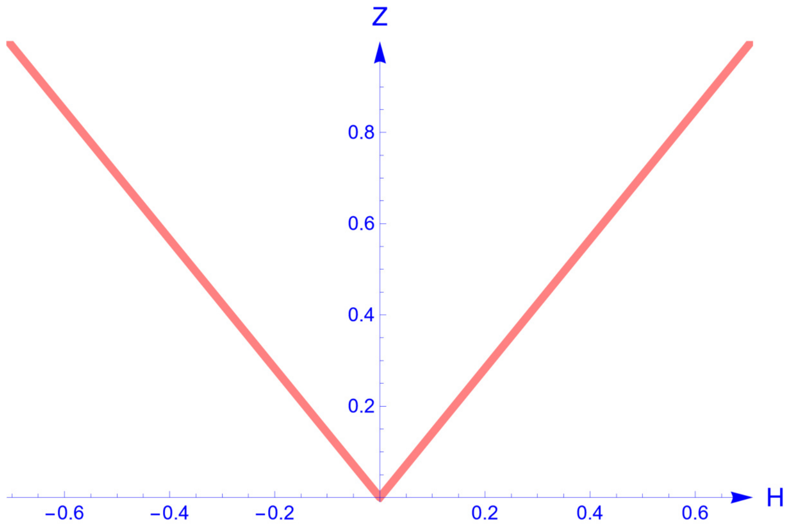

Figure 460. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$. The function is not differentiable in 0.

60.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

The partial derivatives do not exist in any neighbourhood of $(0,0)$. This criterion cannot be applied.

60.5 Differentiability

At least one of the directional derivatives does not exist, thus the function is not differentiable. So it is futile to continue.

60.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

60.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

60.8 Overview

$$f(x, y) = |x| + |y|.$$

continuous	yes
partial derivatives exist	no
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 61.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

61.1 Continuity

We restrict the function to the continuous curves with equations $y = \lambda x$. We observe then that

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{\lambda x^2}{\lambda^2 x^2 + x^2} = \dfrac{\lambda}{\lambda^2+1} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

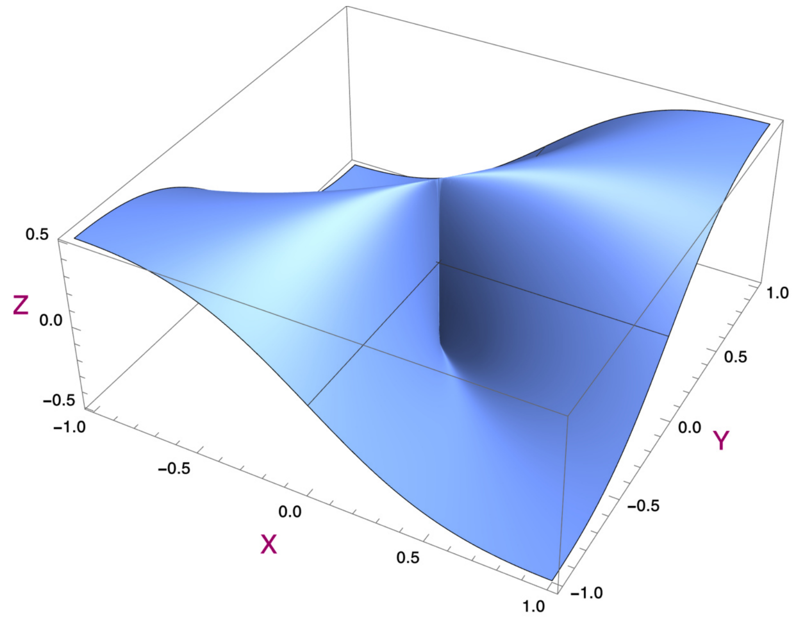

Figure 461. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

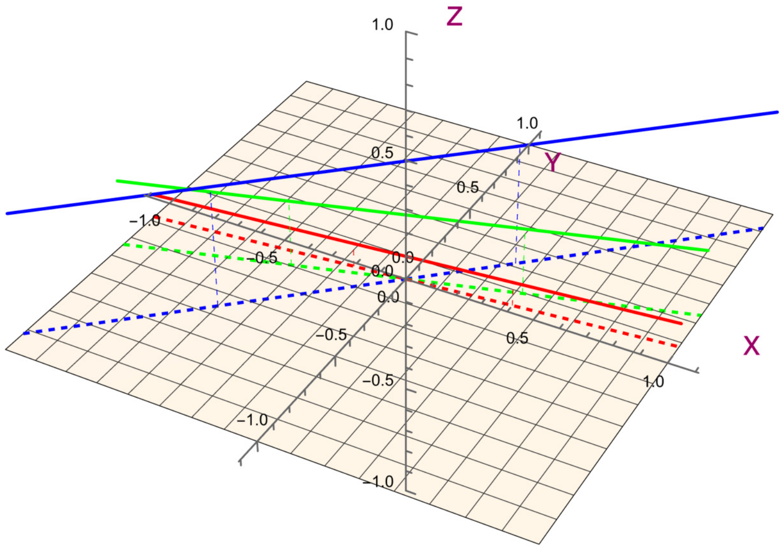

Figure 462. We have restricted the function here to $y = 1/10\,x$ and $y = 3/10\,x$ and $y = 9/10\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

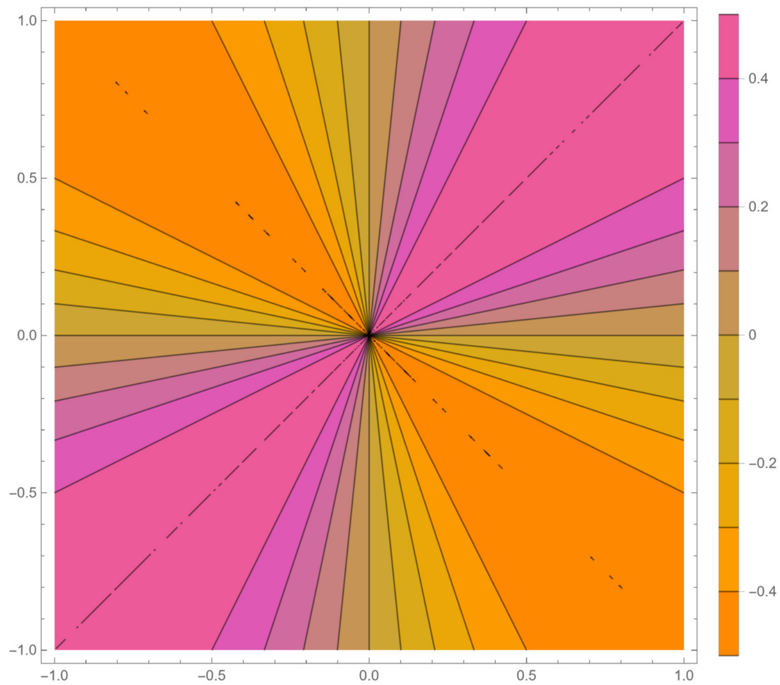

Figure 463. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

61.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0.$$

So the partial derivative to x does not exist.

We see that the partial derivative to y exists by symmetry considerations.

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

61.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

We have however the following discontinuous function. It is not differentiable.

$$f(0+hu,0+hv) = \begin{cases} \dfrac{uv}{u^2+v^2} & \text{if } h \neq 0, \\ 0 & \text{if } h = 0. \end{cases}$$

So the directional derivatives do not always exist.

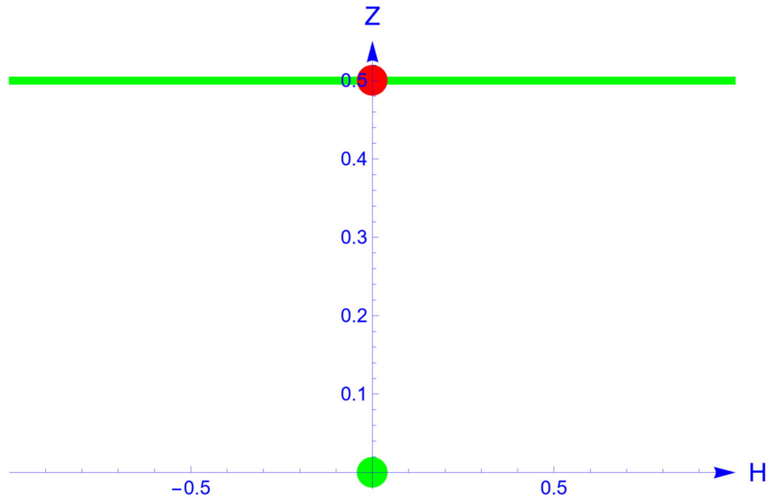

Figure 464. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$.

61.4 Alternative proof of continuity (optional)

The function is not continuous. So this section is irrelevant.

61.5 Differentiability

We have that the function is not continuous. Thus the function is not differentiable.

61.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

61.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

61.8 Overview

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	no
partial derivatives exist	yes
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 62.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \left(\sqrt[3]{x} + \sqrt[3]{y}\right)^3.$$

62.1 Continuity

We could reason that the function is composed by continuous functions and that there is nothing to be proved any more. We are nevertheless going to give a classic proof.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \left(\sqrt[3]{x} + \sqrt[3]{y}\right)^3 - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\left(\sqrt[3]{x}+\sqrt[3]{y}\right)^3\right| &\leq \left(\sqrt[3]{\sqrt{x^2+y^2}}+\sqrt[3]{\sqrt{x^2+y^2}}\right)^3\\ &\leq \left(2\sqrt[3]{\sqrt{x^2+y^2}}\right)^3\\ &\leq 8\sqrt{x^2+y^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon/8$. We can find a δ, so we conclude that the function is continuous.

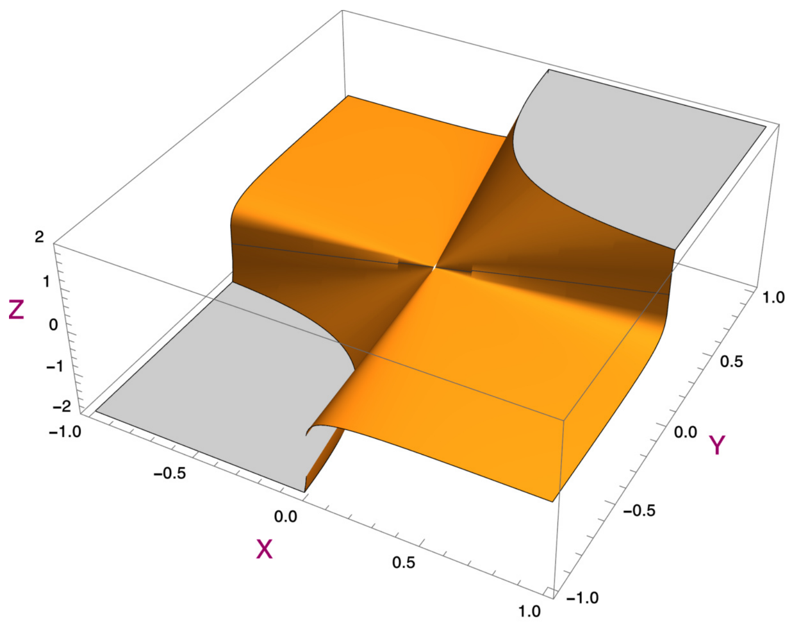

Figure 465. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

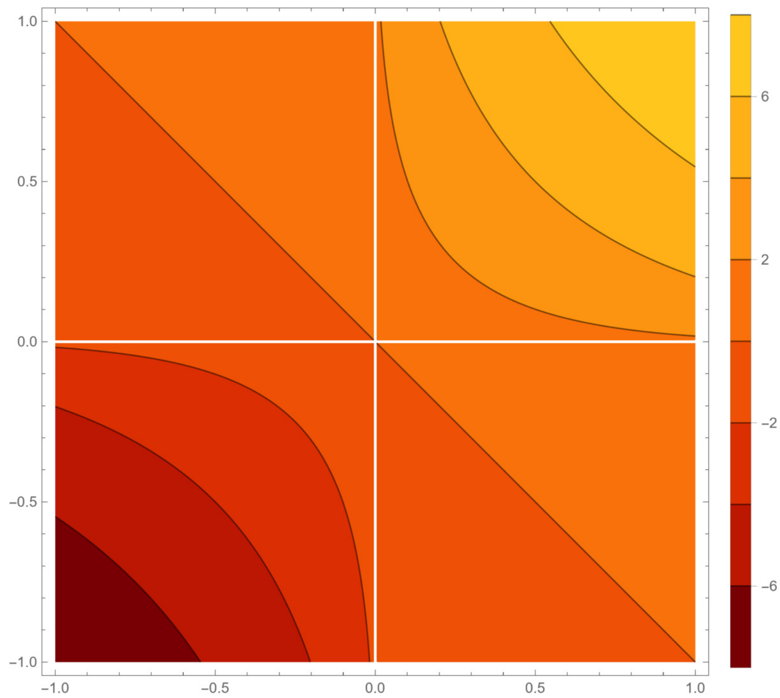

Figure 466. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0, 0)$.

62.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).
We observe then that

$$f|_{y=0}(x, y) = \left(\sqrt[3]{x}\right)^3 = x.$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0, 0) &= \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} \\ &= \lim_{h\to 0} 1 \\ &= 1.\end{aligned}$$

So the partial derivative to x does exist.

By symmetry reasons we can do similar calculations for y. So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 1 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 1.$$

62.3 Directional derivatives

Let (u, v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2 + v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u, v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h \to 0} \frac{f(0 + hu, 0 + hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h \to 0} \frac{f(0 + hu, 0 + hv) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{\left(\sqrt[3]{hu} + \sqrt[3]{hv}\right)^3}{h} \\ &= \lim_{h \to 0} \frac{h\left(\sqrt[3]{u} + \sqrt[3]{v}\right)^3}{h} \\ &= \left(\sqrt[3]{u} + \sqrt[3]{v}\right)^3. \end{aligned}$$

So the directional derivatives do always exist.

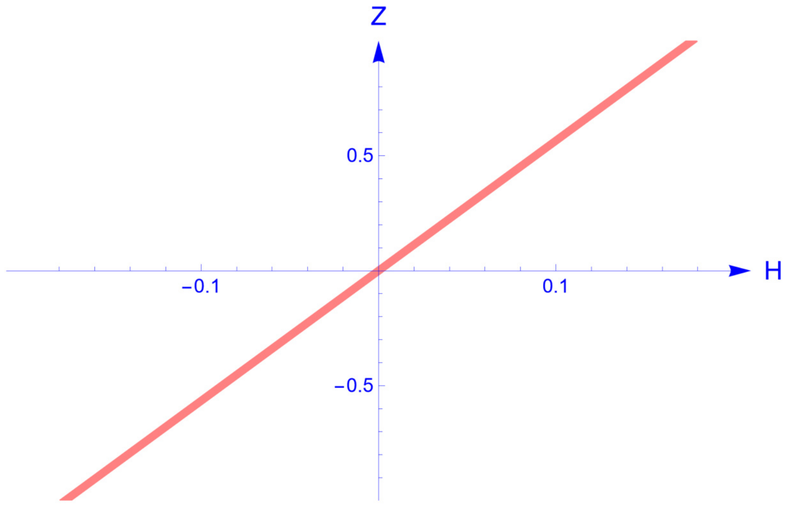

Figure 467. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

62.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.
Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{\left(\sqrt[3]{x} + \sqrt[3]{y}\right)^2}{x^{2/3}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We can consult figures of the absolute values of the first partial derivative at the end of this section.

We investigate the boundedness of this partial derivative to x. To see this, we take $x = y^2$ and see that the main part of the definition is

$$f(y^2, y) = \frac{\left(y^{2/3} + \sqrt[3]{y}\right)^2}{y^{4/3}} = \frac{\left(\sqrt[3]{y} + 1\right)^2}{y^{2/3}}.$$

We can observe unboundedness in any neighbourhood of $(0,0)$. We cannot apply this criterion for continuity. So we have not an alternative proof with this criterion for the continuity.

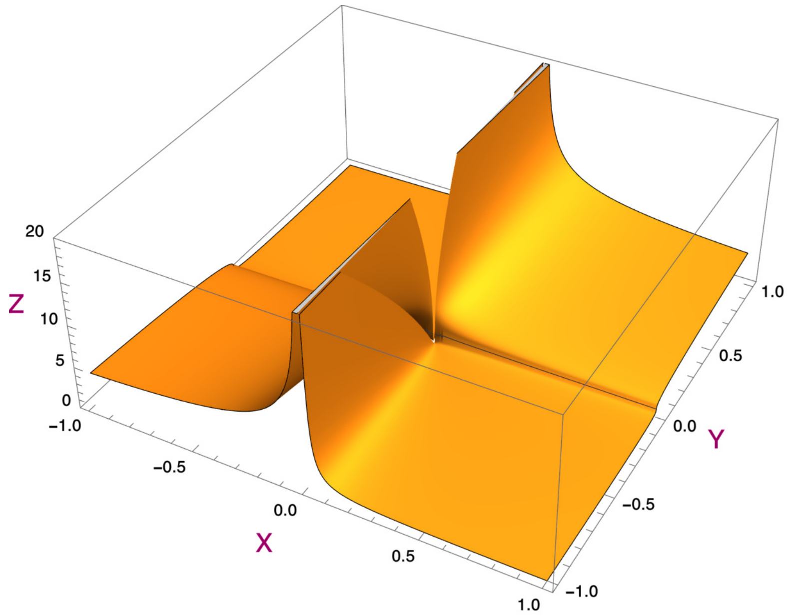

Figure 468. We see here the absolute value of the first partial derivative $\left|\frac{\partial f}{\partial x}\right|$. We can observe the unboundedness from this picture.

62.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

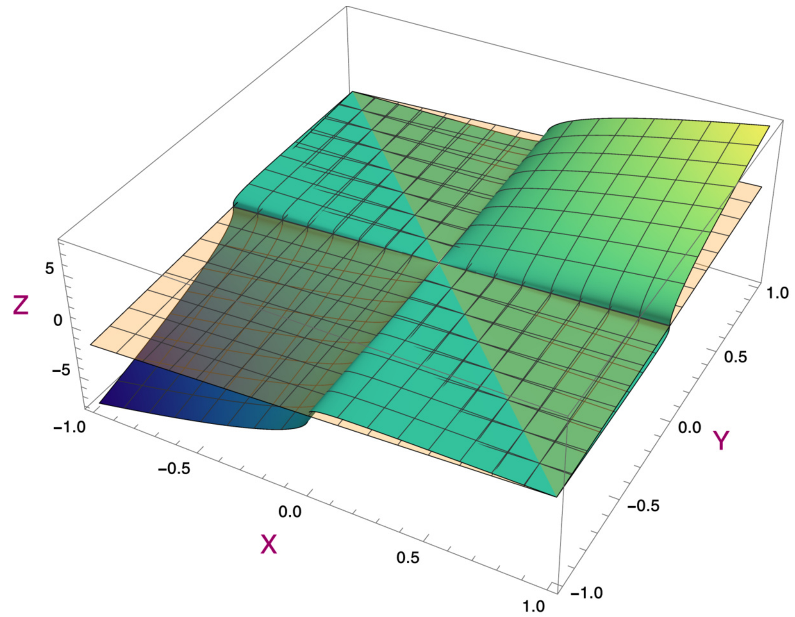

Figure 469. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function nicely but there remain a few doubts. The vertical tangential behaviour on the axes give reason for concern. Further calculations will have to decide.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

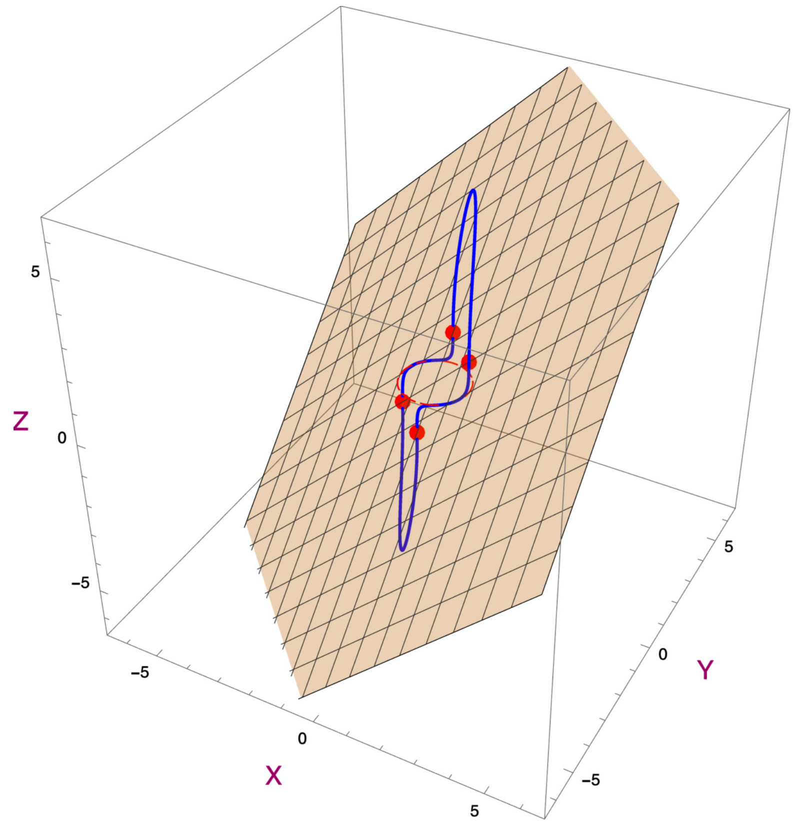

Figure 470. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ do not sweep out a nice ellipse lying in one plane! This is extremely bad news for differentiability.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{\left(\sqrt[3]{h}+\sqrt[3]{k}\right)^3 - h - k}{\sqrt{h^2+k^2}} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We restrict the function $q(h,k)$ to the continuous curves with equations $k = \lambda h$. We observe then that

$$q|_{k=\lambda h}(h,k) = \begin{cases} q(h,\lambda h) = \dfrac{3h\left(\sqrt[3]{\lambda}+1\right)\sqrt[3]{\lambda}}{\sqrt{h^2(\lambda^2+1)}} & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

We see that these restricted functions have many different limits. But if $q(h,k)$ is continuous, all these limit values should be $q(0,0) = 0$. So this function $q(h,k)$ is not continuous in $(0,0)$. The function $f(x,y)$ is not differentiable in $(0,0)$.

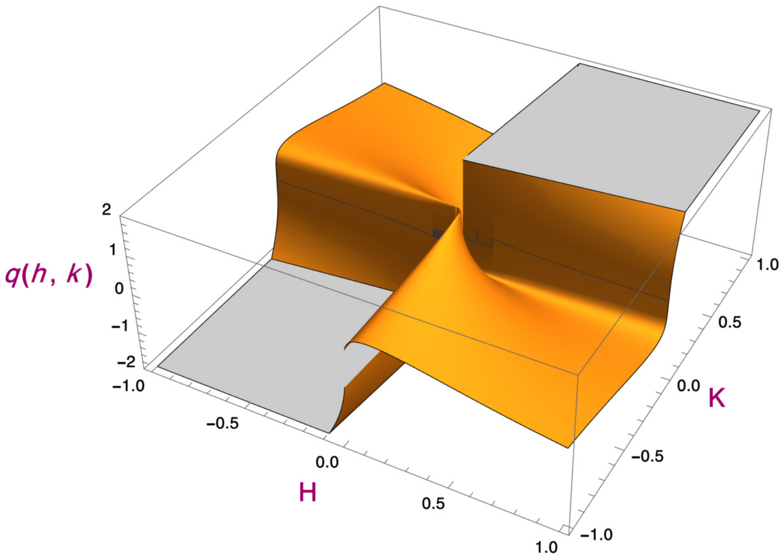

Figure 471. We see here a three dimensional figure of the graph of the function $q(h, k)$. The vertical line above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

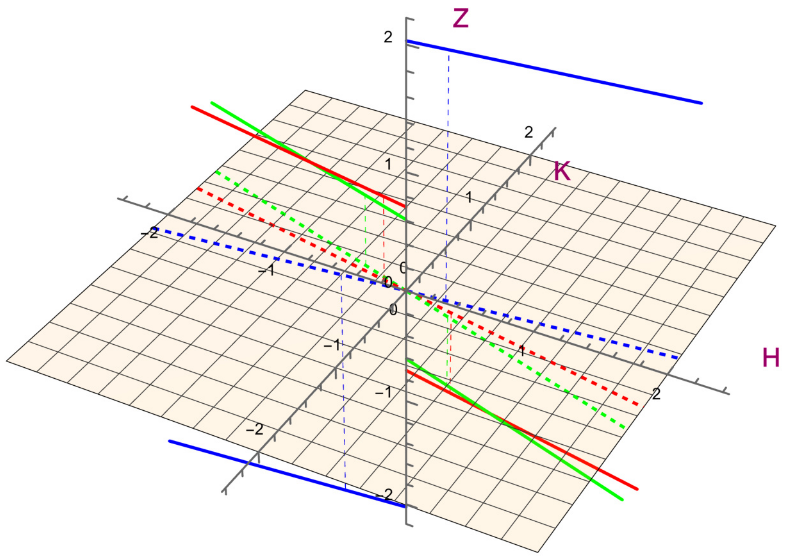

Figure 472. We have restricted the function $q(h, k)$ here to $k = -2\,h$ and $k = -1/3\,h$ and $k = -h$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

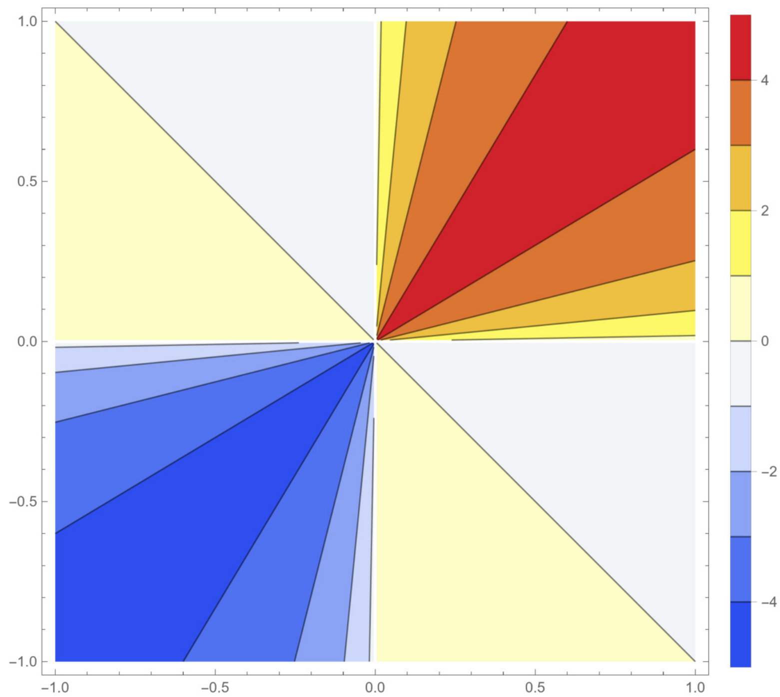

Figure 473. We see here a figure of the contour plot of the function $q(h,k)$. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

62.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

62.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

62.8 Overview

$$f(x,y) = \left(\sqrt[3]{x} + \sqrt[3]{y}\right)^3.$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

62.9 One step further

We are interested in the magical curves $k = \lambda h$ which we used in the proof of non differentiability.

Let us define a plane curve $\beta(t) = (t, \lambda t)$. This curve goes through $(0,0)$ and is differentiable. Now we take the space curve that projects to $\beta(t)$: $\alpha(t) = (t, \lambda t, f(t, \lambda t)) = \left(t, \lambda t, \left(\sqrt[3]{\lambda t} + \sqrt[3]{t}\right)^3\right)$.
This curve lies on the surface defined by the function f. We see that this curve is differentiable, but remark that this is not guaranteed because f is not differentiable!

We calculate the derivative in t.

$$\alpha'(t) = \left(1, \lambda, 1 + 3\lambda^{1/3} + 3\lambda^{2/3} + \lambda\right).$$

We draw the curve, the tangent line and the candidate tangent plane.

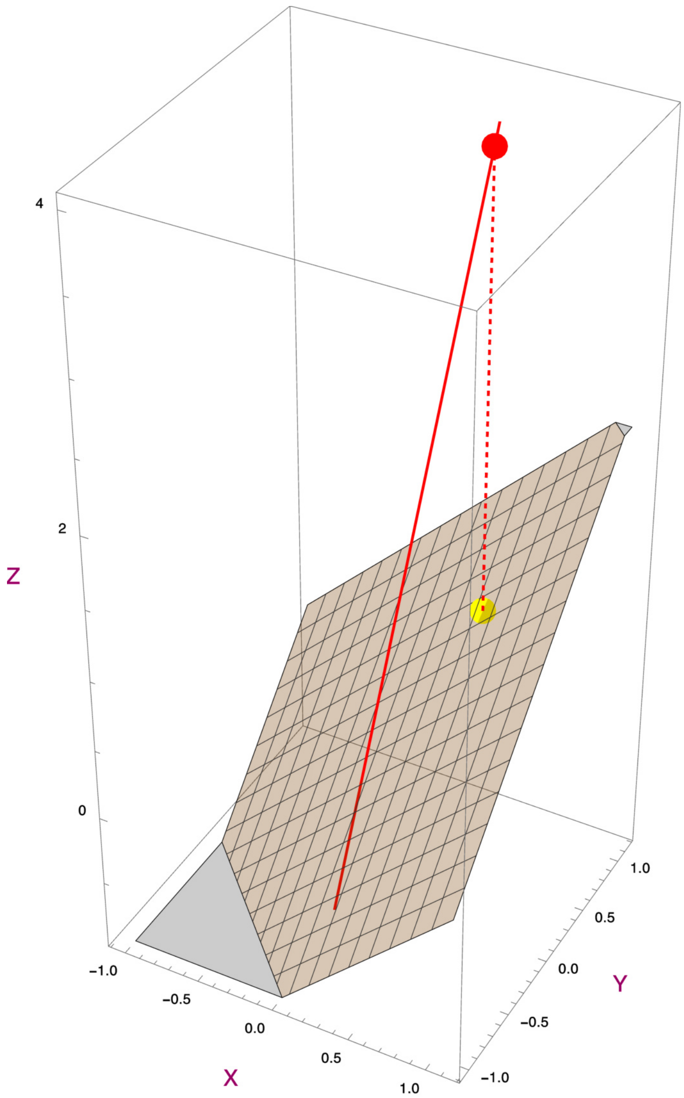

Figure 474. We see here a figure of the candidate tangent plane. The tangent line is drawn in red and completely obscures the curve which is also the same line in this case. We see that the tangent line is not in the candidate tangent plane. It intersects the tangent plane transversally and not tangentially. So the candidate tangent plane is not a tangent plane. The yellow point lies on the candidate tangent plane. We have drawn that yellow point in order to see more clearly that the tangent line is not on the candidate tangent plane. We conclude that the candidate tangent plane is not a tangent plane. The function is not differentiable. The figure is made with $\lambda = 1$.

Exercise 63.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{(x+y)^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

63.1 Continuity

We restrict the function to the continuous curves with equations $y = \lambda x$. We observe then that

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{(\lambda x + x)^2}{\lambda^2 x^2 + x^2} = \dfrac{(\lambda+1)^2}{\lambda^2+1} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

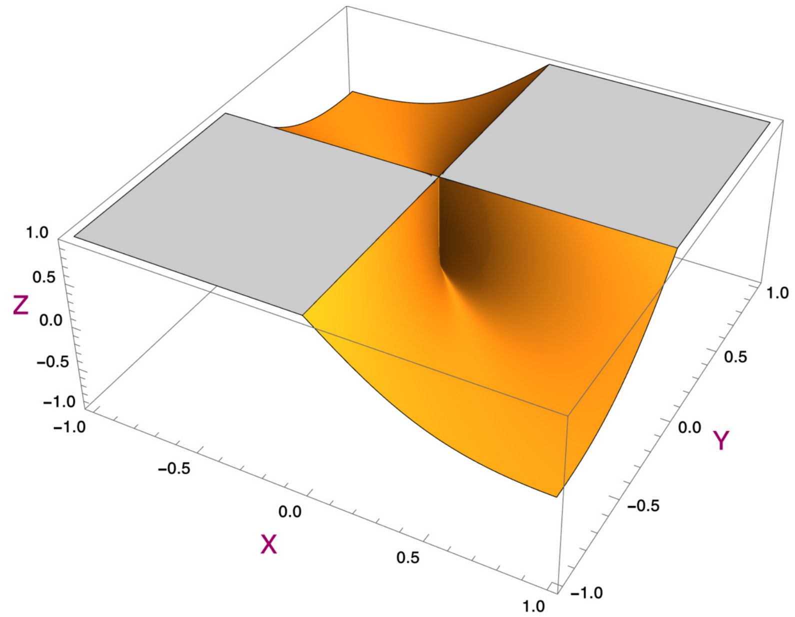

Figure 475. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

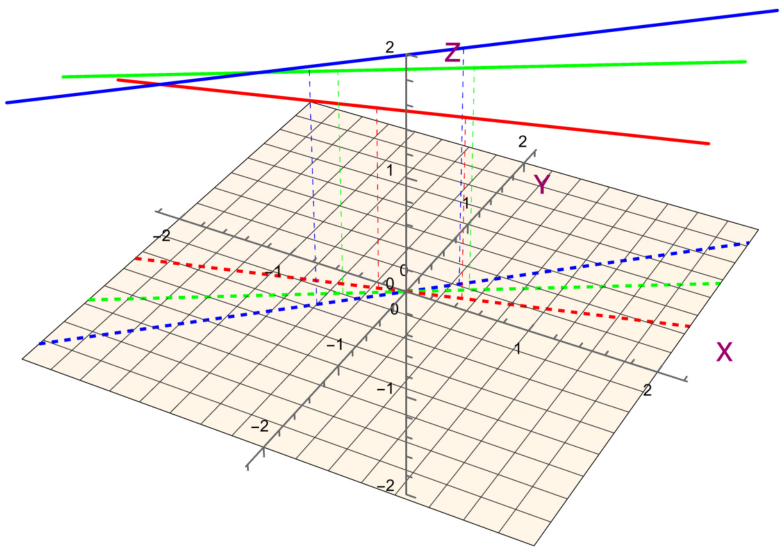

Figure 476. We have restricted the function here to $y = 3/10\,x$ and $y = 6/10\,x$ and $y = 9/10\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

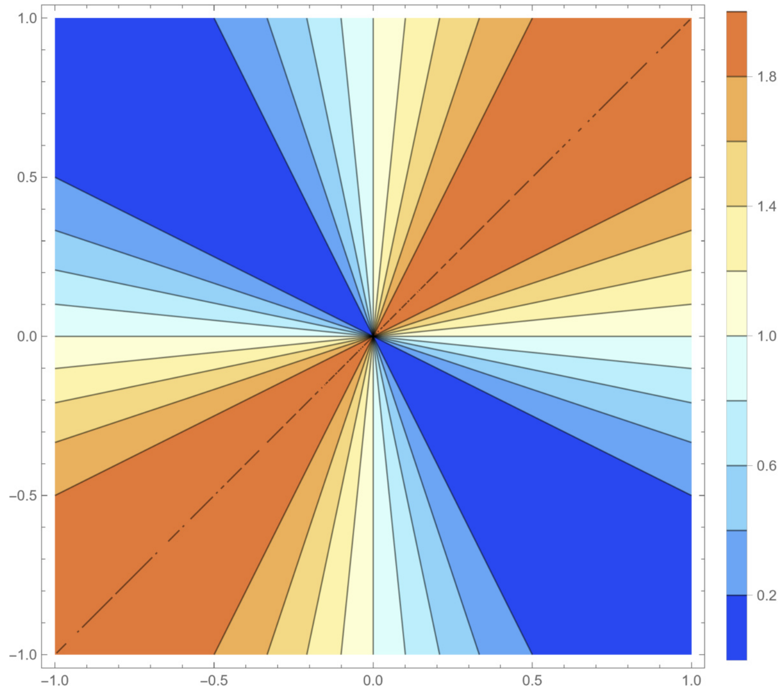

Figure 477. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

63.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 1 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

The function restricted to $y = 0$ is not continuous and therefore not differentiable.

By symmetry reasons the same holds for the derivative to y.

We conclude that the partial derivative to x does not exist and that the partial derivative to y does not exist.

63.3 Directional derivatives

Let (u, v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2 + v^2} = 1$. Then the directional derivative in $(0, 0)$ in the direction (u, v), notation $D_{(u,v)}(0, 0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h \to 0} \frac{f(0 + h\,u, 0 + h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1, 0)$ and $(0, 1)$.

We see that the function is not continuous.

So the directional derivatives do not always exist.

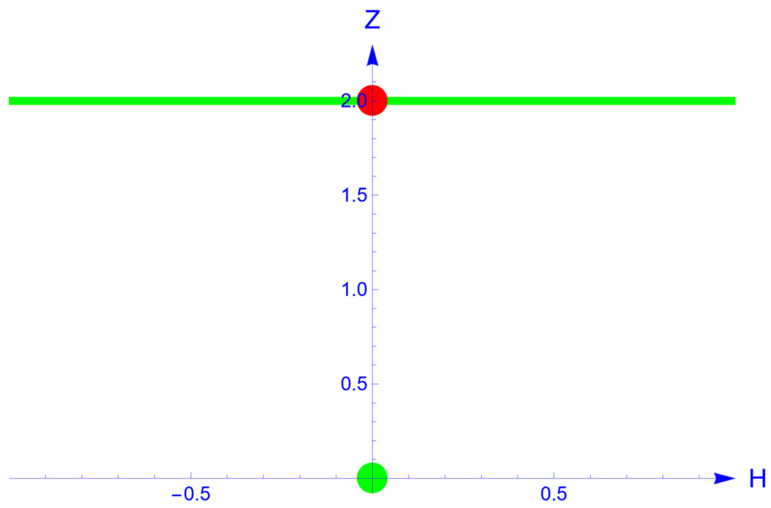

Figure 478. We see here a figure of the graph of the function restricted to the line through $(0, 0)$ with direction $(u, v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

63.4 Alternative proof of continuity (optional)

Irrelevant. The function is not continuous.

63.5 Differentiability

The function is not continuous. Thus the function is not differentiable.

63.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

63.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

63.8 Overview

$$f(x,y) = \begin{cases} \dfrac{(x+y)^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	no
partial derivatives exist	no
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 64.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{\log\left(x^2 y^2\right)}{x^2+y^2} & \text{if } x\,y \neq 0; \\ 0 & \text{if } x\,y = 0. \end{cases}$$

64.1 Continuity

We restrict the function to the continuous curves with equations $y = \lambda x$. We observe then that

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{\log\left(\lambda^2 x^4\right)}{\lambda^2 x^2 + x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We see that

$$\lim_{x\to 0} \frac{\log\left(\lambda^2 x^4\right)}{\lambda^2 x^2 + x^2} = -\infty.$$

We see that these restricted functions tend to $-\infty$ if $\lambda \neq 0$. But if $f(x,y)$ is continuous, it is bounded in at least one neighbourhood of $(0,0)$. So this function $f(x,y)$ is not continuous.

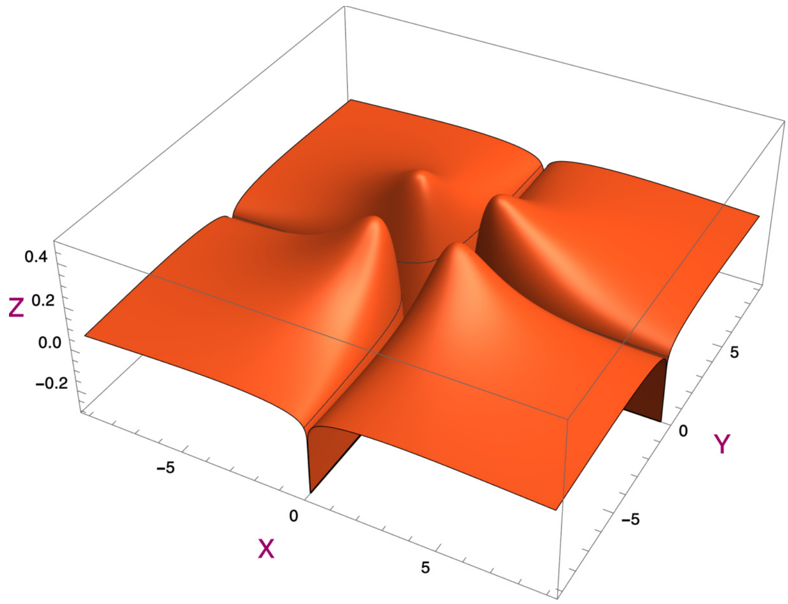

Figure 479. We see here a three dimensional figure of the graph of the function. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

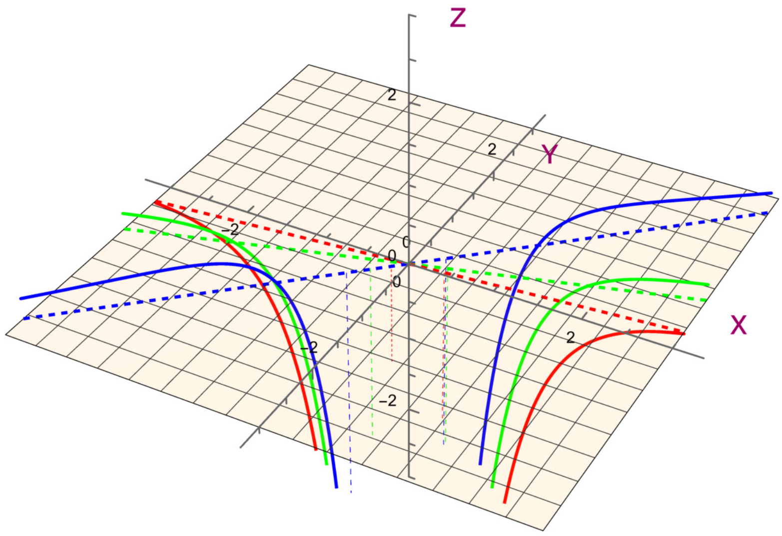

Figure 480. We have restricted the function here to $y = 1/10\,x$ and $y = 3/10\,x$ and $y = 9/10\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that are unbounded in any neighbourhood of $(0,0)$.

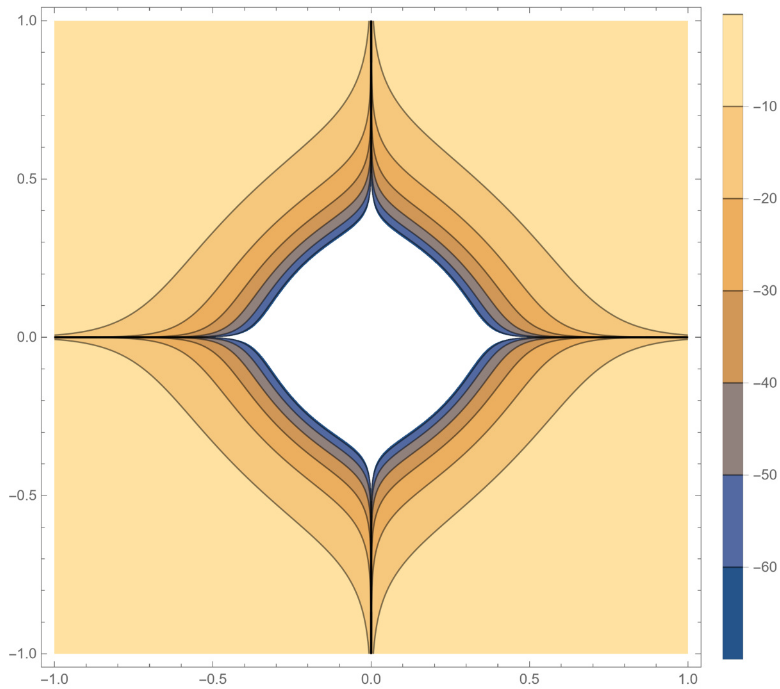

Figure 481. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed. We see that the function is unbounded in any neighbourhood of $(0,0)$. There are not enough colours to show that, so the neighbourhood of $(0,0)$ is just depicted by white area.

64.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

By symmetry reasons, the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

64.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0}\frac{f(0+h\,u,0+h\,v)-f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned}D_{(u,v)}(0,0) &= \lim_{h\to 0}\frac{f(0+h\,u,0+h\,v)-f(0,0)}{h}\\ &= \lim_{h\to 0}\frac{\log(h^4u^2v^2)}{h^3(u^2+v^2)}.\end{aligned}$$

This limit is not a real number, so the direction derivatives do not exist if $u\,v\neq 0$. We calculated these exceptional cases before.

So the directional derivatives do not always exist.

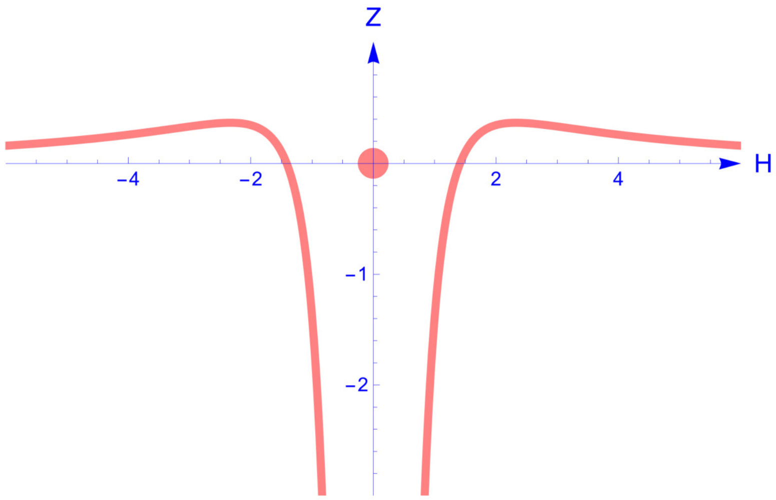

Figure 482. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$. These functions are unbounded. So the derivatives do not exist.

64.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

We doubt that the partial derivatives do all exist. We take a point $(a,0)$ on the X-axis and investigate the function $f(a,h)$ with $a \neq 0$. We see that this function is not continuous and thus not differentiable.

$$f(a,h) = \begin{cases} \dfrac{\log\left(a^2 h^2\right)}{a^2+h^2} & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

We consult a figure for this.

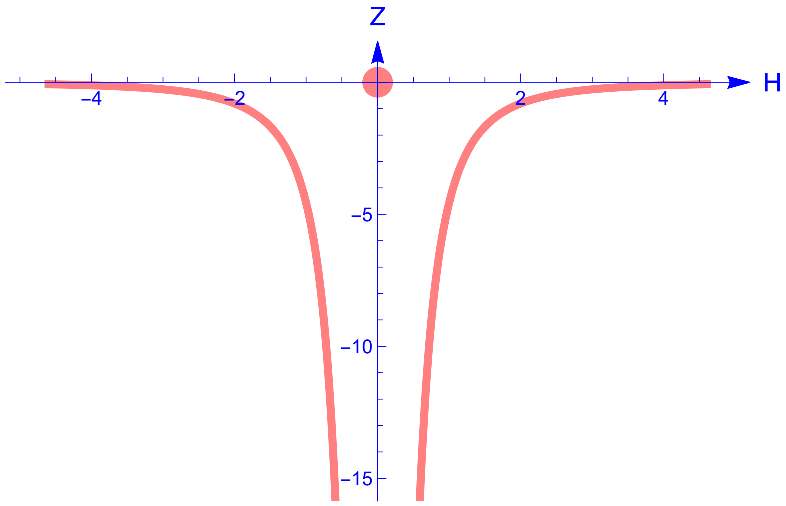

Figure 483. We see here a figure of the graph of the function $f(a,h)$. We have drawn the function here for the value $a = 1/2$ which is exemplary for the values of a close to 0. This does not look like a continuous or differentiable function.

64.5 Differentiability

We have that the function is not continuous. Thus the function is not differentiable.

64.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

64.8 Overview

continuous	no
partial derivatives exist	yes
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 65.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + y^2 \sin\left(\frac{1}{y}\right) & \text{if } x\,y \neq 0, \\ x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \text{ and } y = 0, \\ y^2 \sin\left(\frac{1}{y}\right) & \text{if } y \neq 0 \text{ and } x = 0, \\ 0 & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

65.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

We are going to find a δ valid for the interior parts of the quadrants, and then investigate the X-axis and the Y-axis separately.
When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| x^2 \sin\left(\frac{1}{x}\right) + y^2 \sin\left(\frac{1}{y}\right) - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|x^2 \sin\left(\frac{1}{x}\right) + y^2 \sin\left(\frac{1}{y}\right)\right| &\leq x^2 \left|\sin\left(\frac{1}{x}\right)\right| + y^2 \left|\sin\left(\frac{1}{y}\right)\right| \\ &\leq x^2 + y^2 \\ &\leq \sqrt{x^2+y^2}^{\,2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon^{1/2}$. We can find a δ, so we conclude that the function is continuous.

We look now for a δ for the points on the X-axis.

$$\begin{aligned}\left|x^2 \sin\left(\frac{1}{x}\right)\right| &\leq x^2 \left|\sin\left(\frac{1}{x}\right)\right| \\ &\leq x^2 \\ &\leq \sqrt{x^2+y^2}^{\,2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon^{1/2}$. We can find a δ, so we conclude that the function is continuous.

The reasoning on the Y-axis is similar.

We conclude the continuity of the function. We can take a δ which is valid for all three sets at the same time, e.g. the minimum of the three delta's.

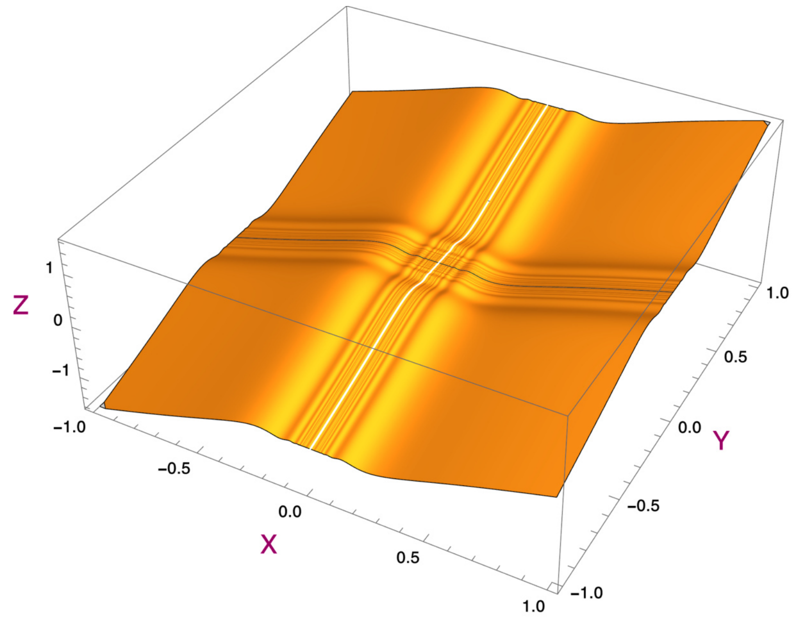

Figure 484. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

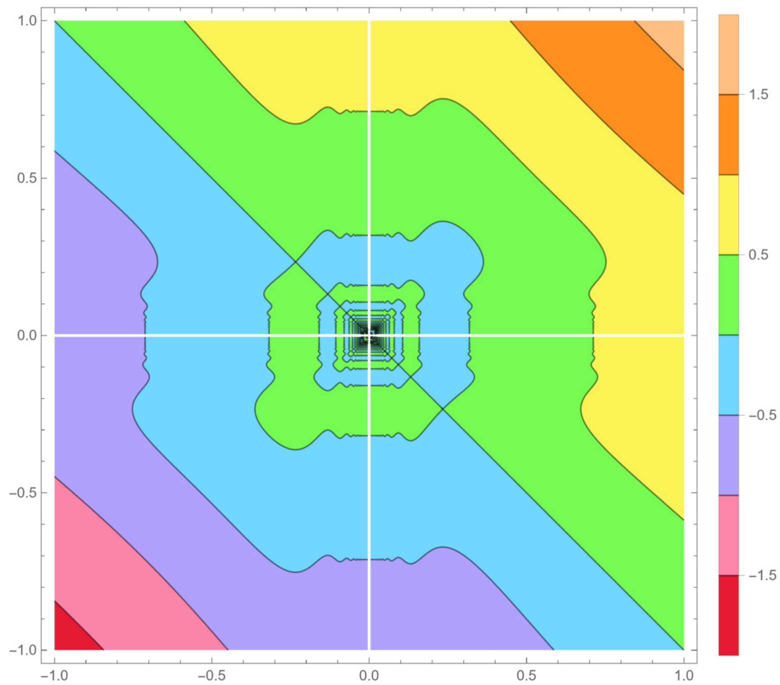

Figure 485. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

65.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$
\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} \\
&= \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) \\
&= 0.
\end{aligned}
$$

We can for example use the squeeze theorem. So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).

This runs in a completely similar way. So the partial derivative to y does exist.

We conclude that the partial derivative to x does exist and that the partial derivative to y does exist.

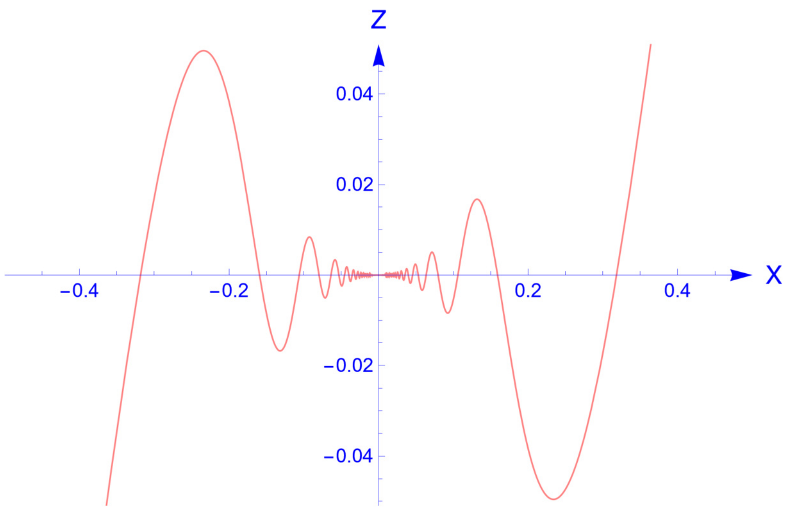

Figure 486. We see here a figure of the graph of the function restricted to the horizontal X-axis through $(0,0)$.

65.3 Directional derivatives

Let (u, v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2 + v^2} = 1$. Then the directional derivative in $(0, 0)$ in the direction (u, v), notation $D_{(u,v)}(0, 0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0 + h\,u, 0 + h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1, 0)$ and $(0, 1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0 + h\,u, 0 + h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} h\left(u^2 \sin\left(\frac{1}{h\,u}\right) + v^2 \sin\left(\frac{1}{hv}\right)\right) \\ &= 0. \end{aligned}$$

We can for example use the squeeze theorem.

So the directional derivatives do always exist.

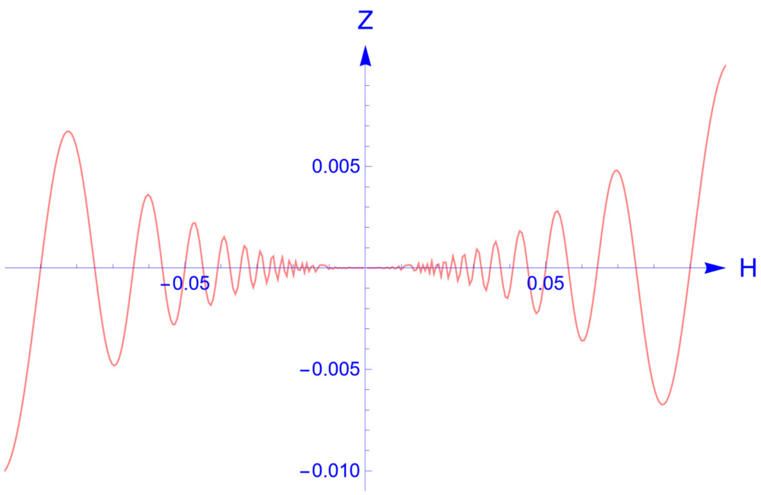

Figure 487. We see here a figure of the graph of the function restricted to the line through $(0, 0)$ with direction $(u, v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

65.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$.

We try now to see that the partial derivatives do exist. We start our investigation with the partial derivatives in $(a,0)$ with $a \neq 0$. We can explicitly calculate the partial derivative to x.
The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} 2\,x\,\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We conclude that the partial derivative to x does exist in $(a,0)$.

We start investigating the partial derivative to y in $(a,0)$ with $a \neq 0$. The function is then composed of classical functions which are known to be infinitely differentiable. Let us start with writing the function $f(a,h)$. This function is equal to

$$f(a,h) = \begin{cases} a^2 \sin\left(\frac{1}{a}\right) + h^2 \sin\left(\frac{1}{h}\right) & \text{if } h \neq 0, \\ a^2 \sin\left(\frac{1}{a}\right) & \text{if } h = 0. \end{cases}$$

So

$$\frac{\partial f}{\partial y}(a,0) = \lim_{h\to 0} \frac{a^2 \sin\left(\frac{1}{a}\right) + h^2 \sin\left(\frac{1}{h}\right) - a^2 \sin\left(\frac{1}{a}\right)}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0.$$

So the derivative to y in $(a,0)$ does exist and equals 0.

By the symmetry of the function, we conclude that the derivative to y does also exist.

We summarise.

$$\frac{\partial f}{\partial x}(x,y)=\begin{cases}2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right) & \text{if } x\neq 0,\\ 0 & \text{if } x=0.\end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

$$\begin{aligned}\left|\frac{\partial f}{\partial x}\right| &\leq \left|2\,x\,\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)\right|\\ &\leq \left|2\,|x|\,\left|\sin\left(\frac{1}{x}\right)\right|+\left|\cos\left(\frac{1}{x}\right)\right|\right|\\ &\leq 2\,|x|+1\\ &\leq 2\sqrt{x^2+y^2}+1\\ &\leq 3.\end{aligned}$$

We have chosen here the restriction to the neighbourhood defined by $\sqrt{x^2+y^2}<1$.

The calculation for the absolute value of the partial derivative $\left|\frac{\partial f}{\partial y}\right|$ is completely similar.

So the partial derivative $\left|\frac{\partial f}{\partial y}\right|$ is bounded.

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

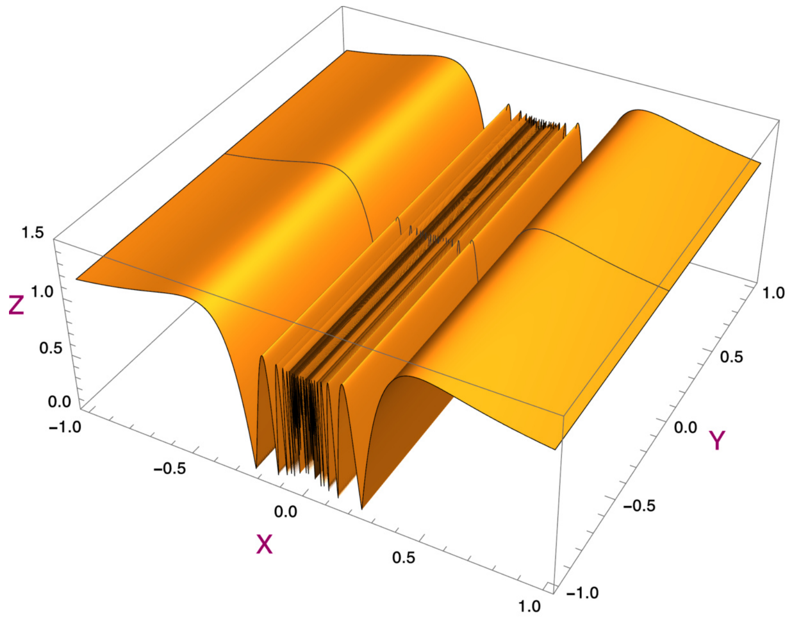

Figure 488. We see here the absolute value of the first partial derivative $\left|\frac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

65.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

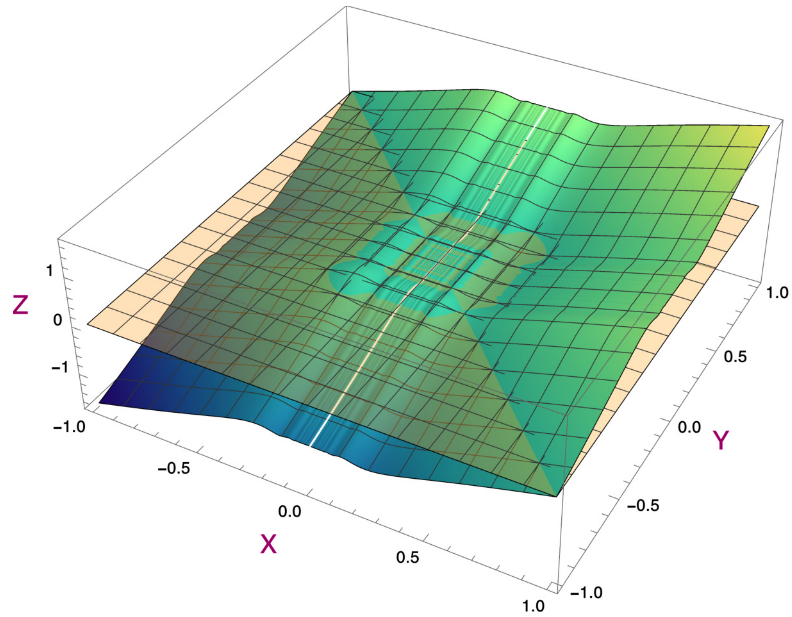

Figure 489. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function nicely.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

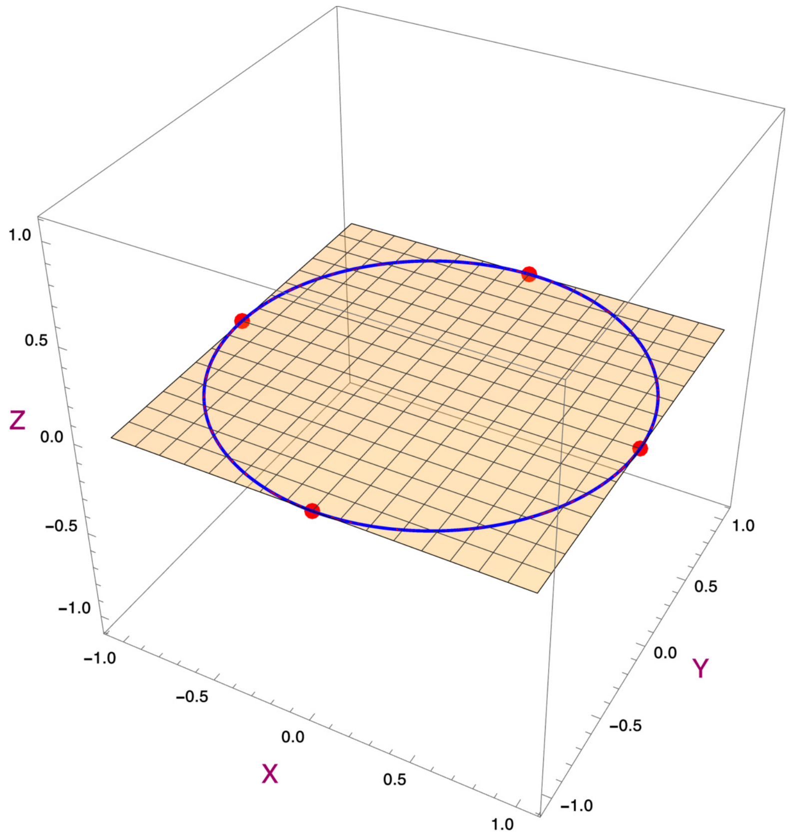

Figure 490. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h^2 \sin\left(\dfrac{1}{h}\right) + k^2 \sin\left(\dfrac{1}{k}\right)}{\sqrt{h^2+k^2}} & \text{if } (h,k) \neq (0,0) \text{ and } h \neq 0 \text{ and } k \neq 0; \\ 0 & \text{if } (h,k) = (0,0) \text{ or } h = 0 \text{ or } k = 0 \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left| \frac{h^2 \sin\left(\dfrac{1}{h}\right) + k^2 \sin\left(\dfrac{1}{k}\right)}{\sqrt{h^2+k^2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{h^2 \sin\left(\frac{1}{h}\right) + k^2 \sin\left(\frac{1}{k}\right)}{\sqrt{h^2+k^2}} \right| &\leq \frac{h^2 \left|\sin\left(\frac{1}{h}\right)\right| + k^2 \left|\sin\left(\frac{1}{k}\right)\right|}{\sqrt{h^2+k^2}} \\
&\leq \frac{h^2+k^2}{\sqrt{h^2+k^2}} \\
&\leq \frac{\sqrt{h^2+k^2}^2}{\sqrt{h^2+k^2}} \\
&\leq \sqrt{h^2+k^2}.
\end{aligned}$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous. The function is thus differentiable.

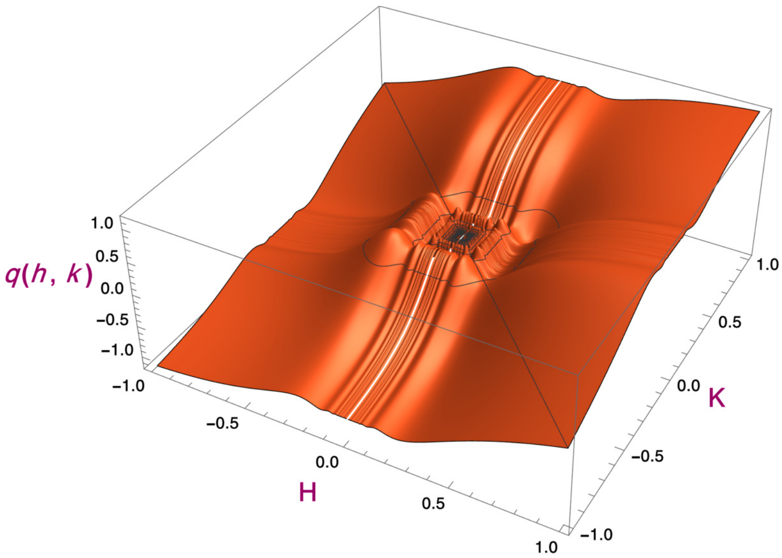

Figure 491. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

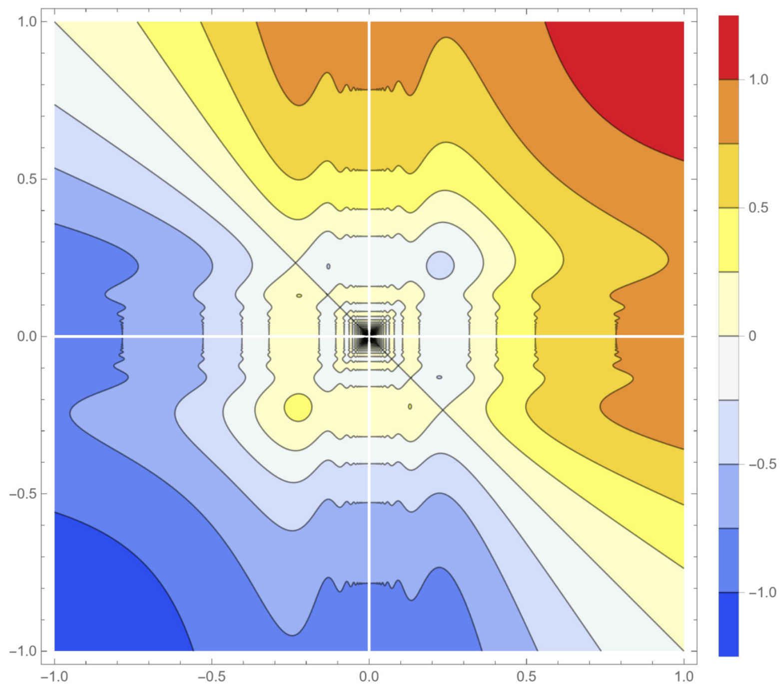

Figure 492. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

65.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We can see that the partial derivatives are not bounded in any neighbourhood of $(0,0)$. So the function is not Lipschitz continuous. We cannot give an alternative proof following this criterion.

65.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

We have that the partial derivative to x is

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} 2\,x\,\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We restrict this function to the X-axis. But this function has many limit or accumulation points in $x = 0$. But no one makes the function continuous. We see that this function is not continuous.

So we cannot use this alternative criterion in order to prove differentiability.

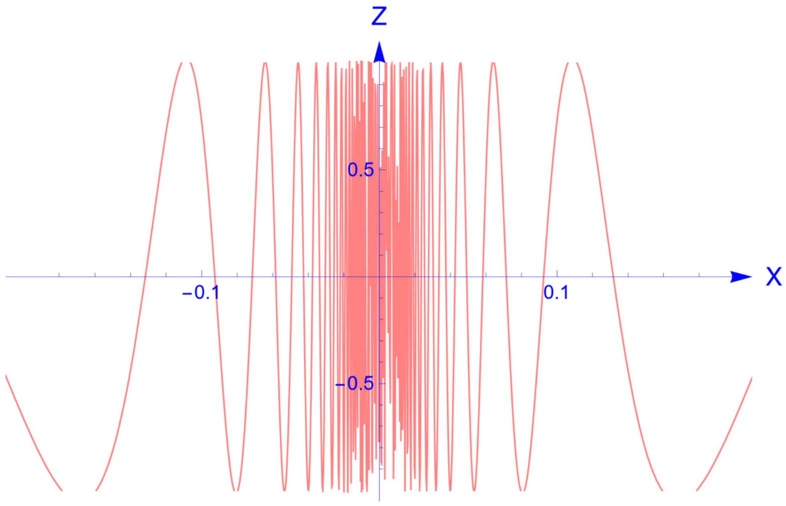

Figure 493. We see here a figure of the graph of the function $\frac{\partial f}{\partial x}(x, y)$ restricted to the X-axis. This does not seem to be a graph of a continuous function. There are many accumulation or limit points in $x = 0$. But no one makes the function continuous.

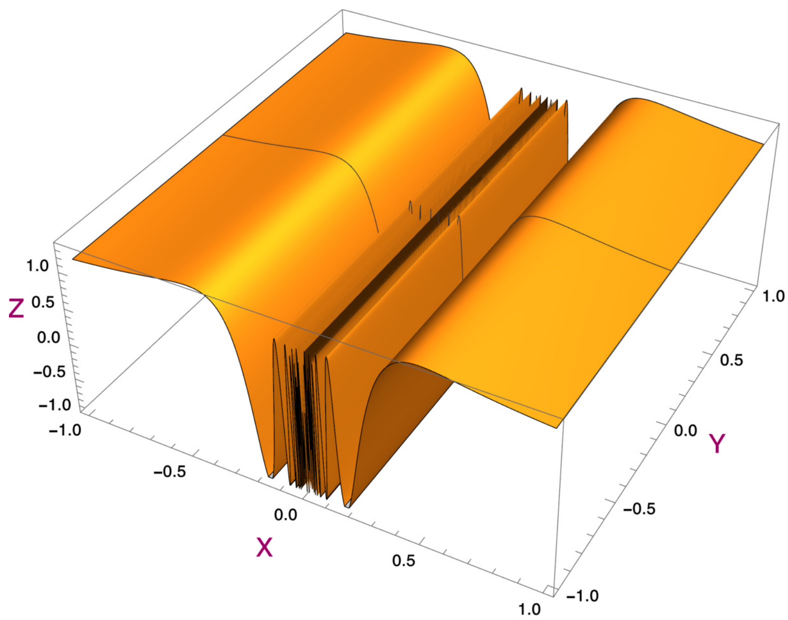

Figure 494. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial x}(x, y)$. This does not look like a continuous function in $(0, 0)$.

65.8 Overview

$$f(x,y) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + y^2 \sin\left(\frac{1}{y}\right) & \text{if } x\,y \neq 0, \\ x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \text{ and } y = 0, \\ y^2 \sin\left(\frac{1}{y}\right) & \text{if } y \neq 0 \text{ and } x = 0, \\ 0 & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

Exercise 66.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} x+y & \text{if } xy \neq 0; \\ 0 & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

66.1 Continuity

Remark the subtlety in the definition. The main definition $x+y$ is a perfect function in the sense that it is everywhere differentiable. It exists everywhere in the real plane, but it is forced to be something else on the X-axis and the Y-axis. We must take care and consider this fact.
We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements.
We are going to prove first the ϵ criterion for the points (x,y) with $xy \neq 0$. Then we will do it for the X-axis and the Y-axis. Try to find for the points (x,y) with $xy \neq 0$ a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$|x+y-0| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$|x+y| \leq \sqrt{x^2+y^2} + \sqrt{x^2+y^2}$$
$$\leq 2\sqrt{x^2+y^2}.$$

It is sufficient to take $\delta = \epsilon/2$. We can find a δ, so we conclude that the function is continuous on this set.

We work now on the X-axis. Because the function is identically 0 on this axis, we can take every δ that is allowed. We take $\delta = 1$. We can find a δ, so we conclude that the function is continuous on this set. We do a similar reasoning on the Y-axis.

We take then the minimum of the three δ values. We can find a δ, so the function is continuous.

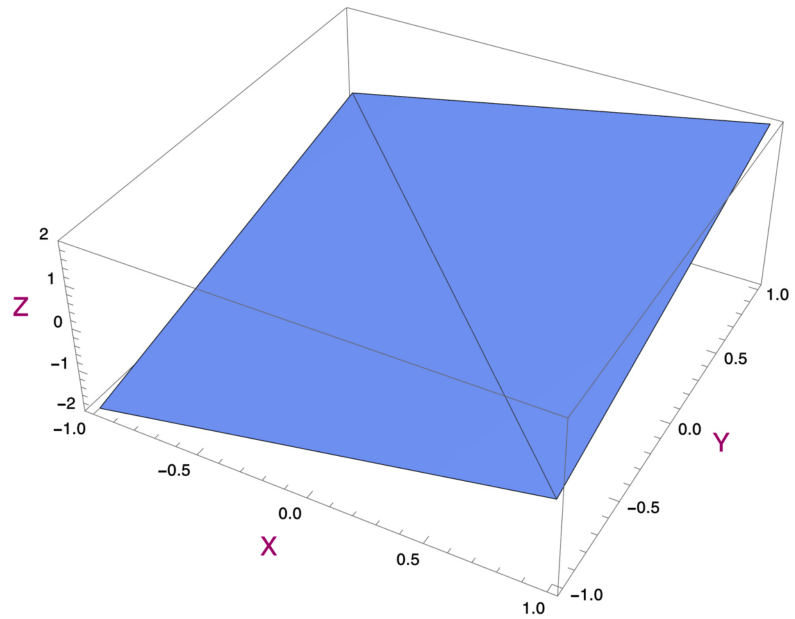

Figure 495. We see here a three dimensional figure of the graph of the function. This looks like a continuous function. We see here the plane $z = x + y$ but our imagination must see also the two exceptional lines above the X-axis and Y-axis. They make the function quite special. It is difficult to draw them.

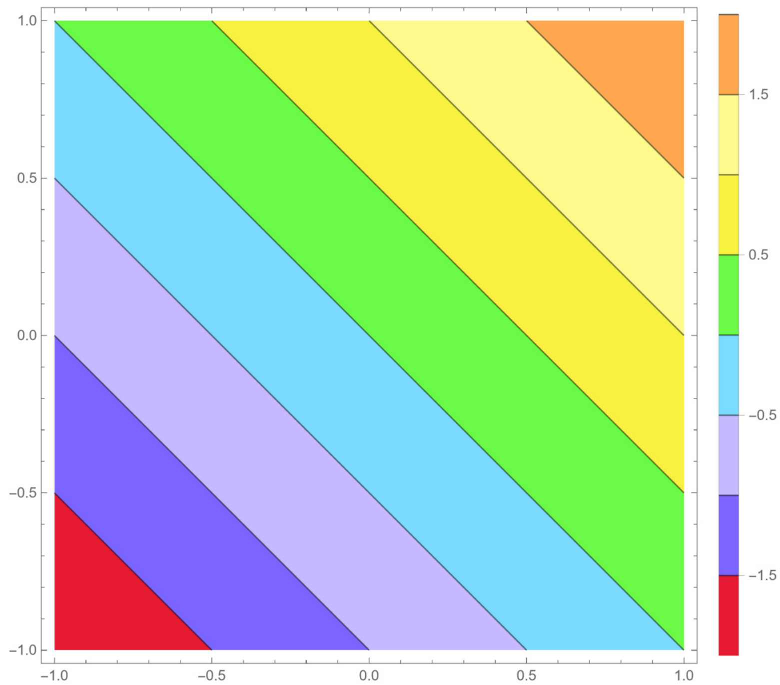

Figure 496. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

66.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

The calculation for the second partial to y is completely similar.

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

66.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0}\frac{f(0+hu,0+hv)-f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned}D_{(u,v)}(0,0) &= \lim_{h\to 0}\frac{f(0+hu,0+hv)-f(0,0)}{h}\\ &= \lim_{h\to 0} u+v\\ &= u+v.\end{aligned}$$

This calculation is only valid for the directions not pointing to the X-axis or the Y-axis. We did these special cases before.

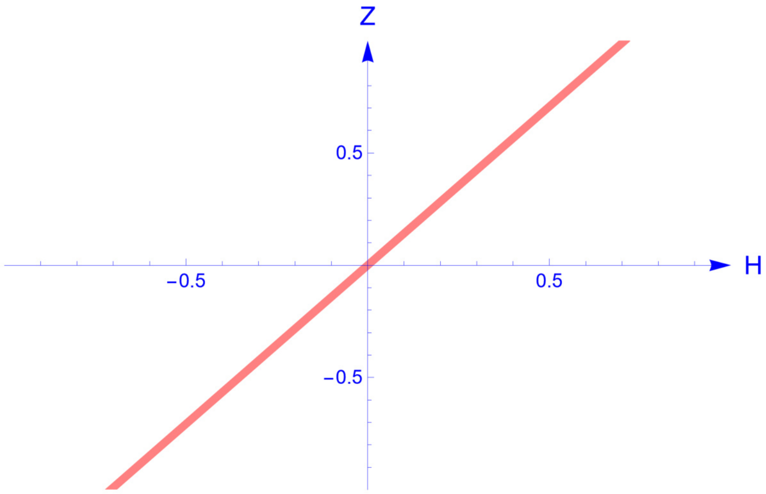

Figure 497. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$.

66.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

But it is clear that in the points $(a,0)$ with $a \neq 0$, the function in the Y-direction is not continuous. Let us investigate this further.

$$f(a,h) = \begin{cases} a+h & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

This function is not continuous if $a \neq 0$, so the function is not differentiable. So the partial derivative does not exist.

We illustrate in the next picture the behaviour of the function.

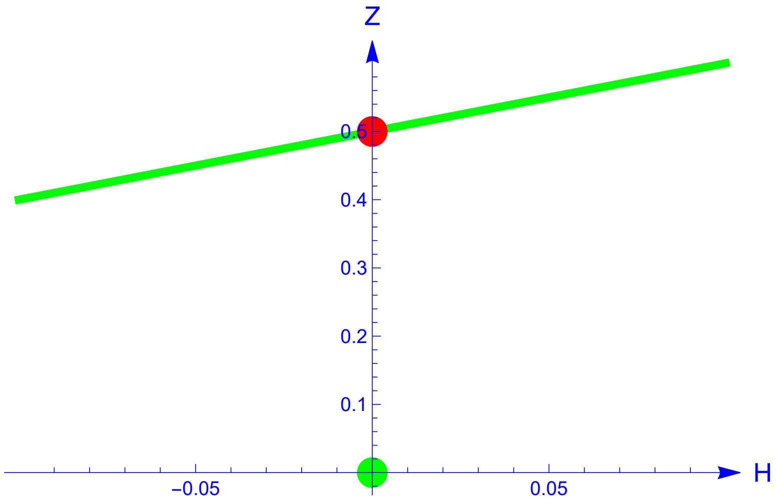

Figure 498. We see here a two dimensional figure of the graph of the function $f(a,h)$. We have drawn the function here for the value $a = 1/2$ which is exemplary for the values of a close to 0. This does not look like a continuous or differentiable function.

66.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

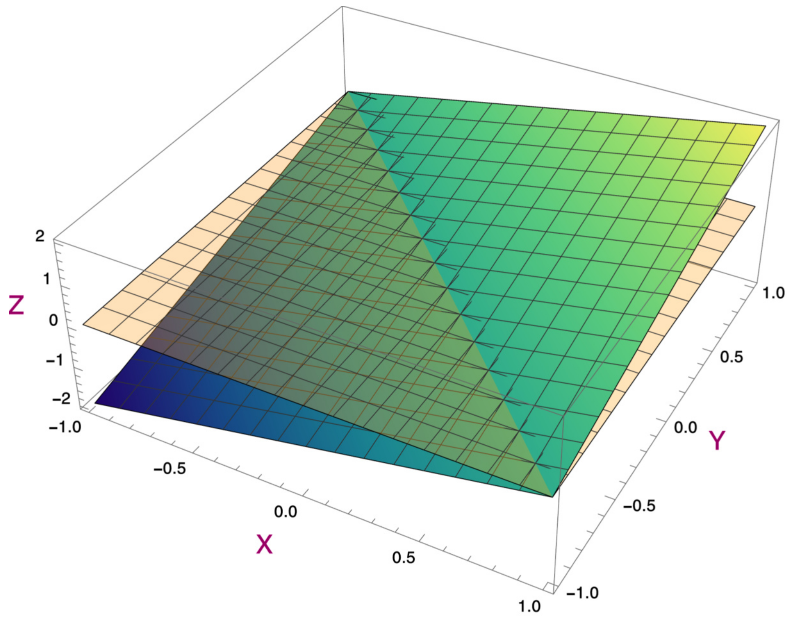

Figure 499. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane does not fit the function at all. It is indeed no tangent plane following our calculations.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

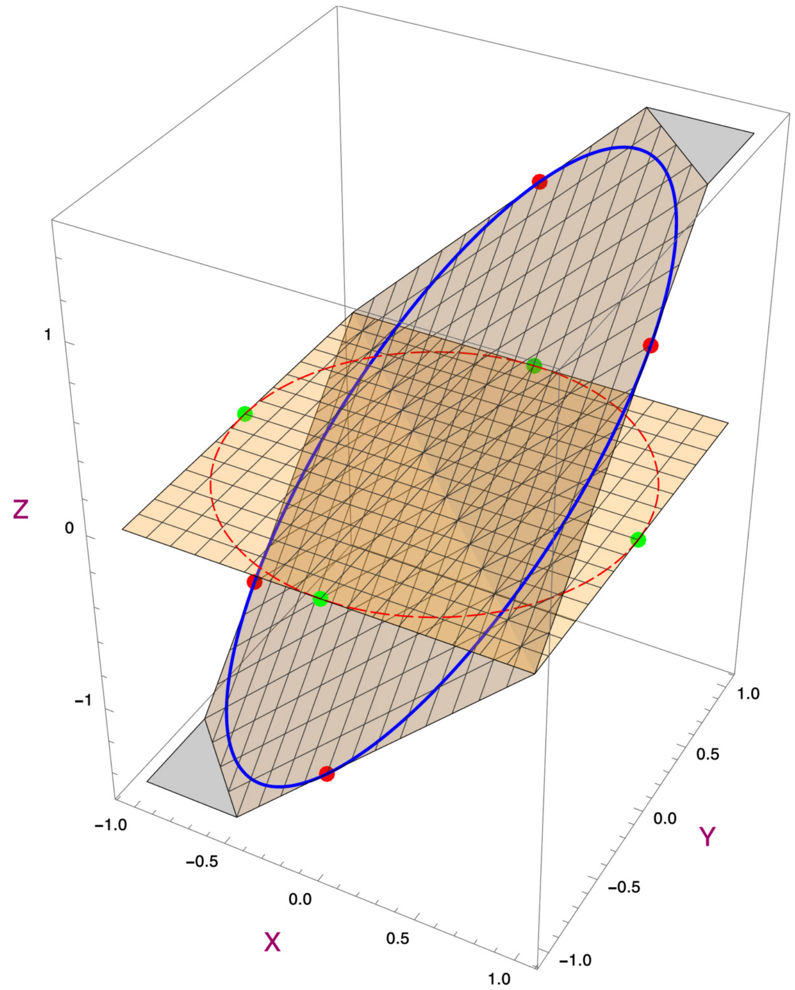

Figure 500. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. But these points are not valid because of the definition of the function on the axes. They must be replaced by the green points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ do not sweep out a nice ellipse in the candidate tangent plane because of the green points. This is extremely bad news for the differentiability.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h+k}{\sqrt{h^2+k^2}} & \text{if } h\,k \neq 0 \text{ and } (h,k) \neq (0,0); \\ 0 & \text{if } h = 0 \text{ or } k = 0 \text{ or } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.
We can remark here also that the linear terms in the numerator of a are not cancelled as they normally, i.e. if there was no special definition on the coordinate axes, are cancelled!

Discussion of the continuity of *q(h,k)* in (0,0).

We restrict the function $q(h,k)$ to the continuous curves with equations $k = \lambda\,h$. We observe then that if $\lambda \neq 0$

$$q\big|_{k=\lambda h}(h,k) = \begin{cases} q(h,\lambda\,h) = \operatorname{sgn}(h)\,\dfrac{\lambda+1}{\sqrt{\lambda^2+1}} & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

We see that these restricted functions have different limits. But if $q(h,k)$ is continuous, all these limit values should be $q(0,0) = 0$. So this function $q(h,k)$ is not continuous in $(0,0)$. The function $f(x,y)$ is not differentiable in $(0,0)$.

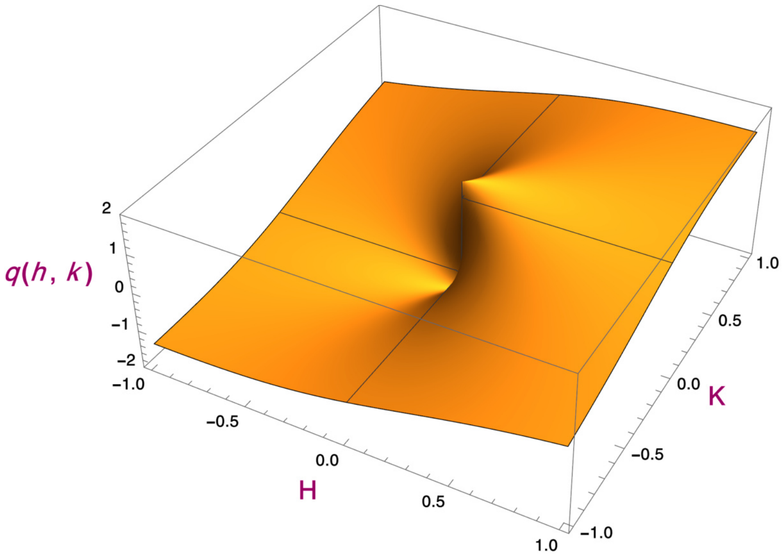

Figure 501. We see here a three dimensional figure of the graph of the function $q(h,k)$. The vertical line above $(0,0)$ looks suspicious. This does not seem to be a graph of a continuous function.

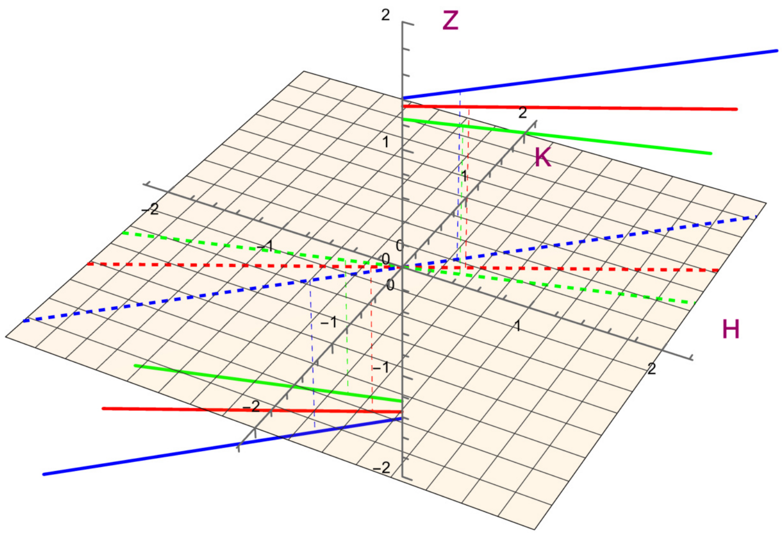

Figure 502. We have restricted the function $q(h,k)$ here to $k = 1/2\,h$ and $k = 3/10\,h$ and $k = 9/10\,h$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

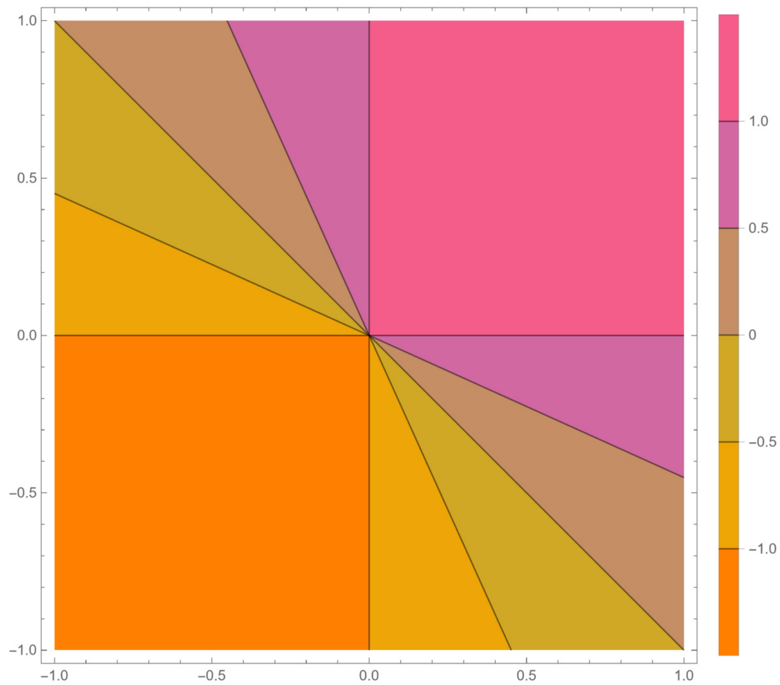

Figure 503. We see here a figure of the contour plot of the function $q(h,k)$. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

66.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

66.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

66.8 Overview

$$f(x,y) = \begin{cases} x+y & \text{if } x\,y \neq 0; \\ 0 & \text{if } x=0 \text{ or } y=0. \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

66.9 One step further

We want to investigate further what happens with those curves $k = \lambda h$, $\lambda \neq 0$, which we used in the investigation of the differentiability. We draw now in the X-Y plane a plane curve $\beta(t) = (t, \lambda t)$. The space curve $\alpha(t) = (t, \lambda t, f(t, \lambda t)) = (t, \lambda t, t + \lambda t)$ lies on the surface defined by f and projects to $\beta(t)$. We remark that this curve is differentiable and note that this is not guaranteed because f is not differentiable. The curve $\alpha(t)$ is a line and the tangent line is the same as the curve. The tangent line is now transversal and not tangential to the candidate tangent plane which is the X-Y plane. So the candidate tangent plane is not a tangent plane. The function is not differentiable in $(0,0)$.

Remark also that if the function is differentiable in $(0,0)$, then its directional derivative can be written in the form

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\, u + \frac{\partial f}{\partial y}(0,0)\, v.$$

In our case this takes the form

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\, u + \frac{\partial f}{\partial y}(0,0)\, v = 0\, u + 0\, v = 0.$$

This is not the case for this function. We have calculated in section 67.3 the formula $u + v$ for (u,v) not pointing to the X-direction or the Y-direction. So it is impossible that the function is differentiable.

Exercise 67.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} x^2\left(1-\cos\left(\frac{y}{x}\right)\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

67.1 Continuity

We remark first that we have the identity

$$x^2\left(1-\cos\left(\frac{y}{x}\right)\right) = 2\,x^2\,\sin^2\left(\frac{y}{2\,x}\right).$$

We think that the function definition in the second form will sometimes be easier to deal with.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left|2\,x^2\,\sin^2\left(\frac{y}{2\,x}\right) - 0\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\left| 2\,x^2 \sin^2\left(\frac{y}{2\,x}\right) \right| \le \left| 2\,x^2 \sin^2\left(\frac{y}{2\,x}\right) \right|$$
$$\le 2\,x^2$$
$$\le 2\,\sqrt{x^2+y^2}^{\,2}.$$

It is sufficient to take $\delta = (\epsilon/2)^{1/2}$. We can find a δ, so we conclude that the function is continuous.

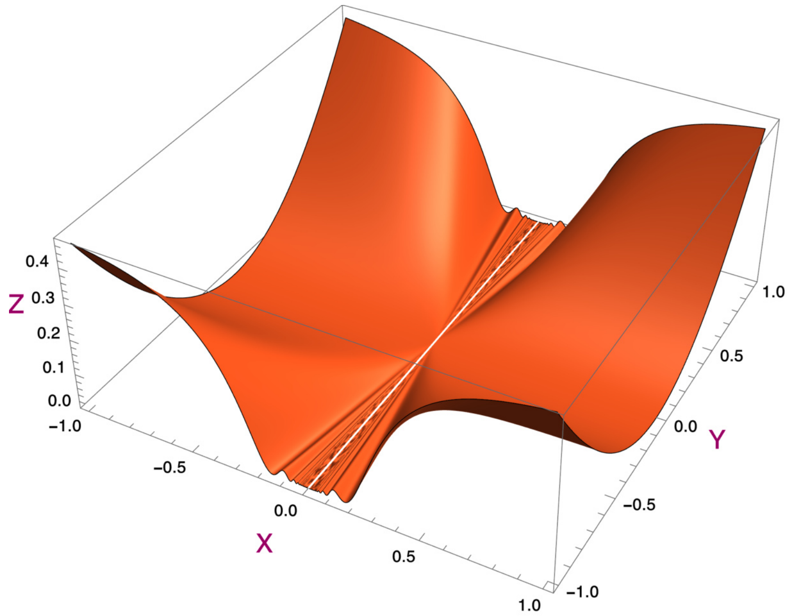

Figure 504. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

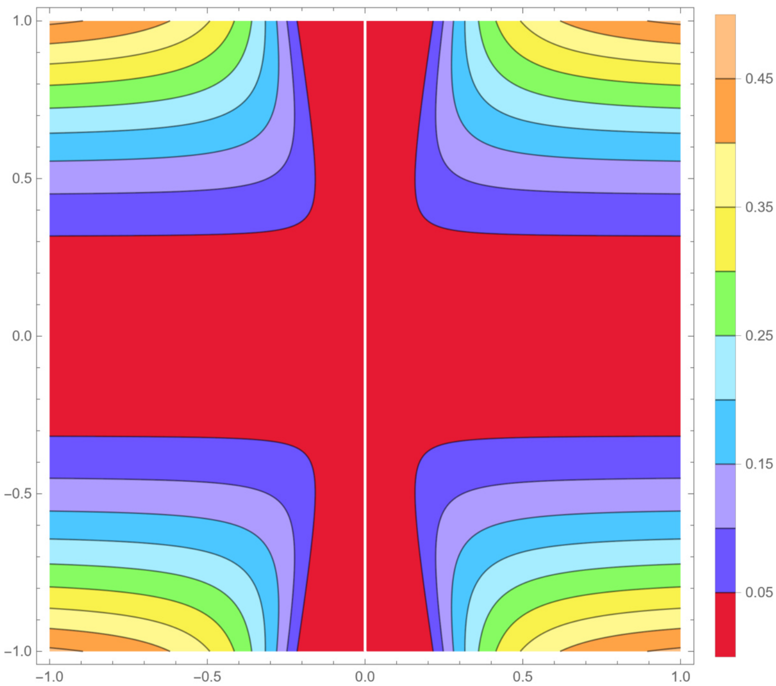

Figure 505. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

67.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

67.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\ &= \lim_{h\to 0} -hu^2 \left(\cos\left(\frac{v}{u}\right) - 1\right) \\ &= 0. \end{aligned}$$

So the directional derivatives do always exist.

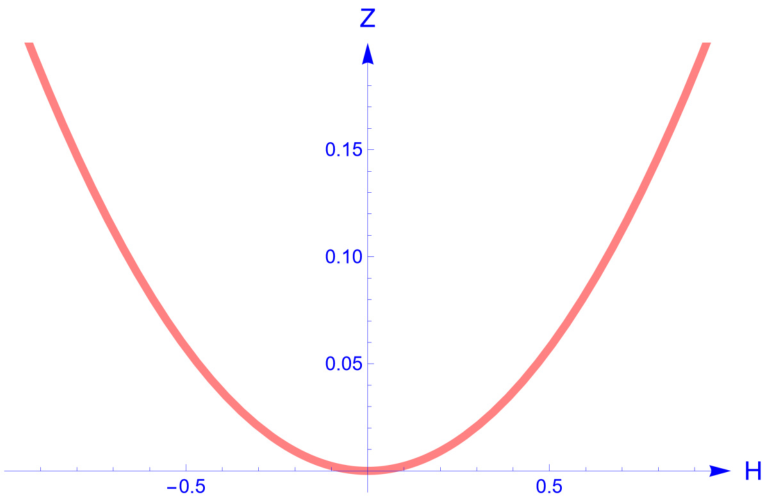

Figure 506. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$.

67.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

We ask first if the partial derivatives exist in a neighbourhood of $(0,0)$. We investigate a point $(0,b)$ with $b \neq 0$ in the X-direction. So the function $f(h,b) = 2\,h^2 \sin^2\left(\frac{b}{2\,h}\right)$ has to have a derivative in $h = 0$.

$$\lim_{h\to 0} \frac{f(h,b) - f(0,b)}{h} = \lim_{h\to 0} 2\,h \sin^2\left(\frac{b}{2\,h}\right) = 0.$$

So the partials exist everywhere because the only points left to investigate were the points $(0,b)$ with $b \neq 0$.

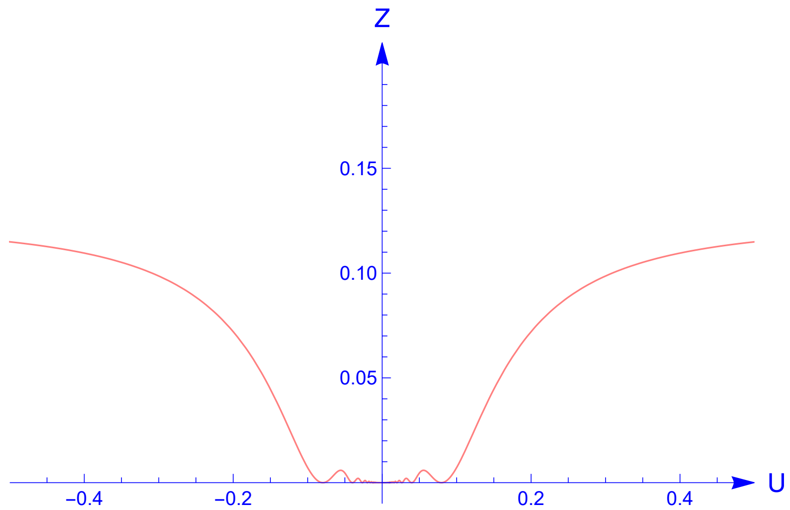

Figure 507. We see here a two dimensional figure of the graph of the function $f(h,b)$. We have drawn the function here for the value $b = 1/2$ which is exemplary for the values of b close to 0. This looks like a differentiable function.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} 2x\left(1-\cos\left(\frac{y}{x}\right)\right) - y\sin\left(\frac{y}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} x\sin\left(\frac{y}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

$$\begin{aligned} \left|\frac{\partial f}{\partial x}(x,y)\right| &\leq \left|2x\left(1-\cos\left(\frac{y}{x}\right)\right) - y\sin\left(\frac{y}{x}\right)\right| \\ &\leq 2|x|\left|\left(1-\cos\left(\frac{y}{x}\right)\right)\right| + |y|\left|\sin\left(\frac{y}{x}\right)\right| \\ &\leq 2|x| + |y| \\ &\leq 2\sqrt{x^2+y^2} + \sqrt{x^2+y^2} \\ &\leq 3\sqrt{x^2+y^2} \\ &\leq 3. \end{aligned}$$

We have chosen here the restriction to the neighbourhood defined by $\sqrt{x^2+y^2} < 1$ for the last inequality.

Let us try to prove that $\left|\frac{\partial f}{\partial y}\right|$ is bounded.

$$\begin{aligned}
\left|\frac{\partial f}{\partial y}(x,y)\right| &\le \left|x \sin\left(\frac{y}{x}\right)\right| \\
&\le |x| \left|\sin\left(\frac{y}{x}\right)\right| \\
&\le |x| \\
&\le \sqrt{x^2+y^2}. \\
&\le 1.
\end{aligned}$$

We have chosen here the restriction to the neighbourhood defined by $\sqrt{x^2+y^2} < 1$ for the last inequality.

Because the two partial derivatives are bounded in a neighbourhood of $(0,0)$, we have an alternative proof for the continuity.

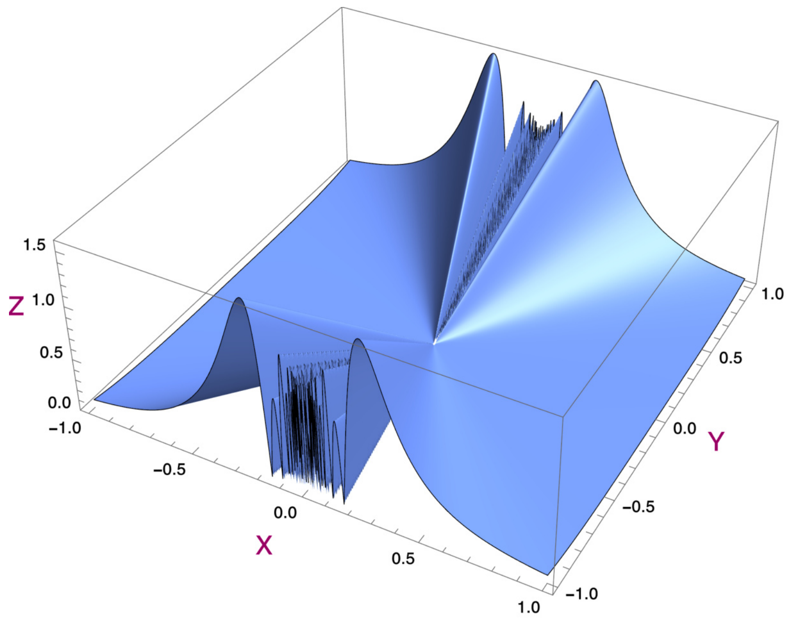

Figure 508. We see here the absolute value of the first partial derivative $\left|\frac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

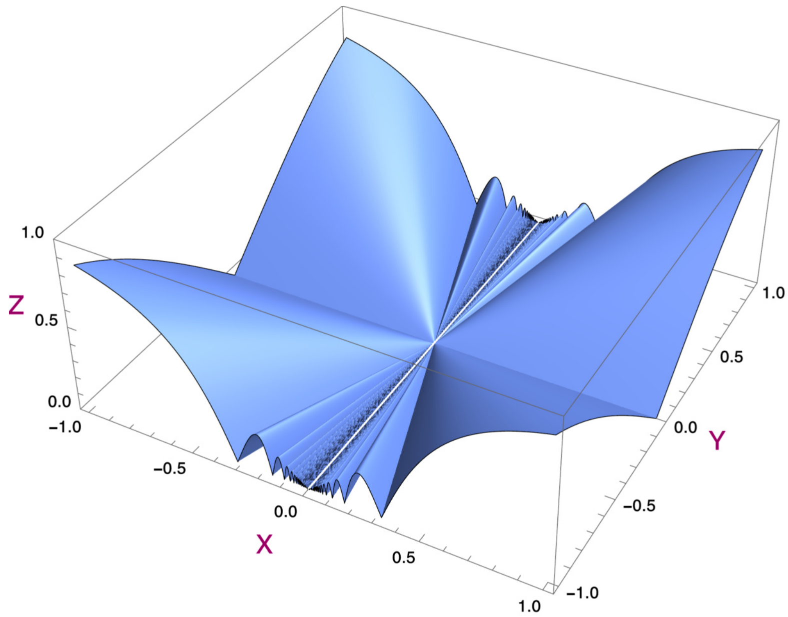

Figure 509. We see here the absolute value of the second partial derivative $\left|\frac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture.

67.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

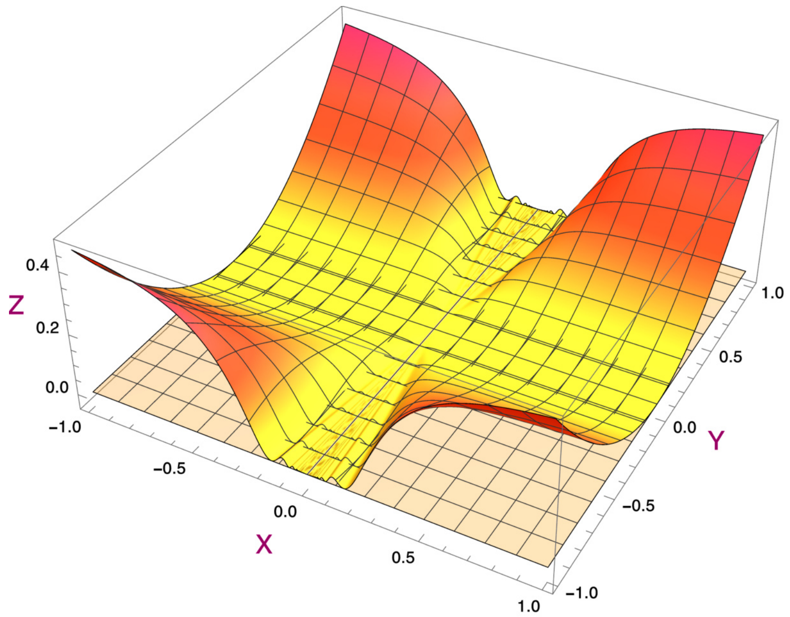

Figure 510. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

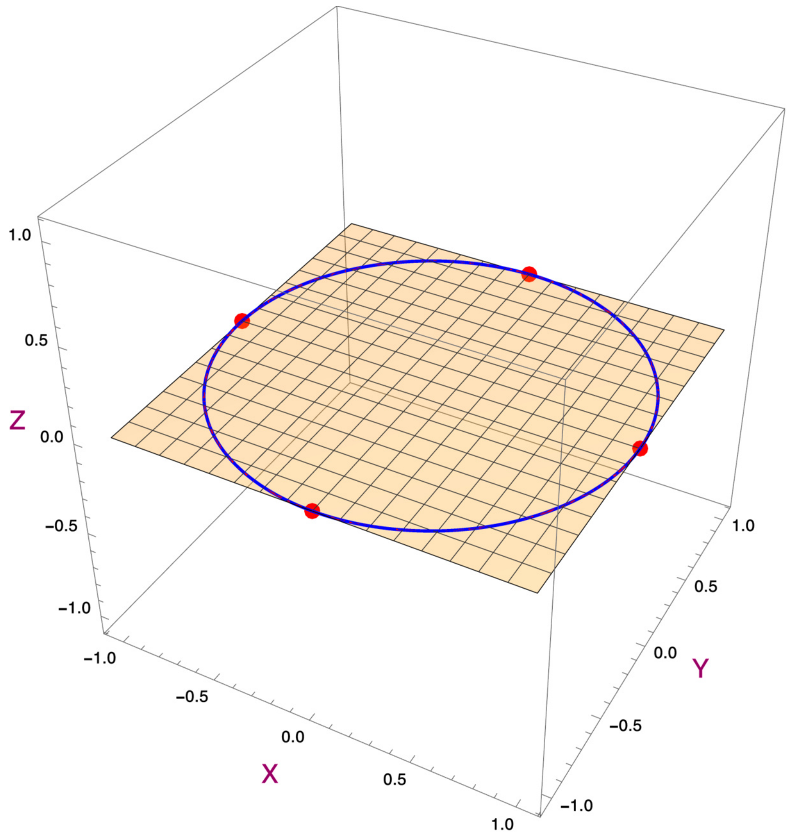

Figure 511. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} -\dfrac{h^2\left(\cos\left(\dfrac{k}{h}\right) - 1\right)}{\sqrt{h^2+k^2}} & \text{if } h \neq 0 \text{ and } (h,k) \neq (0,0); \\ 0 & \text{if } h = 0 \text{ or } (h,k) = 0 \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2+k^2} < \delta$, we have that

$$\left| -\frac{h^2\left(\cos\left(\frac{k}{h}\right) - 1\right)}{\sqrt{h^2+k^2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have

the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left| -\frac{h^2\left(\cos\left(\frac{k}{h}\right)-1\right)}{\sqrt{h^2+k^2}}\right| &\leq \frac{h^2\left|\left(\cos\left(\frac{k}{h}\right)+1\right)\right|}{\sqrt{h^2+k^2}}\\ &\leq \frac{2\,h^2}{\sqrt{h^2+k^2}}\\ &\leq \frac{2\sqrt{h^2+k^2}^{\,2}}{\sqrt{h^2+k^2}}\\ &\leq 2\sqrt{h^2+k^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon/2$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous.

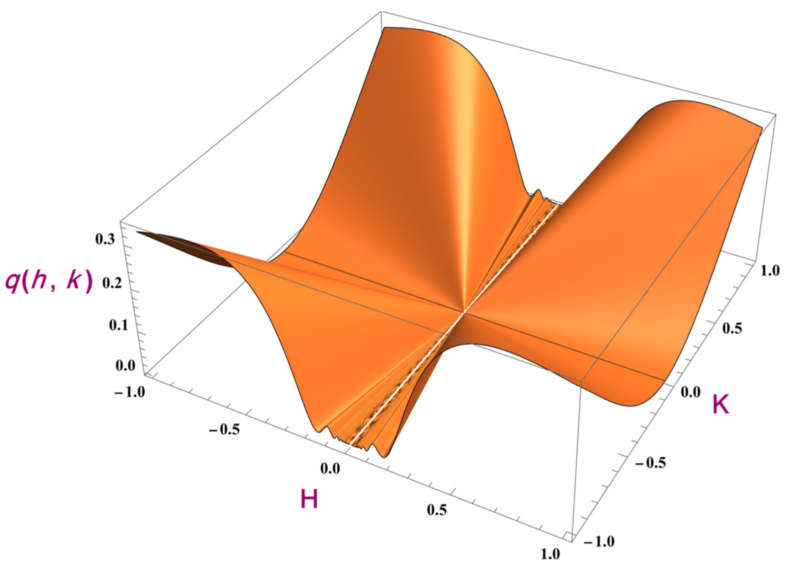

Figure 512. We see here a three dimensional figure of the graph of the function $q(h,k)$. This looks like a continuous function.

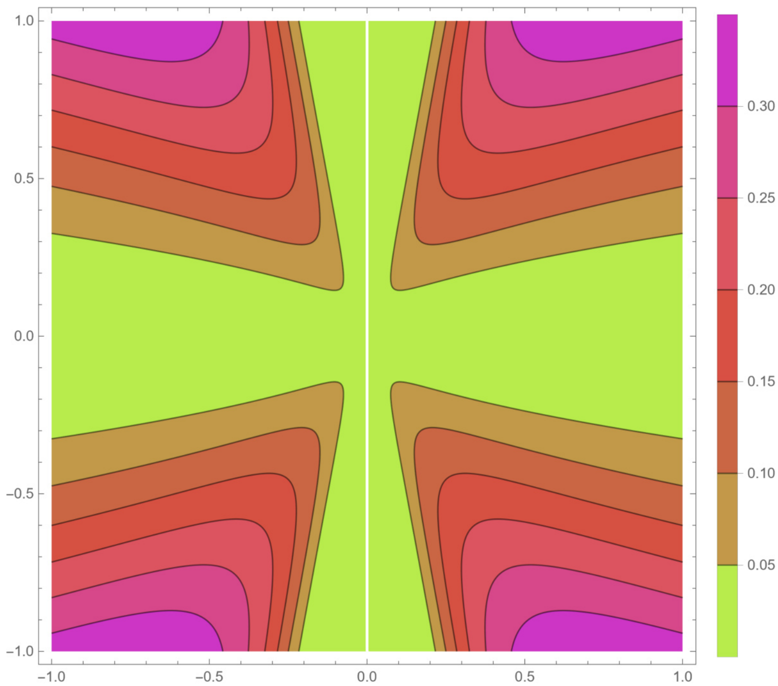

Figure 513. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

67.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\, u + \frac{\partial f}{\partial y}(0,0)\, v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the func-

tion $f(x, y)$ is locally Lipschitz continuous in $(0, 0)$. We cite here the criterion that we will use.

A function is differentiable in (a, b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.
2. The directional derivatives have the following form: $D_{(u,v)}(a, b) = \nabla f(a, b) \cdot (u, v) = \frac{\partial f}{\partial x}(a, b)\, u + \frac{\partial f}{\partial y}(a, b)\, v$.
3. The function is locally Lipschitz continuous in (a, b). This means that there exists at least one neighbourhood of (a, b) and a number $K > 0$ such that for all (x_1, y_1) and (x_2, y_2) in the neighbourhood, we have $|f(x_1, y_1) - f(x_2, y_2)| < K\,||(x_1, y_1) - (x_2, y_2)||$. Remark that the same K must be valid for all (x_1, y_1) and (x_2, y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be “locally flat" and thus “locally linear" which is partly expressed by this Lipschitz continuity condition.

Let us try to prove the Lipschitz condition. We are going to use a well known method.

$$\begin{aligned} &|f(x_1, y_1) - f(x_2, y_2)| \\ &\qquad = |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\ &\qquad \le |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)|. \end{aligned}$$

We focus now on the first term $|f(x_1, y_1) - f(x_1, y_2)|$. We fix now x_1 and look upon $f(x_1, y)$ as a function in one variable y. So it springs to mind that we can estimate this by using Lagrange’s intermediate value theorem. So we have

$$|f(x_1, y_1) - f(x_1, y_2)| = \left| \frac{\partial f}{\partial y}(x_1, \xi) \right| \; |y_1 - y_2|$$

where ξ is a number lying in the open interval (y_1, y_2) if e.g. $y_1 \leq y_2$. Remark that we already have proven that the partial derivatives are bounded in a neighbourhood, we can take that bound found for $\frac{\partial f}{\partial y}$, say M_2 and that neighbourhood in this case also. So $\left|\frac{\partial f}{\partial y}(x_1, \xi)\right| \leq M_2$. We work in a completely analogous way for the second term $|f(x_1, y_2) - f(x_2, y_2)|$. We have in that case $\left|\frac{\partial f}{\partial x}(\xi, y_2)\right| \leq M_1$ where ξ is now a number in the open interval (x_1, x_2) if e.g. $x_1 \leq x_2$.

We take up our inequality again

$$
\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)| \\
&\quad \leq |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\quad \leq |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)| \\
&\quad \leq M_2\,|(y_1 - y_2)| + M_1\,|(x_1 - x_2)| \\
&\quad \leq M_2\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + M_1\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&\quad \leq (M_1 + M_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.
\end{aligned}
$$

So this function is locally Lipschitz continuous. We have an alternative proof for the differentiability.

67.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

We know that the partial derivative to x exists and is equal to

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} 2x\left(1-\cos\left(\frac{y}{x}\right)\right) - y\sin\left(\frac{y}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We want to see if it is continuous or not.
We know that the partial derivative to y exists and is equal to

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} x\sin\left(\frac{y}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We want to see if it is continuous or not.

Let us start with investigating $\frac{\partial f}{\partial x}$.

We have to distinguish three cases. We have to investigate the continuity of the function $\frac{\partial f}{\partial x}$ first in points (x,y) where $x \neq 0$. Then we have to investigate the function in $(0,0)$. We have thirdly to investigate the function in points $(0,b)$ where $b \neq 0$.

We remark in the first case that the function $\frac{\partial f}{\partial x}$ is composed of classical functions which are known to be infinitely differentiable. So the partial derivative $\frac{\partial f}{\partial x}$ exists there and is continuous.

We are going to investigate the function $\frac{\partial f}{\partial x}$ secondly in $(0,0)$.

Discussion of the continuity of the first partial derivative in $(0,0)$.
We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove the inequality $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if the inequality $||(x,y)-(0,0)|| < \delta$ holds, it follows that $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have

the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
&\left|2\,x\,\left(1-\cos\left(\frac{y}{x}\right)\right)-y\,\sin\left(\frac{y}{x}\right)\right| \\
&\quad \le 2\,|x|\,\left|\left(1+\cos\left(\frac{y}{x}\right)\right)\right| + |y|\,\left|\sin\left(\frac{y}{x}\right)\right| \\
&\quad \le 2\,|x| + |y| \\
&\quad \le 2\sqrt{x^2+y^2}+\sqrt{x^2+y^2} \\
&\quad \le 3\sqrt{x^2+y^2}.
\end{aligned}$$

It is sufficient to take $\delta = \epsilon/3$. We can find a δ, so we conclude that the function is continuous in $(0,0)$.

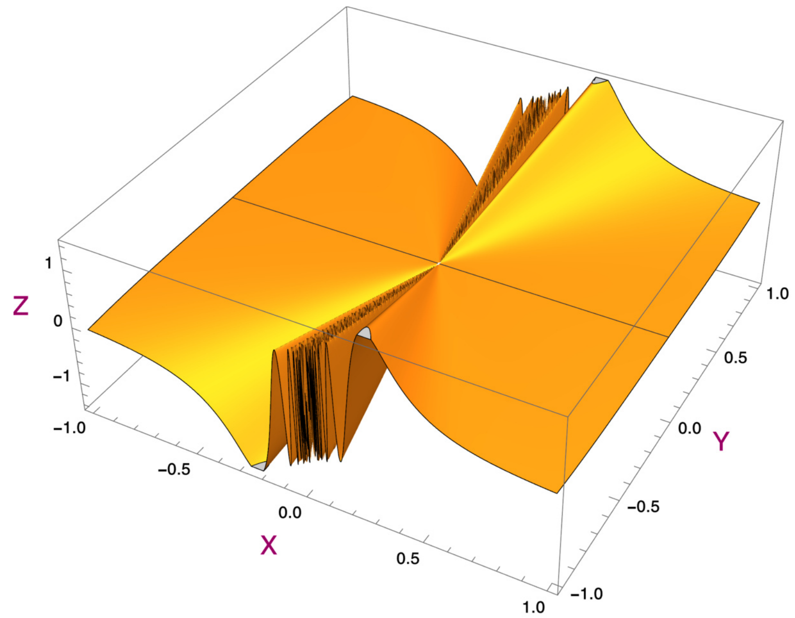

Figure 514. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial x}(x,y)$. This looks like a continuous function in $(0,0)$.

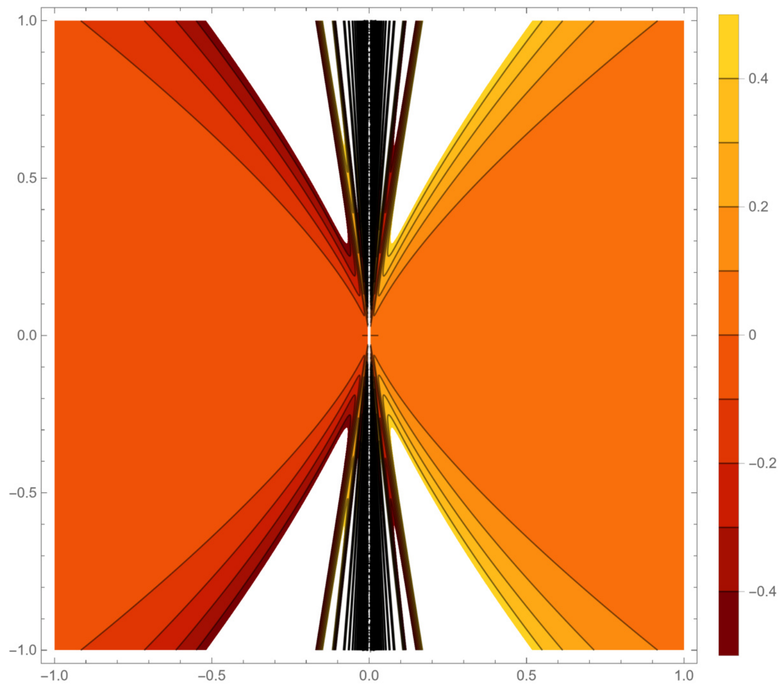

Figure 515. We see here a figure of the contour plot of the $\frac{\partial f}{\partial x}(x, y)$. Only level curves of level around 0 come close to $(0, 0)$.

We are thirdly going to investigate the continuity of the function $\frac{\partial f}{\partial x}$ in points $(0, b)$.

The function $\frac{\partial f}{\partial x}$ is now

$$\frac{\partial f}{\partial x}(x, y) = \begin{cases} 4\,x\,\sin^2\left(\frac{y}{2\,x}\right) - 2\,y\,\sin\left(\frac{y}{2\,x}\right)\cos\left(\frac{y}{2\,x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We transform now $x = u$ and $y = v + b$ in order to work technically easier.

$$g(u, v) = \begin{cases} 4\,u\,\sin^2\left(\frac{b+v}{2\,u}\right) - 2\,(b + v)\,\sin\left(\frac{b+v}{2\,u}\right)\cos\left(\frac{b+v}{2\,u}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We see that the first term is certainly continuous. We look at the second term and expand to the factor $b + v$.

$$-2\,(b+v)\,\sin\left(\frac{b+v}{2u}\right)\cos\left(\frac{b+v}{2u}\right) = -2\,b\,\sin\left(\frac{b+v}{2\,u}\right)\cos\left(\frac{b+v}{2\,u}\right) - 2v\,\sin\left(\frac{b+v}{2\,u}\right)\cos\left(\frac{b+v}{2\,u}\right).$$

We see that the second term is certainly continuous in 0. So we have to investigate the first term. After dropping the non zero constant coefficient, we are left with investigating the function

$$h(u,v) = \sin\left(\frac{b+v}{2\,u}\right)\cos\left(\frac{b+v}{2\,u}\right).$$

We rewrite this as

$$h(u,v) = \sin\left(\frac{b+v}{2u}\right)\cos\left(\frac{b+v}{2\,u}\right) = \frac{1}{2}\sin\left(\frac{b+v}{u}\right).$$

It is obvious that this function is not continuous. We can be explicit. We define a sequence $(u_n = \frac{b\,(n+1)}{2\,\pi\,n^2}, v_n = \frac{b}{n})$, $n \in \mathbf{N}_0$, such that $h(u_n, v_n) = 0$ and $\lim_{n\to\infty} h(u_n, v_n) = 0$. We can define another sequence $(u_n = \frac{2\,(b\,n+b)}{\pi\,n\,(4\,n+1)}, v_n = \frac{b}{n})$, $n \in \mathbf{N}_0$, such that $h(u_n, v_n) = 1/2$ and we have then $\lim_{n\to\infty} h(u_n, v_n) = 1/2$. This is impossible if h is continuous.

We see that we cannot apply this criterion.

67.8 Overview

$$f(x,y) = \begin{cases} x^2\left(1-\cos\left(\frac{y}{x}\right)\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

67.9 One step further

We want to know if this function is further uneventful from the point of view of differentiability. Let us take a look at the second order partial derivative

$$\frac{\partial^2 f}{\partial x^2}(x, y) = \frac{(y^2 - 2x^2)\cos\left(\frac{y}{x}\right) + 2x\left(x - y\sin\left(\frac{y}{x}\right)\right)}{x^2}.$$

Let us take a look of a three dimensional plot of this second order partial derivative of the function.

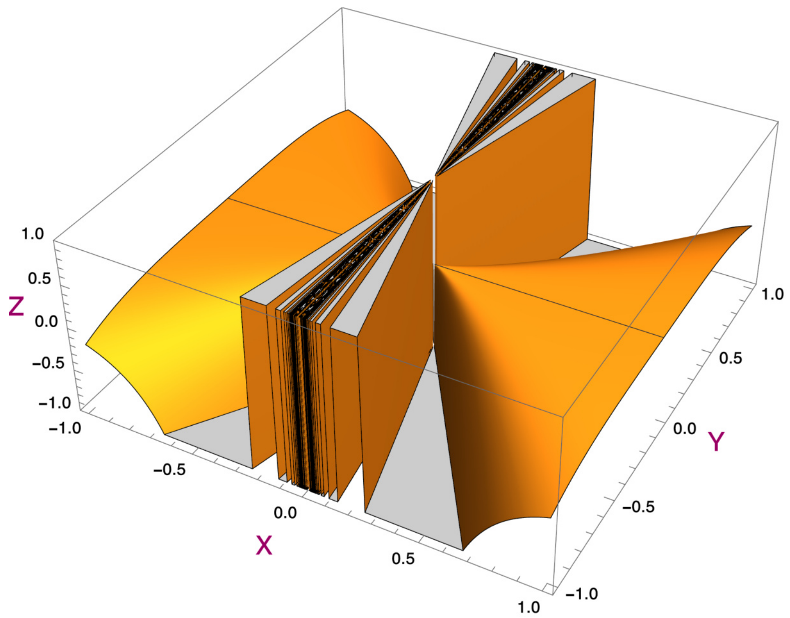

Figure 516. We see here a figure of the second order partial derivative $\frac{\partial^2 f}{\partial x^2}(x, y)$. It seems quite improbable that this second order partial derivative is continuous or even exists. We stop however our investigations here and leave this to the initiative of the interested reader.

Exercise 68.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{\sin(x-y)}{\sqrt{|x|}+\sqrt{|y|}} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

68.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{\sin(x-y)}{\sqrt{|x|}+\sqrt{|y|}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\left|\frac{\sin(x-y)}{\sqrt{|x|}+\sqrt{|y|}}\right| \leq \left|\frac{x-y}{\sqrt{|x|}+\sqrt{|y|}}\right| \leq \frac{|x|+|y|}{\sqrt{|x|}+\sqrt{|y|}}.$$

It is enough to prove that $|x| < \epsilon\sqrt{|x|}$ and $|y| < \epsilon\sqrt{|y|}$ because if we add left hand sides and right hand sizes of the two inequalities, we have $|x| + |y| < \epsilon\sqrt{|x|} + \epsilon\sqrt{|y|}$. This is equivalent with $\frac{|x|+|y|}{\sqrt{|x|}+\sqrt{|y|}} < \epsilon$ if $|x| + |y| \neq 0$. Now $|x| < \epsilon\sqrt{|x|}$ is equivalent with $\sqrt{|x|} < \epsilon$ if $x \neq 0$. So it is enough that $\sqrt{|x|} \leq \sqrt{\sqrt{|x|^2 + |y|^2}} < \epsilon$. If $x = 0$ or $y = 0$, we can reason in a similar way. So we can take $\delta = \epsilon^2$.

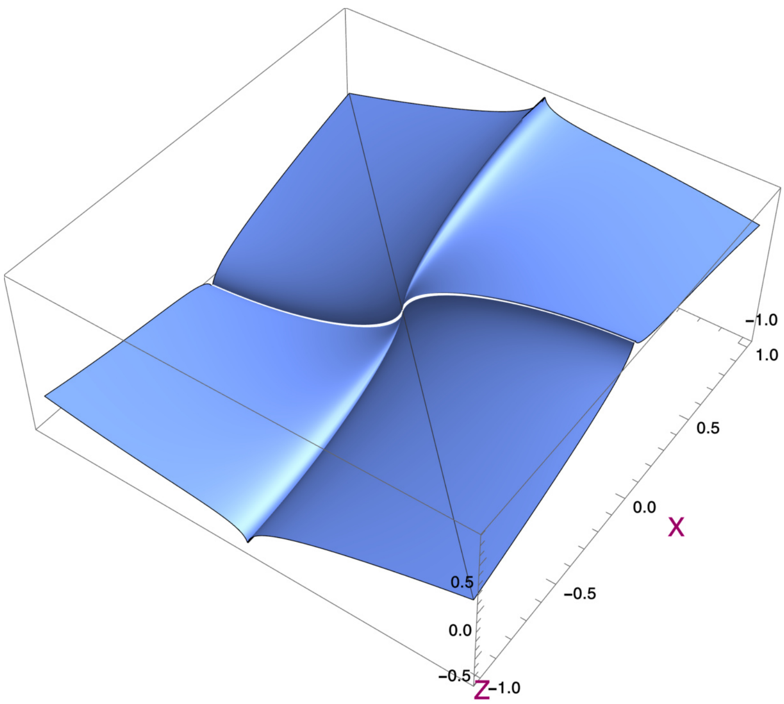

Figure 517. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

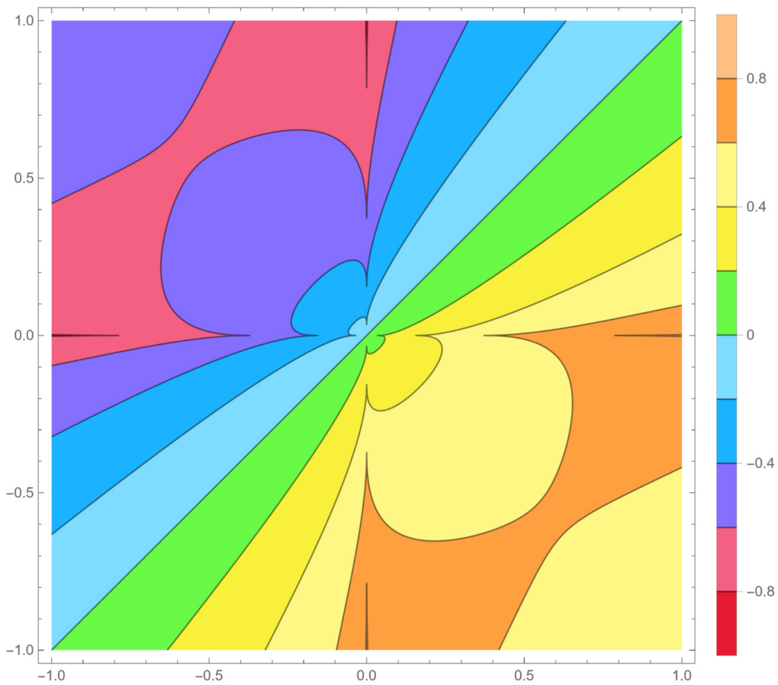

Figure 518. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

68.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).
We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = \dfrac{\sin(x)}{\sqrt{|x|}} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\sin(h)}{h} \frac{1}{\sqrt{|h|}}. \end{aligned}$$

So the partial derivative to x does not exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = \dfrac{\sin(-y)}{\sqrt{|y|}} & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$
$$= \lim_{h\to 0} -\frac{\sin(h)}{h}\frac{1}{\sqrt{|h|}}.$$

So the partial derivative to y does not exist.

We conclude that the partial derivative to x and to y do not exist.

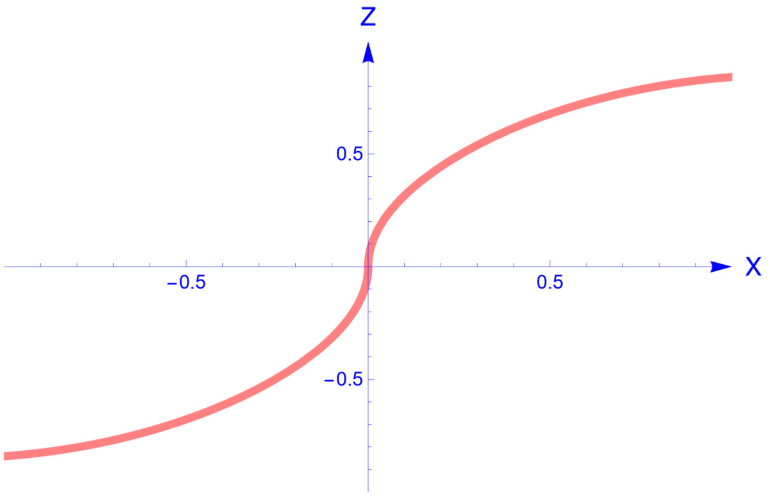

Figure 519. We see here a figure of the graph of the function restricted to the horizontal X-axis through $(0,0)$. This is $f(x,0)$. The vertical tangent in $x = 0$ makes this function not differentiable.

68.3 Directional derivatives

Let (u, v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2 + v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u, v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0 + h\,u, 0 + h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0 + h\,u, 0 + h\,v) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\sin(h\,(u - v))}{h\left(\sqrt{|h\,u|} + \sqrt{|h\,v|}\right)} \\ &= \lim_{h\to 0} \frac{\sin(h\,(u - v))}{h\,(u - v)} \frac{u - v}{\left(\sqrt{|h\,u|} + \sqrt{|h\,v|}\right)}. \end{aligned}$$

We can observe that we have no finite limit if $u - v \neq 0$.

So the directional derivatives do not always exist.

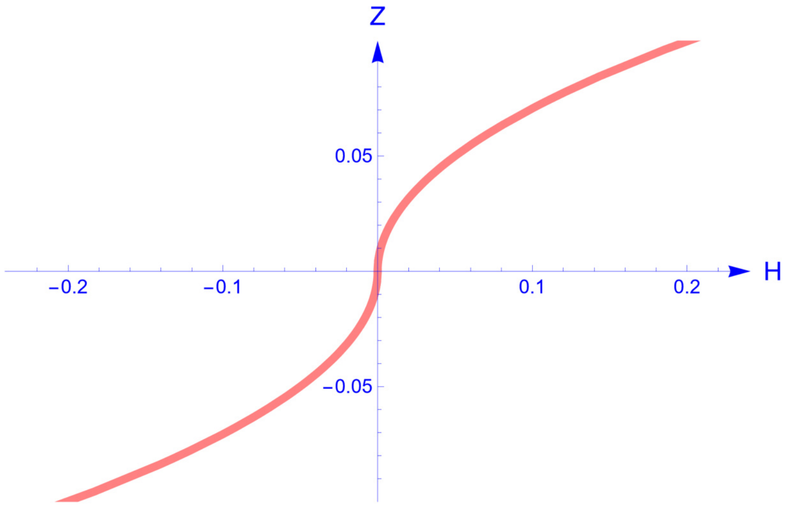

Figure 520. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(\sqrt{3}/2, 1/2\right)$. The vertical tangent in $h = 0$ prevents differentiability. We have plotted here the function $f(hu, hv)$.

68.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

But the partials in $(0,0)$ do not even exist. So there is no alternative proof possible for the continuity.

68.5 Differentiability

At least one of the directional derivatives does not exist, thus the function is not differentiable.

68.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

68.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

68.8 Overview

$$f(x,y) = \begin{cases} \dfrac{\sin(x-y)}{\sqrt{|x|}+\sqrt{|y|}} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	yes
partial derivatives exist	no
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 69.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{x\,y - \sin(x)\,\sin(y)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

69.1 Continuity

Before starting with this exercise, it is necessary to look first at the function definition, which is a little bit hard to handle. We are going to use Taylor series out of necessity. We know from the theory of one variable of the Taylor approximation with remainder

$$\sin(\alpha) = \alpha - \frac{1}{2}\sin(\xi)\,\alpha^2.$$

It is generally known as the second order approximation. We have that ξ is a number between 0 and α. We do now the fourth order approximation.

$$\sin(\alpha) = \alpha - \frac{\alpha^3}{3!} + \frac{1}{24}\sin(\xi)\,\alpha^4.$$

We have that ξ is a number between 0 and α.

We remark that the numerator of the function definition after substituting with the second order approximations can be written as follows

$$\begin{aligned} &x\,y - \sin(x)\,\sin(y) \\ &= xy - \left(x - \frac{1}{2}x^2\,\sin(\xi_1)\right)\left(y - \frac{1}{2}y^2\sin(\xi_2)\right) \\ &= -\frac{1}{4}x^2\,y^2\,\sin(\xi_1)\,\sin(\xi_2) + \frac{1}{2}x^2\,y\,\sin(\xi_1) + \frac{1}{2}x\,y^2\,\sin(\xi_2). \end{aligned}$$

We are going to use this simplified form in the calculation of the continuity in $(0,0)$.

We try to avoid the big O notation because it is not generally known in a first course in calculus.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{x\,y - \sin(x)\,\sin(y)}{x^2 + y^2} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$
\begin{aligned}
&\left|\frac{x\,y - \sin(x)\,\sin(y)}{x^2+y^2}\right| \\
&\le \left|\frac{-\frac{1}{4}x^2y^2\sin(\xi_1)\sin(\xi_2) + \frac{1}{2}x^2y\sin(\xi_1) + \frac{1}{2}\,x\,y^2\,\sin(\xi_2)}{x^2+y^2}\right| \\
&\le \frac{\frac{1}{4}\,x^2\,y^2|\sin(\xi_1)|\;|\sin(\xi_2)| + \frac{1}{2}\,x^2\,|y|\,|\sin(\xi_1)| + \frac{1}{2}\,|x|\,y^2\,|\sin(\xi_2)|}{x^2+y^2} \\
&\le \frac{\frac{1}{4}\,x^2\,y^2 + \frac{1}{2}\,x^2\,|y| + \frac{1}{2}\,|x|\,y^2}{x^2+y^2} \\
&\le \frac{\frac{1}{4}\,|x|\,|y|\,(|x|\,|y| + 2\,|x| + 2\,|y|)}{x^2+y^2} \\
&\le \frac{\frac{1}{4}\sqrt{x^2+y^2}^{\,2}\left(\sqrt{x^2+y^2}^{\,2} + 2\sqrt{x^2+y^2} + 2\sqrt{x^2+y^2}\right)}{x^2+y^2} \\
&\le \frac{\frac{1}{4}\sqrt{x^2+y^2}^{\,3}\,(\sqrt{x^2+y^2} + 2 + 2)}{x^2+y^2} \\
&\le \frac{\frac{1}{4}\sqrt{x^2+y^2}^{\,3}\;5}{x^2+y^2} \\
&\le \frac{\frac{5}{4}\sqrt{x^2+y^2}^{\,3}}{\sqrt{x^2+y^2}^{\,2}} \\
&\le \frac{5}{4}\sqrt{x^2+y^2}.
\end{aligned}
$$

We used in the first line the Taylor expansion described above. We have also chosen the restriction $\sqrt{x^2+y^2} < 1$.

It is sufficient to take $\delta = \min\{1, \frac{4}{5}\,\epsilon\}$. We can find a δ, so we conclude that the function is continuous.

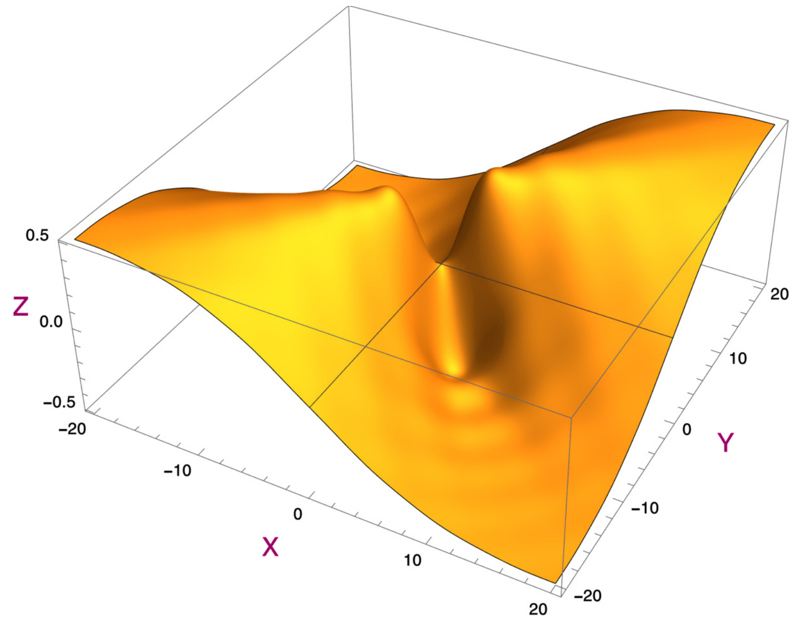

Figure 521. We see here a three dimensional figure of a more global view of the graph of the function. This looks like a continuous function.

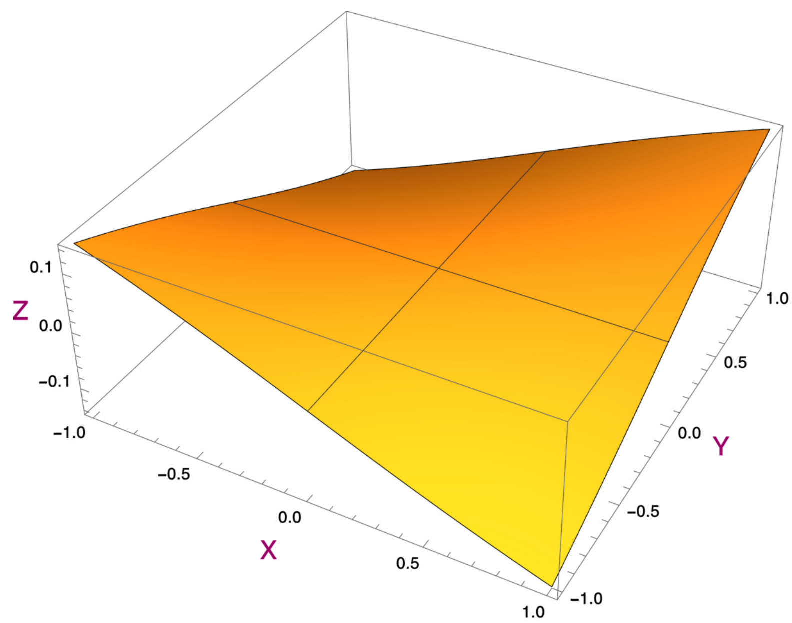

Figure 522. We see here a three dimensional figure of a more local view of the graph of the function. This looks like a continuous function.

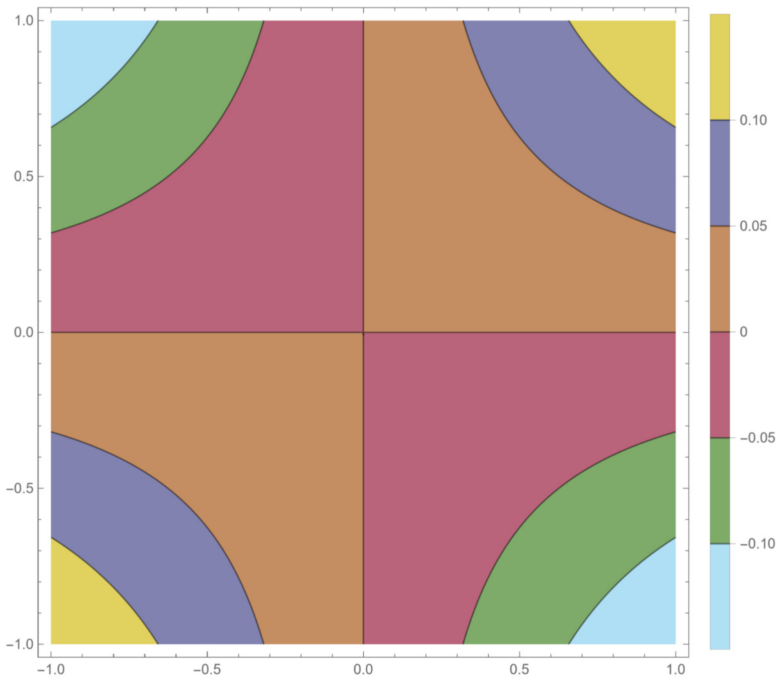

Figure 523. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

69.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).
We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to x does exist.

By the symmetry in the function definition, we can state the same for the partial derivative to x.

69.3 Directional derivatives

Let (u, v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2 + v^2} = 1$. Then the directional derivative in $(0, 0)$ in the direction (u, v), notation $D_{(u,v)}(0, 0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h \to 0} \frac{f(0 + h\,u, 0 + h\,v) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1, 0)$ and $(0, 1)$.

We remember that we have the function

$$f(x,y) = -\frac{x^2 y^2 \sin(\xi_1)\sin(\xi_2)}{4\,(x^2+y^2)} + \frac{x^2 y \sin(\xi_1)}{2\,(x^2+y^2)} + \frac{x y^2 \sin(\xi_2)}{2\,(x^2+y^2)}.$$

In the case of our function we have to calculate the following limit

$$\begin{aligned}
D_{(u,v)}(0,0) &= \lim_{h \to 0} \frac{f(0 + h\,u, 0 + h\,v) - f(0,0)}{h} \\
&= \lim_{h \to 0} -\frac{u\,v\,(h\,u\,v\,\sin(\xi_1)\,\sin(\xi_2) - 2\,u\,\sin(\xi_1) - 2\,v\,\sin(\xi_2))}{4\,(u^2+v^2)} \\
&= 0.
\end{aligned}$$

We see that the number ξ_1 satisfies the inequality $0 \le \xi_1 \le x = h\,u$. Because $x = h\,u \to 0$, we conclude that this value $\xi_1 \to 0$. We can reason in a similar way if $x < 0$. We do a similar procedure with ξ_2.

So the directional derivatives do always exist.

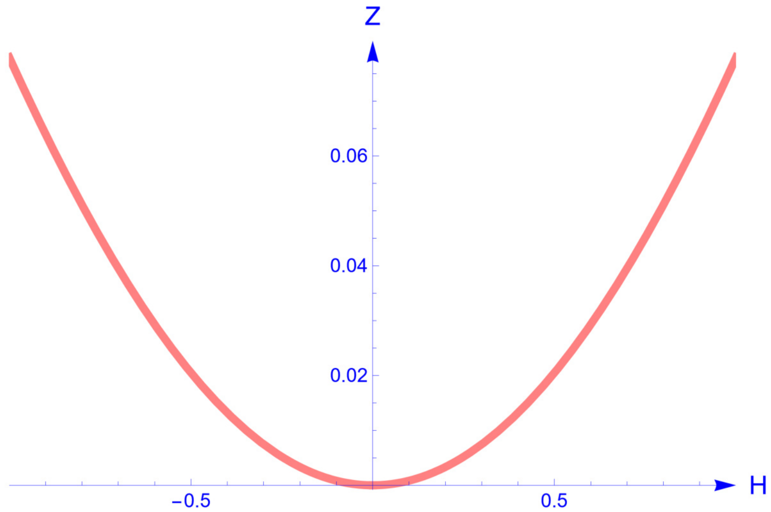

Figure 524. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

69.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.
Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y)$$

$$= \begin{cases} \dfrac{-\,(x^2+y^2)\,\cos(x)\,\sin(y) - x^2\,y \\ \quad + 2\,x\,\sin(x)\,\sin(y) + y^3}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

The partial derivative to y is:

$$\frac{\partial f}{\partial y}(x,y)$$

$$= \begin{cases} \dfrac{x^3 - (x^2+y^2)\,\sin(x)\,\cos(y) - x\,y^2 \\ \quad + 2\,y\,\sin(x)\,\sin(y)}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We can consult figures of the absolute values of these two partial derivatives at the end of this section.

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

The main part of the definition of the first derivative to x is

$$\frac{-\,(x^2+y^2)\,\cos(x)\,\sin(y) - x^2\,y + 2\,x\,\sin(x)\,\sin(y) + y^3}{(x^2+y^2)^2}.$$

We are going to rewrite this formula in order to do some computations. In order to keep the calculations at bay, we are going to do some tricks to keep oversight.

We know also from the theory of Taylor that we can rewrite

$$\cos(u) = 1 - \frac{1}{2!}\,u^2 + \frac{1}{4!}\,u^4 - \frac{1}{6!}\,\cos(\xi)\,u^6$$

with ξ a number between 0 and u and

$$\sin(u) = u - \frac{1}{3!}\,u^3 + \frac{u^5}{5!}\,u^5 - \frac{1}{7!}\,\sin(\xi)\,u^7$$

with ξ a number between 0 and u.

We remember from trigonometry that

$$\sin(x)\,\sin(y) = \frac{1}{2}\,(\cos(x-y) - \cos(x+y)).$$

We use now our Taylor expansions to rewrite

$$\begin{aligned}
&\sin(x)\,\sin(y)\\
&= \frac{1}{2}\left(-\frac{1}{2}\,(x-y)^2 + \frac{1}{2!}\,(x+y)^2 + \frac{1}{4!}\,(x-y)^4 - \frac{1}{24}\,(x+y)^4\right.\\
&\qquad \left. - \frac{1}{6!}\,(x-y)^6\,\cos(\xi_1) + \frac{1}{6!}\,(x+y)^6\cos(\xi_2)\right).
\end{aligned}$$

We remember from trigonometry that

$$\sin(y)\,\cos(x) = \frac{1}{2}\,(\sin(y+x) + \sin(y-x)).$$

We substitute our expansions in this formula and rewrite

$$\begin{aligned}
&\frac{1}{2}\,(\sin(y+x) + \sin(y-x))\\
&= \frac{1}{2!}\left(2\,y + \frac{1}{3!}\,(x-y)^3 - \frac{1}{3!}\,(x+y)^3 - \frac{1}{5!}\,(x-y)^5\right.\\
&\qquad \left. + \frac{1}{5!}\,(x+y)^5 + \frac{1}{7!}(x-y)^7\,\sin(\xi_4) - \frac{1}{7!}\,(x+y)^7\,\sin(\xi_3)\right).
\end{aligned}$$

Now comes the tricky part. We are going to substitute both formulas in the definition formula of the first derivative. We will see that it is *absolutely unnecessary* to calculate all terms. We must only calculate the monomials $x^i\,y^j$ with lowest total degree $i + j$. Otherwise this calculation would be unwieldy. The reason why we rewrote with the help of the trigonometric formulas is that it is easier to predict the total degree of the monomials. We have then that the terms with the lowest total degree have degree 5. If we denote the *finite* sum of the terms with monomials of higher degree with S, then we have for the main part of the definition of the first derivative:

$$\frac{-\left(x^2+y^2\right)\cos(x)\sin(y)-x^2 y+2x\sin(x)\sin(y)+y^3}{\left(x^2+y^2\right)^2} = \frac{\frac{x^4 y}{6}+\frac{x^2 y^3}{3}+\frac{y^5}{6}+S}{\left(x^2+y^2\right)^2}.$$

We have now the following alternative definition of the partial derivative to x.

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{\frac{x^4 y}{6}+\frac{x^2 y^3}{3}+\frac{y^5}{6}+S}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We sketch how we can prove that the partial derivative $\frac{\partial f}{\partial x}$ is bounded. We take a monomial divided by the denominator out of the main definition of the partial derivative. Let us say it is

$$\frac{a\, x^i y^j}{\left(x^2+y^2\right)^2}.$$

We have then the inequalities

$$\begin{aligned}
\left|\frac{a\,x^i\,y^j}{(x^2+y^2)^2}\right| &\le \frac{|a|\,|x|^i\,|y|^j}{(x^2+y^2)^2}\\
&\le \frac{|a|\,\sqrt{x^2+y^2}^{\,i}\,\sqrt{x^2+y^2}^{\,j}}{(x^2+y^2)^2}\\
&\le \frac{|a|\,\sqrt{x^2+y^2}^{\,i+j}}{\sqrt{x^2+y^2}^{\,4}}\\
&\le |a|\,\sqrt{x^2+y^2}^{\,i+j-4}\\
&\le |a|.
\end{aligned}$$

We have used that $i+j-4>0$ and we have restricted to the open neighbourhood $\sqrt{x^2+y^2}<1$. This shows that every term is bounded. We see that the function is indeed locally bounded.

We see by the symmetry in the definition of our function that we do not have to calculate the partial derivative to y. All calculations are similar.

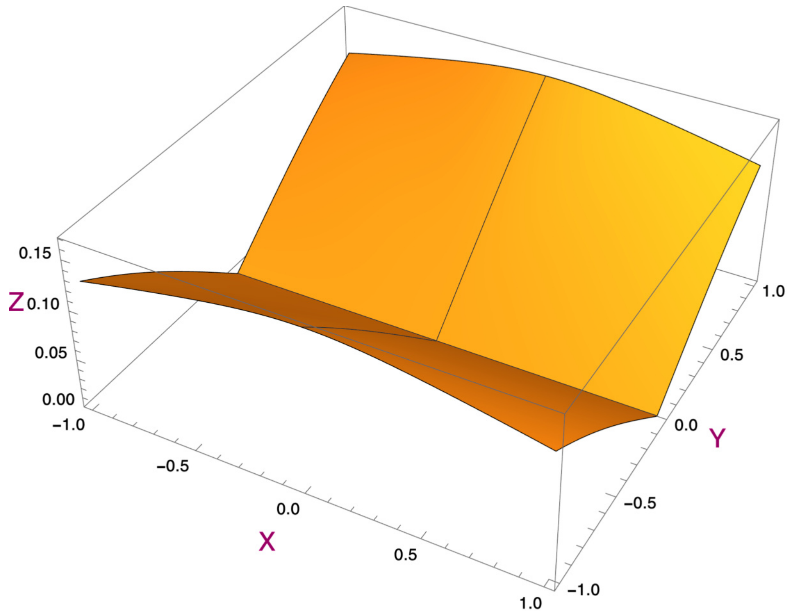

Figure 525. We see here the absolute value of the first partial derivative $\left|\dfrac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

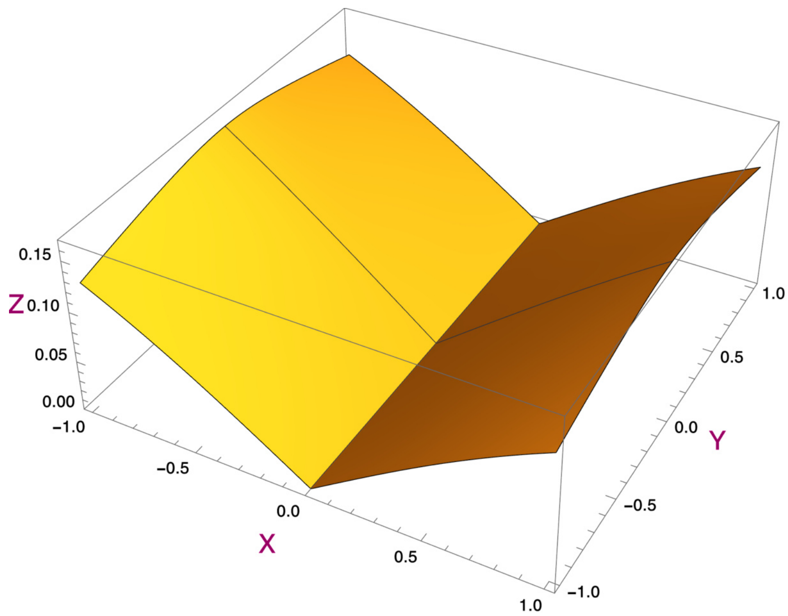

Figure 526. We see here the absolute value of the second partial derivative $\left|\dfrac{\partial f}{\partial y}\right|$. We can observe the boundedness from this picture.

69.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0,0)$ in order to see what is going on.

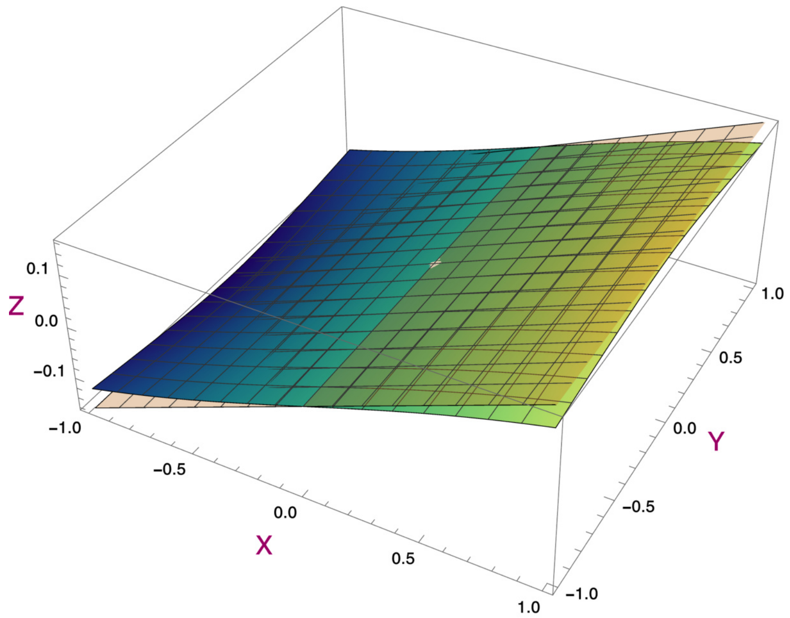

Figure 527. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function very nicely.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we

know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

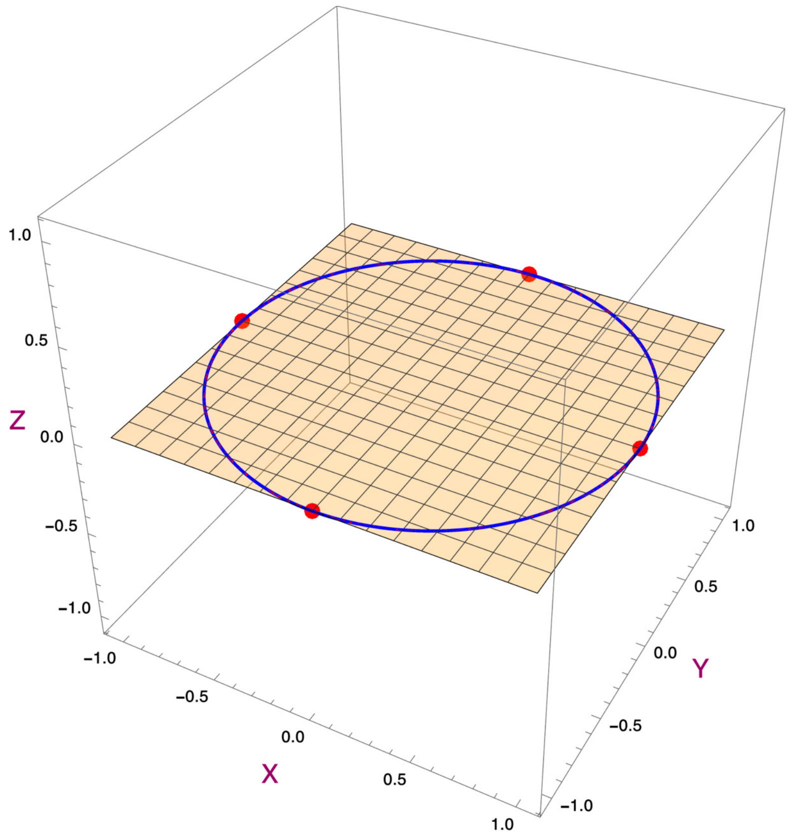

Figure 528. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h\,k - \sin(h)\,\sin(k)}{(h^2+k^2)^{5/2}} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.
This is equivalent by the calculations we made in section 1, that the function

$$q(h,k) = \begin{cases} \dfrac{\begin{array}{l} -\frac{1}{4}h^2k^2\sin(\xi_1)\sin(\xi_2) \\ \quad + \frac{1}{2}h^2k\sin(\xi_1) + \frac{1}{2}\,h\,k^2\,\sin(\xi_2) \end{array}}{(h^2+k^2)^{5/2}} & \text{if } (h,k) \neq (0,0); \\ 0 & \text{if } (h,k) = (0,0) \end{cases}$$

is continuous in $(0,0)$.

Let us shorten this and say that we can work in an analogous way as in the approach we used for the boundedness of the partial derivatives in section 4 of this exercise. We can reuse almost all calculations we did there.

So this function is continuous.

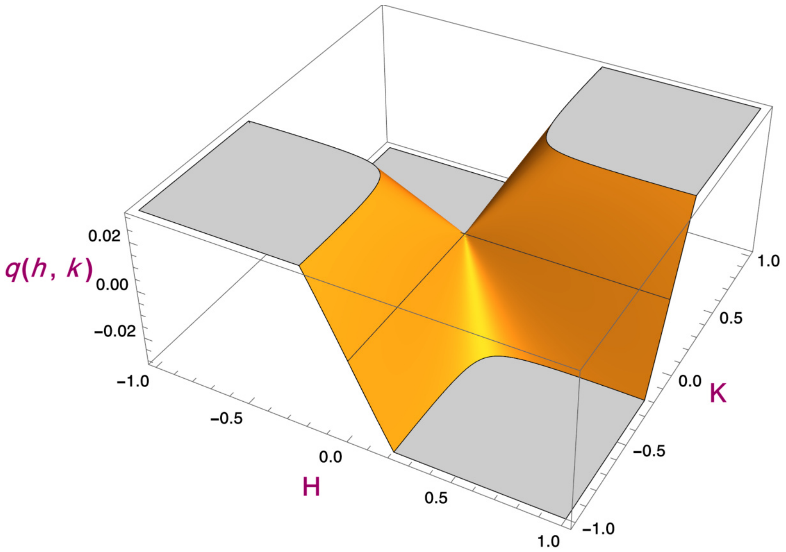

Figure 529. We see here a three dimensional figure of the graph of the function $q(h, k)$. This looks like a continuous function.

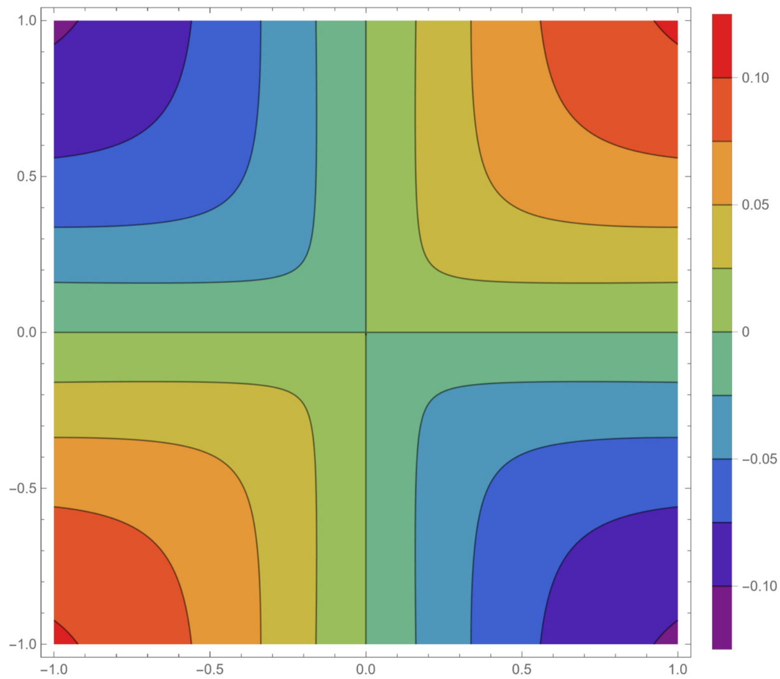

Figure 530. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

69.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\,u + \frac{\partial f}{\partial y}(0,0)\,v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the func-

tion $f(x, y)$ is locally Lipschitz continuous in $(0, 0)$. We cite here the criterion that we will use.

A function is differentiable in (a, b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.

2. The directional derivatives have the following form: $D_{(u,v)}(a, b) = \nabla f(a, b) \cdot (u, v) = \frac{\partial f}{\partial x}(a, b)\, u + \frac{\partial f}{\partial y}(a, b)\, v$.

3. The function is locally Lipschitz continuous in (a, b). This means that there exists at least one neighbourhood of (a, b) and a number $K > 0$ such that for all (x_1, y_1) and (x_2, y_2) in the neighbourhood, we have $|f(x_1, y_1) - f(x_2, y_2)| < K \|(x_1, y_1) - (x_2, y_2)\|$. Remark that the same K must be valid for all (x_1, y_1) and (x_2, y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be “locally flat" and thus “locally linear" which is partly expressed by this Lipschitz continuity condition.

Let us try to prove the Lipschitz condition. We are going to use a well known method.

$$
\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)| \\
&\qquad = |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\qquad \le |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)|.
\end{aligned}
$$

We focus now on the first term $|f(x_1, y_1) - f(x_1, y_2)|$. We fix now x_1 and look upon $f(x_1, y)$ as a function in one variable y. So it springs to mind that we can estimate this by using Lagrange’s intermediate value theorem. So we have

$$
|f(x_1, y_1) - f(x_1, y_2)| = \left| \frac{\partial f}{\partial y}(x_1, \xi) \right| \, |y_1 - y_2|
$$

where ξ is a number lying in the open interval (y_1, y_2) if e.g. $y_1 \leq y_2$. Remark that we already have proven that the partial derivatives are bounded in a neighbourhood, we can take that bound found for $\frac{\partial f}{\partial y}$, say M_2 and that neighbourhood in this case also. So $\left|\frac{\partial f}{\partial y}(x_1, \xi)\right| \leq M_2$. We work in a completely analogous way for the second term $|f(x_1, y_2) - f(x_2, y_2)|$. We have in that case $\left|\frac{\partial f}{\partial x}(\xi, y_2)\right| \leq M_1$ where ξ is now a number in the open interval (x_1, x_2) if e.g. $x_1 \leq x_2$.

We take up our inequality again

$$
\begin{aligned}
&|f(x_1, y_1) - f(x_2, y_2)| \\
&\quad \leq |f(x_1, y_1) - f(x_1, y_2) + f(x_1, y_2) - f(x_2, y_2)| \\
&\quad \leq |f(x_1, y_1) - f(x_1, y_2)| + |f(x_1, y_2) - f(x_2, y_2)| \\
&\quad \leq M_2 \, |(y_1 - y_2)| + M_1 \, |(x_1 - x_2)| \\
&\quad \leq M_2 \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + M_1 \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&\quad \leq (M_1 + M_2) \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.
\end{aligned}
$$

So this function is locally Lipschitz continuous. We have an alternative proof for the differentiability.

69.7 Continuity of the partial derivatives

By section 4, we have to prove that the following function is continuous:

$$
\frac{\partial f}{\partial x}(x, y) = \begin{cases} \dfrac{\dfrac{x^4 y}{6} + \dfrac{x^2 y^3}{3} + \dfrac{y^5}{6} + S}{\left(x^2 + y^2\right)^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}
$$

We can do this calculation in a completely similar way as we explained it in section 70.4. This partial derivative is certainly continuous.

We remark that the computations are completely similar for the other variable y by the symmetry of the function definition.

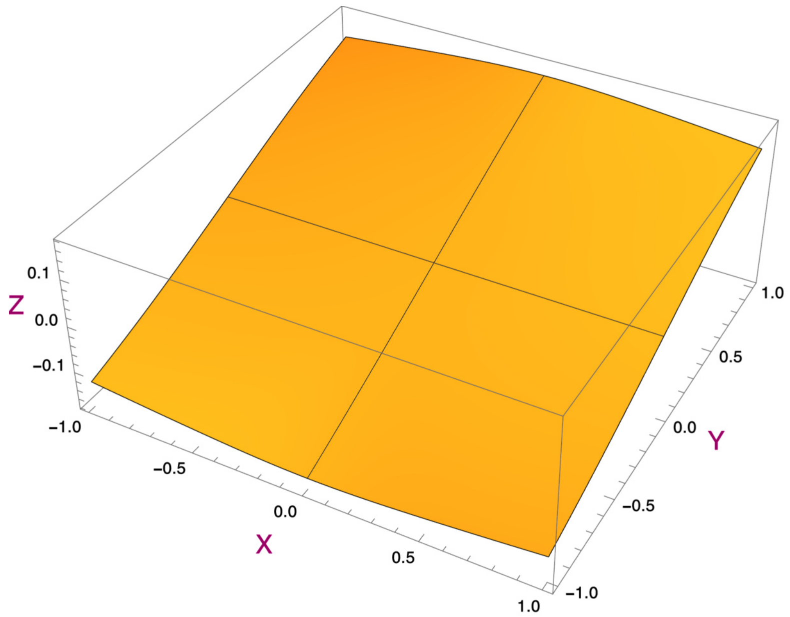

Figure 531. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial x}(x, y)$. This looks like a continuous function.

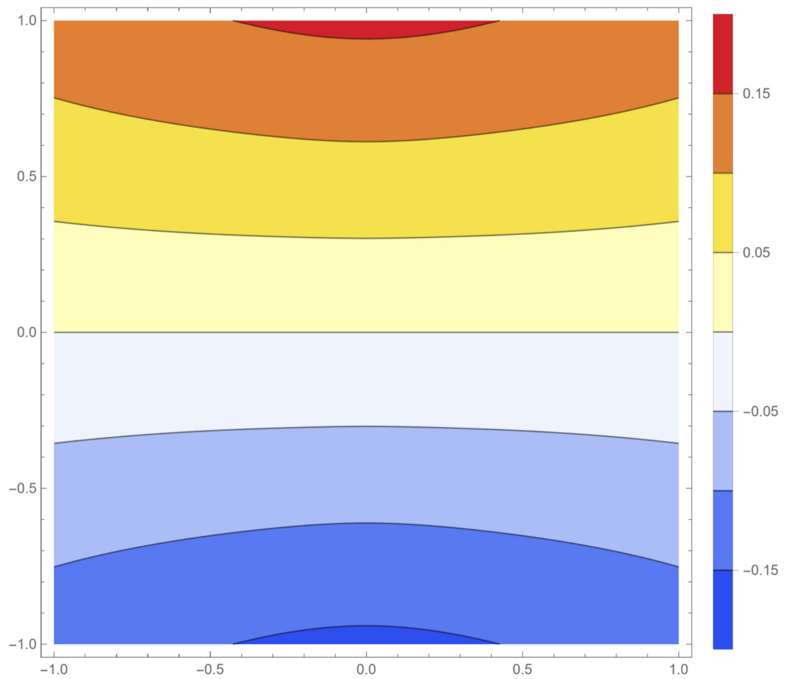

Figure 532. We see here a figure of the contour plot of the $\frac{\partial f}{\partial x}(x, y)$. Only level curves of level around 0 come close to $(0, 0)$.

69.8 Overview

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	yes

69.9 One step further

We want to know if this function is further uneventful from the point of view of differentiability. Let us take a look at the third order partial derivative $\frac{\partial^3 f}{\partial x^3}$.

Let us take a look of a three dimensional plot of this third order partial derivative of the function.

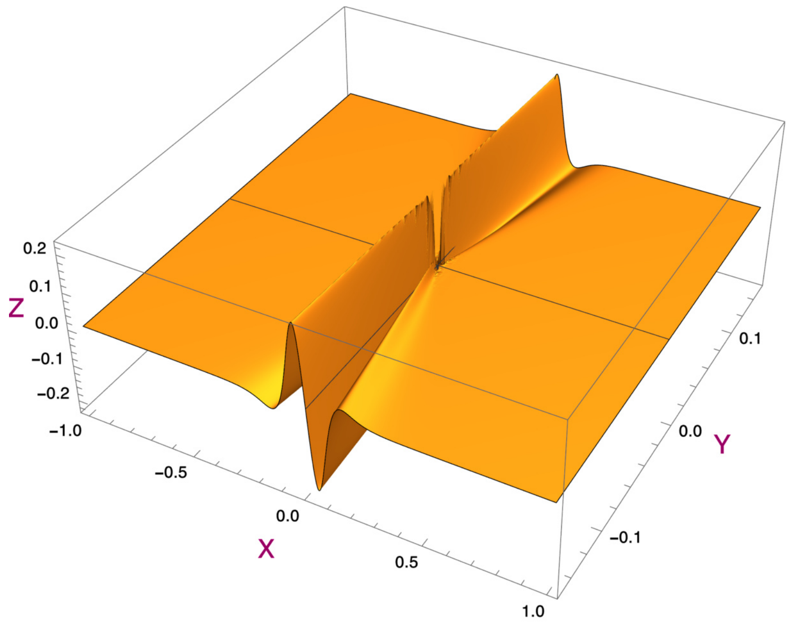

Figure 533. We see here a figure of the third order partial derivative $\frac{\partial^3 f}{\partial x^3}(x, y)$. It seems quite improbable that this third order partial derivative is continuous. We stop however our investigations here and leave this to the initiative of the interested reader. We guess that the calculations will be very unwieldy, so please take care before considering this investigation.

Exercise 70.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{\log\left(|x| + e^{|y|}\right)}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

70.1 Continuity

We restrict the function to the continuous curves with equations $y = \lambda x$. We observe then that

$$f|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = \dfrac{\log\left(e^{|\lambda x|} + |x|\right)}{\sqrt{\lambda^2 x^2 + x^2}} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Let us restrict ourselves to positive values for x and positive values for λ. This will be enough for our purposes.

$$\lim_{x \underset{>}{\to} 0} f(x, \lambda x) = \lim_{x \underset{>}{\to} 0} \frac{\log\left(e^{|\lambda x|} + |x|\right)}{\sqrt{\lambda^2 x^2 + x^2}}$$

$$= \lim_{x \underset{>}{\to} 0} \frac{\log\left(e^{\lambda x} + x\right)}{x\sqrt{\lambda^2 + 1}}$$

$$= \lim_{x \underset{>}{\to} 0} \frac{\lambda e^{\lambda x} + 1}{e^{\lambda x} + x} \frac{1}{\sqrt{\lambda^2 + 1}}$$

$$= \frac{\lambda + 1}{\sqrt{\lambda^2 + 1}}.$$

We see that these restricted functions have different limits. But if $f(x, y)$ is continuous, all these limit values should be $f(0, 0) = 0$. So this function $f(x, y)$ is not continuous.

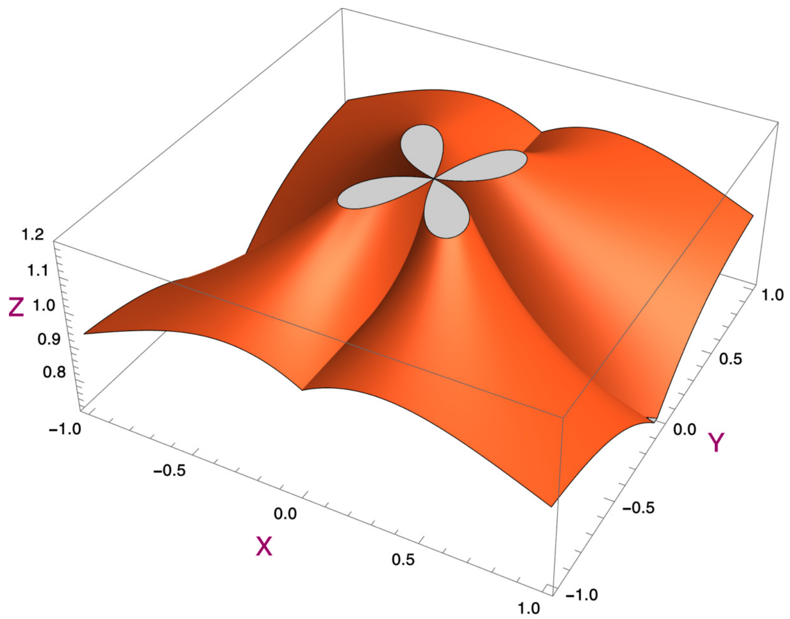

Figure 534. We see here a three dimensional figure of the graph of the function. The vertical line above $(0, 0)$ looks suspicious. This does not seem to be a graph of a continuous function.

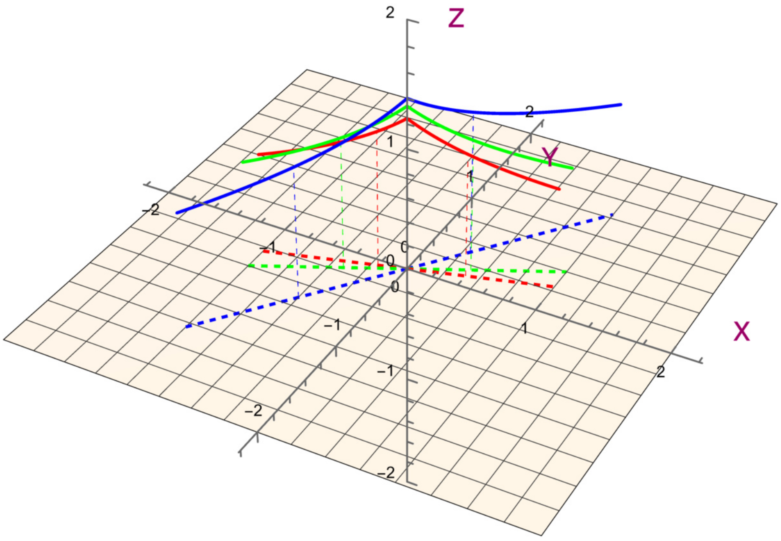

Figure 535. We have restricted the function here to $y = 3/10\,x$ and $y = 1/2\,x$ and $y = 13/10\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have different limits in 0.

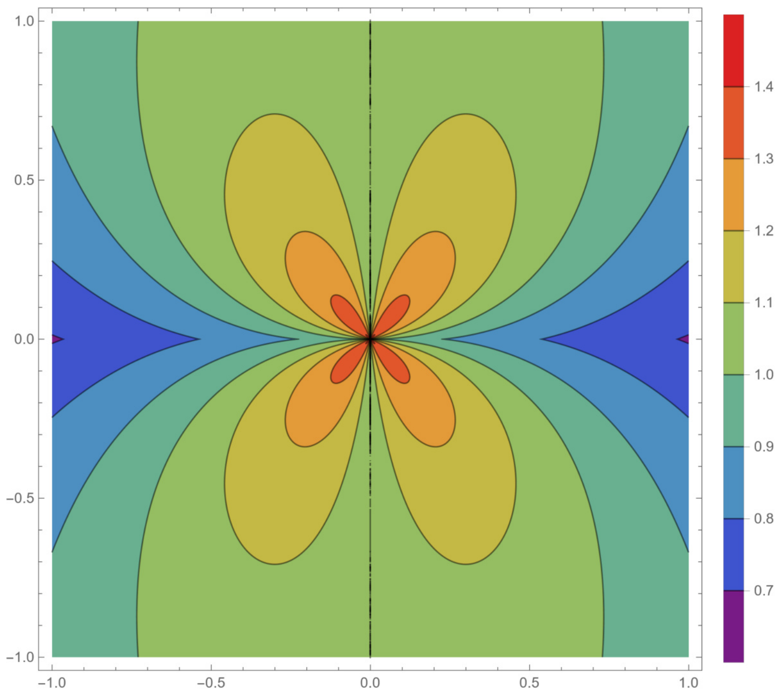

Figure 536. We see here a figure of the contour plot of the function. Many level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

70.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f\big|_{y=0}(x,y) = \begin{cases} f(x,0) = \dfrac{\log(|x|+1)}{\sqrt{x^2}} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Before starting the computation of the partial derivative, we investigate the continuity of the partial functions. Remark that we have

$$\lim_{\substack{h \to 0 \\ <}} \frac{\log(|h|+1)}{\sqrt{h^2}} = \lim_{\substack{h \to 0 \\ <}} \frac{\log(-h+1)}{-h} = \lim_{\substack{h \to 0 \\ <}} \frac{-1}{-(-h+1)} = 1.$$

$$\lim_{\substack{h \to 0 \\ >}} \frac{\log(|h|+1)}{\sqrt{h^2}} = \lim_{\substack{h \to 0 \\ >}} \frac{\log(h+1)}{h} = \lim_{\substack{h \to 0 \\ >}} \frac{1}{\sqrt{(h+1)}} = 1.$$

This function is even for x, so we expect the two calculations being the same.

This function is not continuous in $h = 0$, so the partial derivative to x does not exist.

Discussion of the partial derivative to y in (0,0).

We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = \dfrac{\log\left(e^{|y|}\right)}{\sqrt{y^2}} = 1 & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

This function is not continuous, the second partial derivative does not exist.

We conclude that that the partial derivative to x does not exist and that the partial derivative to y does not exist.

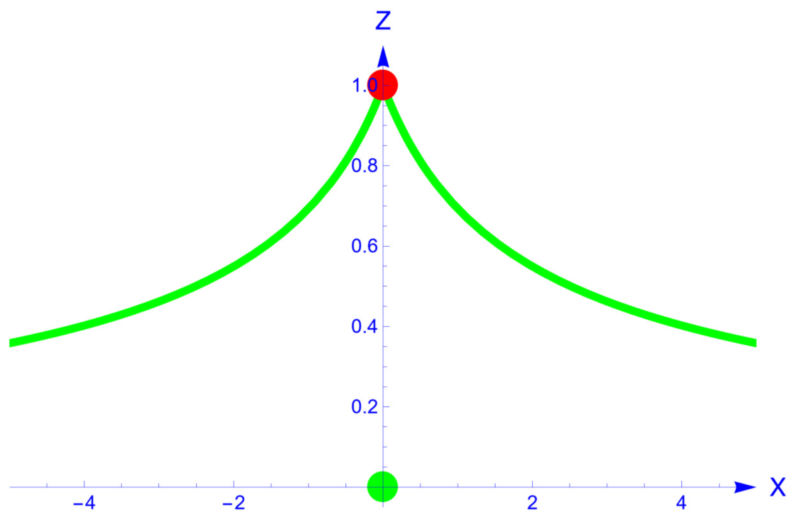

Figure 537. We see here a figure of the graph of the function restricted to the horizontal X-axis through $(0,0)$. This is the graph of $f(x,0)$. This function is not continuous in $x = 0$.

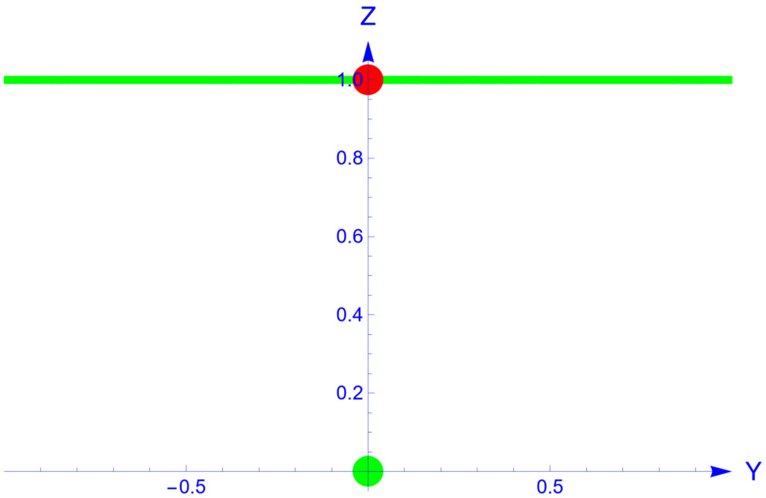

Figure 538. We see here a figure of the graph of the function restricted to the vertical Y-axis through $(0,0)$. This is the function $f(0,y)$.

70.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0)=\lim_{h\to 0}\frac{f(0+h\,u,0+h\,v)-f(0,0)}{h}.$$

Some of these limits do not exist because the function $f(0+h\,u,0+h\,v)$ is not continuous in $h=0$. In order to show that, we calculate the limit for u, v and h positive:

$$\begin{aligned}
\lim_{h \to 0^>} \frac{\log\left(|h\,u| + e^{|h\,v|}\right)}{\sqrt{h^2 u^2 + h^2 v^2}} &= \lim_{h \to 0^>} \frac{\log\left(h\,u + e^{h\,v}\right)}{h\sqrt{u^2 + v^2}} \\
&= \lim_{h \to 0^>} \frac{\log\left(h\,u + e^{h\,v}\right)}{h\sqrt{u^2 + v^2}} \\
&= \lim_{h \to 0^>} \frac{u + e^{h\,v}\,v}{\left(h\,u + e^{h\,v}\right)\sqrt{u^2 + v^2}} \\
&= \frac{u + v}{\sqrt{u^2 + v^2}}.
\end{aligned}$$

So not all directional derivatives exist.

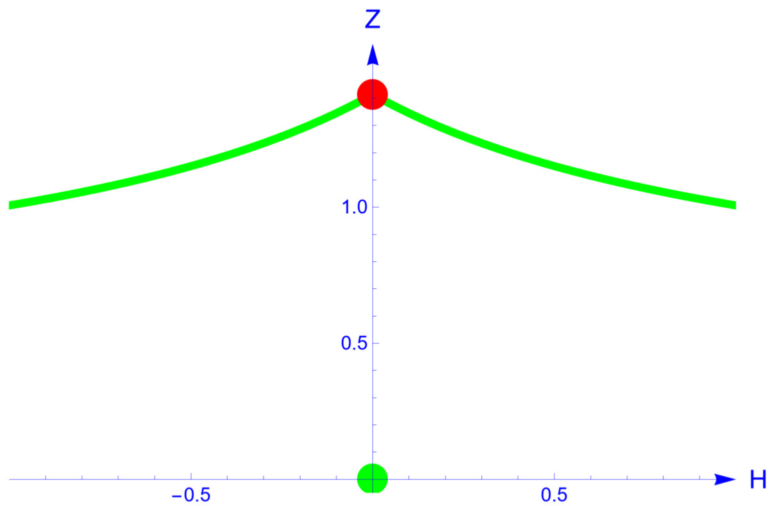

Figure 539. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(h\,u, h\,v)$.

70.4 Alternative proof of continuity (optional)

Irrelevant. The function is not continuous.

70.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

70.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

70.8 Overview

$$f(x,y) = \begin{cases} \dfrac{\log\left(|x| + e^{|y|}\right)}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

continuous	no
partial derivatives exist	no
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 71.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{(x-y)\,y\,e^{-\frac{1}{x^2}}}{(x-y)^2 + 2\,e^{-\frac{2}{x^2}}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

71.1 Continuity

Before starting with the calculations, we want make some remarks. The first observation is that the graphics in this exercise are not to be trusted without using our imagination. The reasons for this is the overflow caused by the exponential in the function definition. We remark that e.g. $x = 10^{-10}$ is a decent machine representable number but that e^{-1/x^2} with $x = 10^{-10}$ is more problematic depending on the hardware of the calculator. We have however chosen to show the graphics in order to understand the trustworthiness and limits of interpreting the calculations by machine graphics.

We multiply the main part of the definition of the function numerator and denominator with $e^{\frac{2}{x^2}}$. So we have that our function

$$f(x,y) = \begin{cases} \dfrac{(x-y)\,y\,e^{-\frac{1}{x^2}}}{(x-y)^2 + 2\,e^{-\frac{2}{x^2}}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

can be written as

$$f(x,y) = \begin{cases} \dfrac{y\left((x-y)\,e^{\frac{1}{x^2}}\right)}{\left(e^{\frac{1}{x^2}}\,(x-y)\right)^2 + 2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

So if $g(z) = \frac{z}{z^2+2}$ and $z = (x - y)\, e^{\frac{1}{x^2}}$, we can write the main part of the function definition as

$$\frac{y\left((x-y)\, e^{\frac{1}{x^2}}\right)}{\left(e^{\frac{1}{x^2}}(x-y)\right)^2 + 2} = y\, g(z).$$

We see a plot of the function $g(z)$ in the following picture.

We see that this function is bounded and uneven. Let us denote the boundedness as $|g(z)| \le M_1$ with M_1 being the two sided bound.

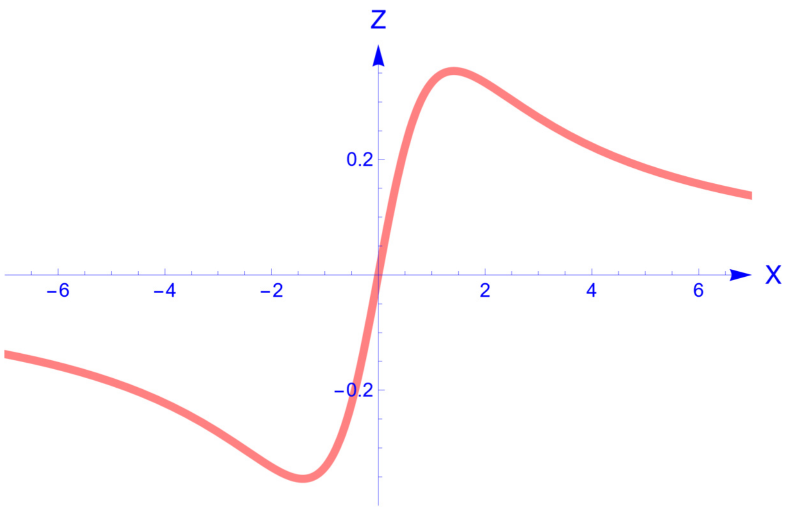

Figure 540. We see here a figure of the graph of the function $g(z)$. This function is everywhere continuous and has horizontal asymptotes. It is clearly a bounded function.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| \frac{(x-y)\, y\, e^{-\frac{1}{x^2}}}{(x-y)^2 + 2\, e^{-\frac{2}{x^2}}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x, y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x, y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{(x-y)\,y\,e^{-\frac{1}{x^2}}}{(x-y)^2+2\,e^{-\frac{2}{x^2}}}\right| &\le g(z)\,y \\ &\le M_1\sqrt{x^2+y^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon/M_1$. We can find a δ, so we conclude that the function is continuous.

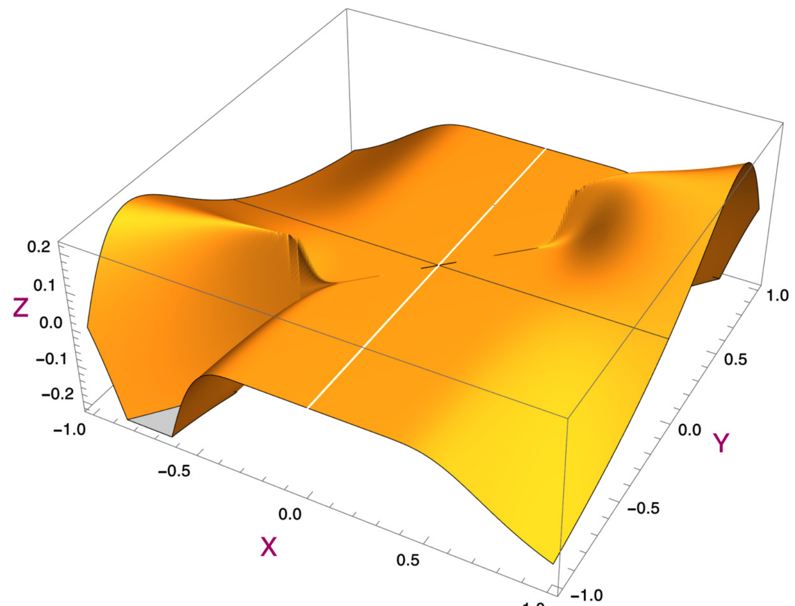

Figure 541. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

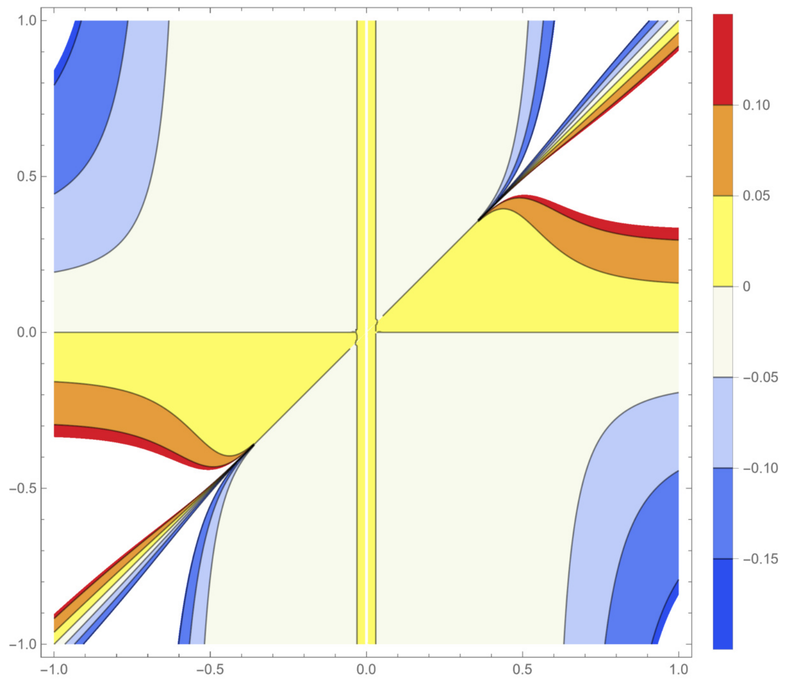

Figure 542. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

71.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).
We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f\,|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases}$$

So

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

So the partial derivative to y does exist.

We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

71.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$
\begin{aligned}
D_{(u,v)}(0,0) &= \lim_{h\to 0}\frac{f(0+hu,0+hv)-f(0,0)}{h}\\
&= \lim_{h\to 0}\frac{g(z(hu,hv))\,hv}{h}\\
&= \lim_{h\to 0} g(z(hu,hv))\,v
\end{aligned}
$$

where $z(x,y)$ is the function that we have introduced at the start of the exercise. We remember that $z(x,y) = (x-y)\,e^{\frac{1}{x^2}}$.

Now we want to know the limit

$$\lim_{h\to 0} g(z(hu,hv)) = \lim_{h\to 0} e^{\frac{1}{(hu)^2}}(hu-hv).$$

We observe by the Taylor expansion of $e^{\frac{1}{\alpha^2}}$ that

$$0 \le \frac{1}{\alpha^2} \le e^{\frac{1}{\alpha^2}}$$

and consequently for $\alpha \ge 0$

$$0 \le \frac{1}{\alpha^2}\,\alpha \le e^{\frac{1}{\alpha^2}}\,\alpha.$$

So

$$\lim_{\substack{\alpha\to 0\\ >}} \alpha\, e^{\frac{1}{\alpha^2}} = \infty.$$

We have for $\alpha < 0$ that this limit equals $-\infty$.

Therefore if $u \neq 0$

$$\lim_{\substack{h\to 0\\ <}} g\left(e^{\frac{1}{hu^2}}\,h\right) = (u-v)\,g(-\infty) = 0$$

and

$$\lim_{\substack{h\to 0\\ >}} g\left(e^{\frac{1}{hu^2}}\,h\right) = (u-v)\,g(\infty) = 0.$$

So the directional derivatives do always exist.

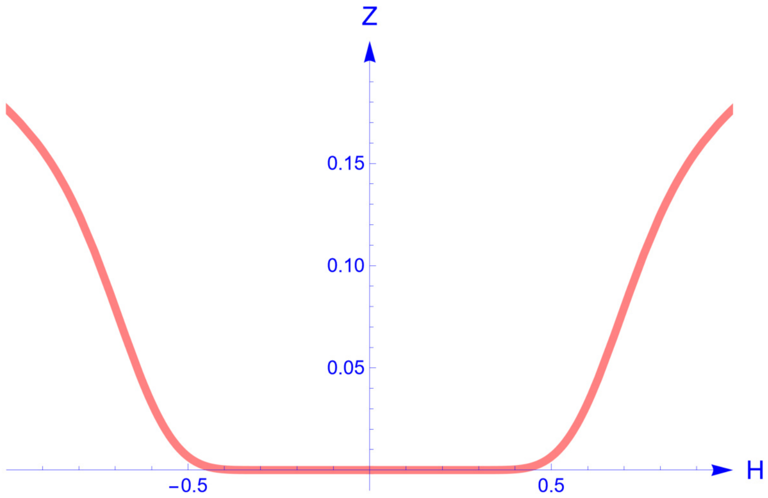

Figure 543. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(\sqrt{3}/2, 1/2\right)$. We see due to the explosive behaviour of the exponential function, we have that the convergence to 0 of $g(z)$ starts very rapidly and goes to zero very fast. We have plotted here the function $f(h\,u, h\,v)$.

71.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\begin{aligned}\frac{\partial f}{\partial x} &= g'(z)\,\frac{\partial z}{\partial x}\,y\\ &= g'(z)\,y\,\left(e^{\frac{1}{x^2}} - \frac{2\,e^{\frac{1}{x^2}}\,(x-y)}{x^3}\right)\\ &= g'(z)\,\frac{e^{\frac{1}{x^2}}\,y\,\left(x^3-2\,x+2\,y\right)}{x^3}\\ &= -\frac{e^{\frac{1}{x^2}}\,y\,\left(e^{\frac{2}{x^2}}\,(x-y)^2-2\right)\,\left(x^3-2\,x+2\,y\right)}{x^3\,\left(e^{\frac{2}{x^2}}\,(x-y)^2+2\right)^2}.\end{aligned}$$

The partial derivative $\frac{\partial f}{\partial x}$ is in a point (α, α) equal to

$$\frac{\partial f}{\partial x}(\alpha, \alpha) = \frac{1}{2}\,\alpha\,e^{\frac{1}{\alpha^2}}$$

which is a function that is unbounded in any neighbourhood of $\alpha = 0$.

Because this partial derivative is unbounded in any neighbourhood of $(0, 0)$, we have no alternative proof following the lines of the criterion for the continuity.

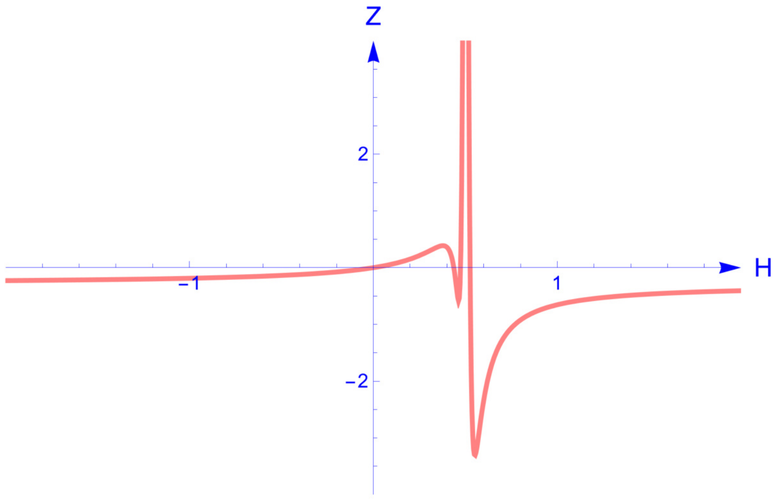

Figure 544. We made a plot of the function $\left|\frac{\partial f}{\partial x}\right|$ restricted to a vertical line through $(\alpha, 0)$. This is the curve $\left|\frac{\partial f}{\partial x}\right|(\alpha, h)$. We have used in this picture $\alpha = 1/2$. If $h = \alpha$ we have a peak above the point $(\alpha, \alpha) = (1/2, 1/2)$ where the derivative has the value $1/2\,\alpha\,e^{1/\alpha^2}$. We can clearly imagine unbounded behaviour when $\alpha \to 0$ caused by these peaks which grow explosively higher and higher when we approach $\alpha = 0$.

71.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

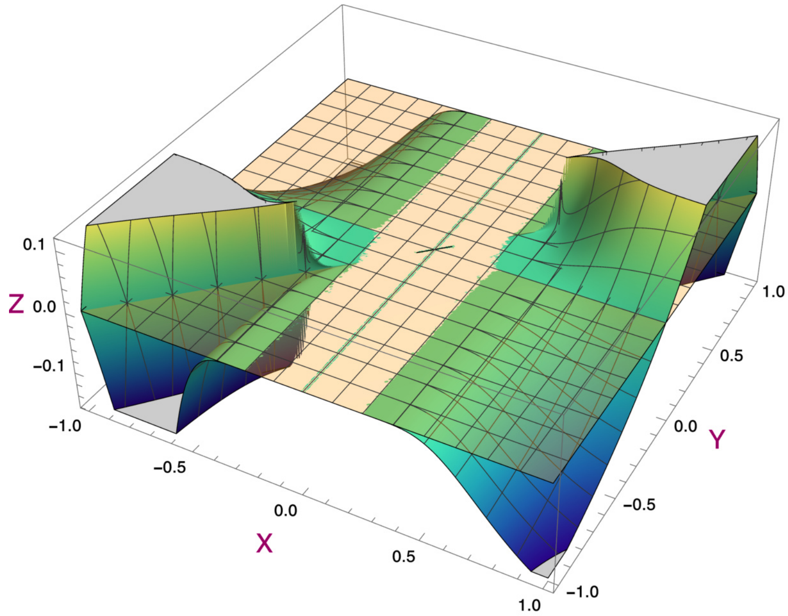

Figure 545. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function very nicely. We must be very careful though. The wild behaviour of the exponential functions could give cause to misleading as we saw in the previous section.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u,v,D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

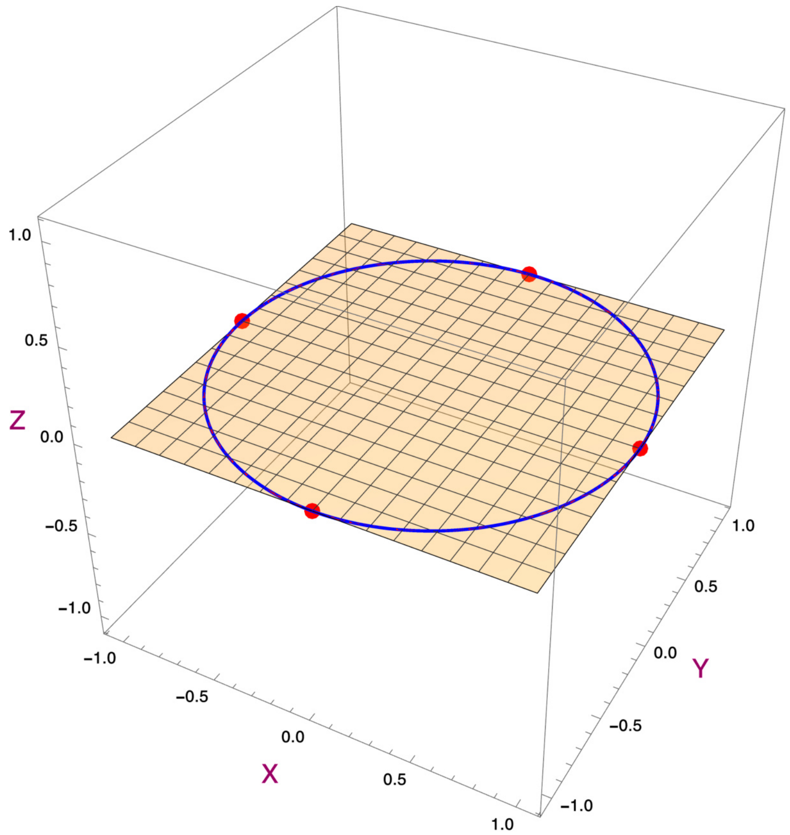

Figure 546. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(u,v)}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h,k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{e^{\frac{1}{h^2}}\,k\,(h-k)}{\left(e^{\frac{2}{h^2}}\,(h-k)^2+2\right)\sqrt{h^2+k^2}} & \text{if } (h,k) \neq (0,0) \text{ and } h \neq 0; \\ 0 & \text{if } (h,k) = (0,0) \text{ or } h = 0 \end{cases}$$

is continuous in $(0,0)$.

Discussion of the continuity of *q(h,k)* in (0,0).

We restrict the function $q(h,k)$ to the continuous curves with equations $k = h - \lambda\,e^{\frac{-1}{h^2}}$. We have chosen these curves because all points of these curves make the value z constant and therefore stand out among all other differentiable curves going through $(0,0)$.

We observe then that

$$q\big|_{k=h-\lambda\,e^{\frac{-1}{h^2}}}(h,k)$$

$$= \begin{cases} q(h, h-\lambda\,e^{\frac{-1}{h^2}}) = \dfrac{e^{-\frac{1}{h^2}}\lambda\left(e^{\frac{1}{h^2}}h-\lambda\right)}{\left(\lambda^2+2\right)\sqrt{\left(-e^{-\frac{1}{h^2}}\lambda+h\right)^2+h^2}} & \text{if } h \neq 0; \\ 0 & \text{if } h = 0. \end{cases}$$

We calculate the left hand limit and the right hand limit.

$$\lim_{\substack{h\to 0\\ <}} \frac{e^{-\frac{1}{h^2}}\lambda\left(e^{\frac{1}{h^2}}h-\lambda\right)}{\left(\lambda^2+2\right)\sqrt{\left(-e^{-\frac{1}{h^2}}\lambda+h\right)^2+h^2}} = -\frac{\lambda}{\sqrt{2}\,(\lambda^2+2)}.$$

$$\lim_{\substack{h\to 0\\ >}} \frac{e^{-\frac{1}{h^2}}\lambda\left(e^{\frac{1}{h^2}}h-\lambda\right)}{\left(\lambda^2+2\right)\sqrt{\left(-e^{-\frac{1}{h^2}}\lambda+h\right)^2+h^2}} = \frac{\lambda}{\sqrt{2}\,(\lambda^2+2)}.$$

We see that these restricted functions have different limits. But if $q(h,k)$ is continuous, all these limit values should be $q(0,0) = 0$. So this function $q(h,k)$ is not continuous in $(0,0)$. The function $f(x,y)$ is not differentiable in $(0,0)$.

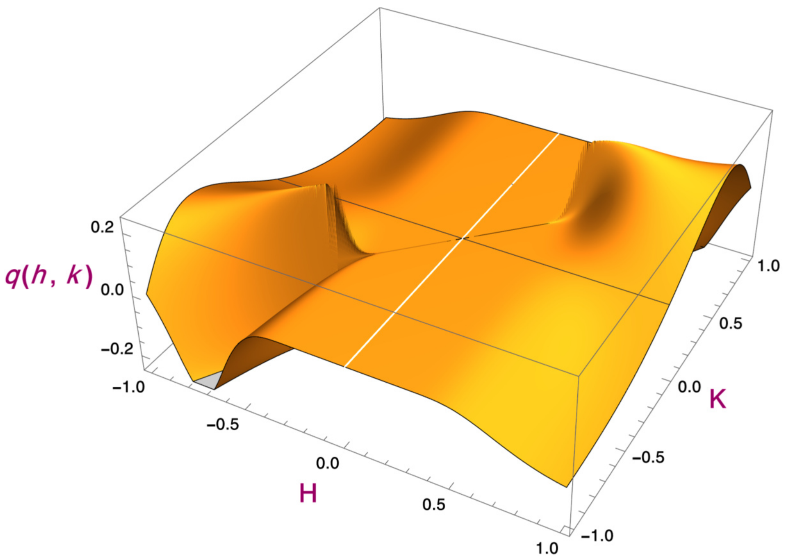

Figure 547. We see here a three dimensional figure of the graph of the function $q(h,k)$. This drawing is not to be trusted. There is a large hole cut out above the line with equation $h = k$.

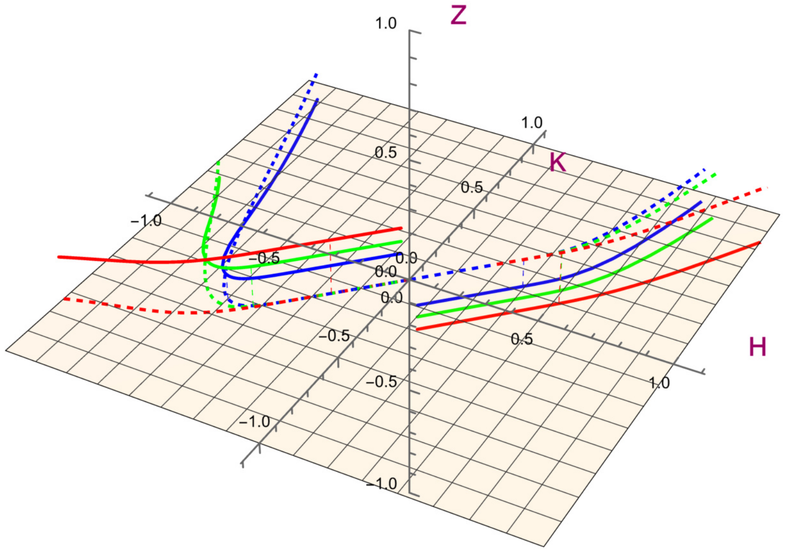

Figure 548. We have restricted the function $q(h,k)$ here to $k = h + 9/10\, e^{\frac{-1}{h^2}}$, $k = h + 36/10\, e^{\frac{-1}{h^2}}$ and $k = h + 56/10\, e^{\frac{-1}{h^2}}$. We should see in this figure that the restrictions of the function to these curves are functions that have different limits in 0. What happens here is that our curves are extremely tangent to the line with equation $h = k$. This tangency is so strong that by underflow it suddenly coincides with $h = k$ and it inherits then suddenly the extreme characteristics of the behaviour of the function q above that line. This has to be avoided. This figure has been made with huge number arithmetic far exceeding the usual machine precision in order to be a good representation.

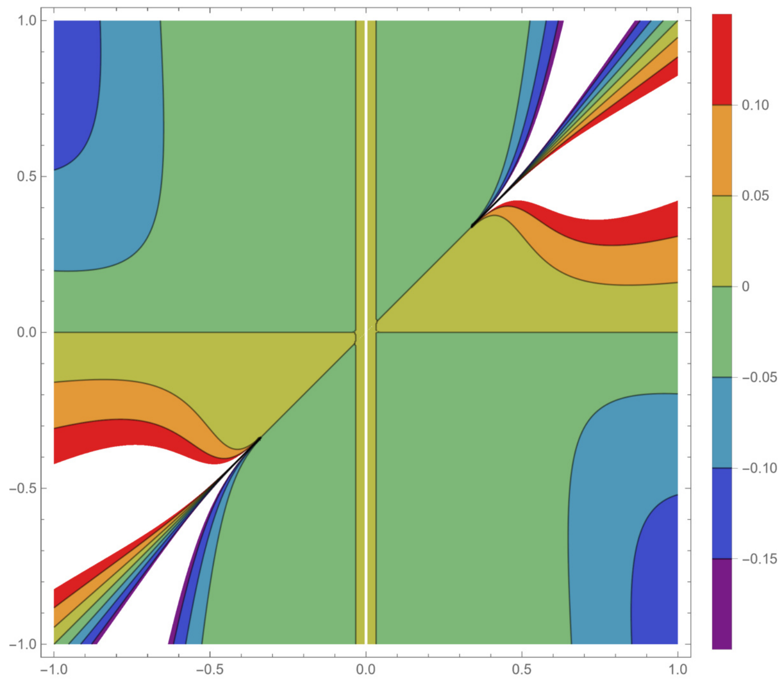

Figure 549. We see here a figure of the contour plot of the function $q(h,k)$. With some wild imagination we can see that many level curves of very different levels approach $(0,0)$ tangent to the line $h = k$ are approaching $(0,0)$ infinitesimally. Then this looks discontinuous indeed.

71.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

71.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

71.8 Overview

$$f(x,y) = \begin{cases} \dfrac{(x-y)\,y\,e^{-\frac{1}{x^2}}}{(x-y)^2 + 2\,e^{-\frac{2}{x^2}}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

Exercise 72.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \sqrt{4 - x^2 - y^2}.$$

72.1 Continuity

We observe that this is a function that is composed of classical differentiable functions in the interior of its domain. This is the standard situation. We do not need to discuss anything further. This function is differentiable, all its partial order derivatives exist and by consequence it behaves in the usual way like any other traditional function. Let us mention that this function describes actually part of a sphere.

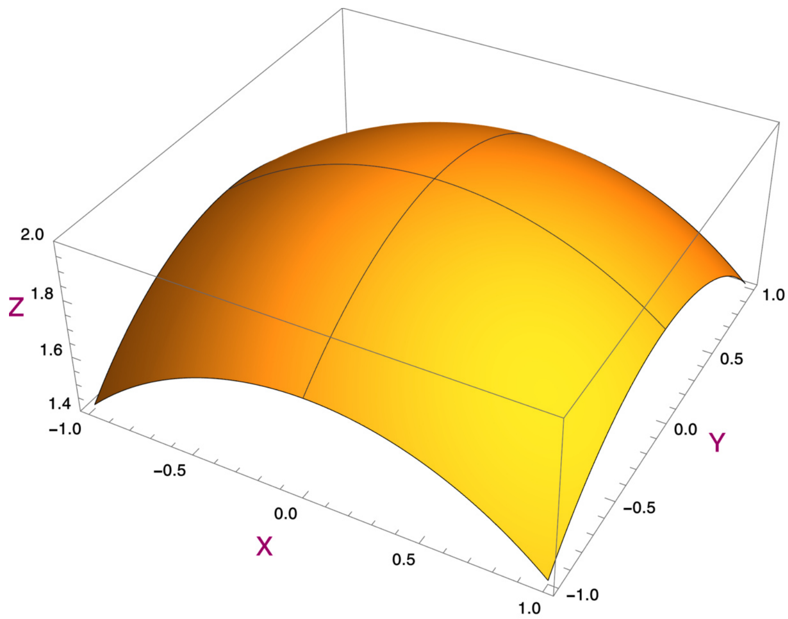

Figure 550. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

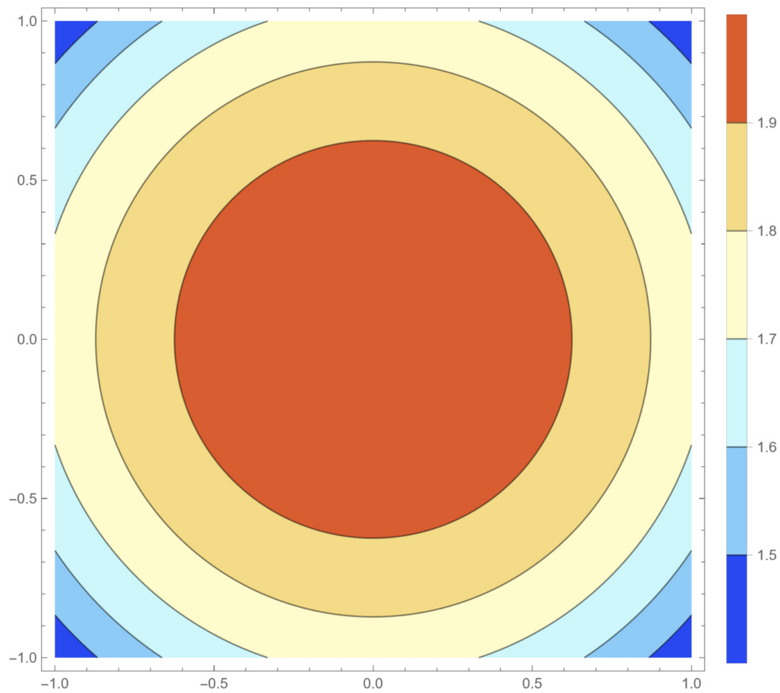

Figure 551. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

72.2 Partial derivatives

This function is differentiable, and by consequence behaves in the usual way like any other traditional function.

72.3 Directional derivatives

This function is differentiable, and by consequence behaves in the usual way like any other traditional function.

72.4 Alternative proof of continuity (optional)

This function is differentiable, and by consequence behaves in the usual way like any other traditional function.

72.5 Differentiability

This function is differentiable, and by consequence behaves in the usual way like any other traditional function.

72.7 Continuity of the partial derivatives

This function is differentiable and all its second order partial derivatives are continuous, and by consequence behaves in the usual way like any other traditional function.

72.8 Overview

$$f(x,y) = \sqrt{4 - x^2 - y^2}.$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	yes

Exercise 73.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} 1 & \text{if } x^2 = y \text{ and } x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

73.1 Continuity

We construct a sequence of points $(1/n, 1/n^2)$, $n \in \mathbf{N_0}$, lying on the parabola in the X-Y plane with equation $y = x^2$. This sequence converges to $(0,0)$. The images of this sequence satisfy $f(1/n, 1/n^2) = 1$. But this sequence converges obviously to 1 and not to 0. If f is continuous, then the images converge to $f(0,0) = 0$ which is obviously not the case. So we conclude that f is not continuous in $(0,0)$.

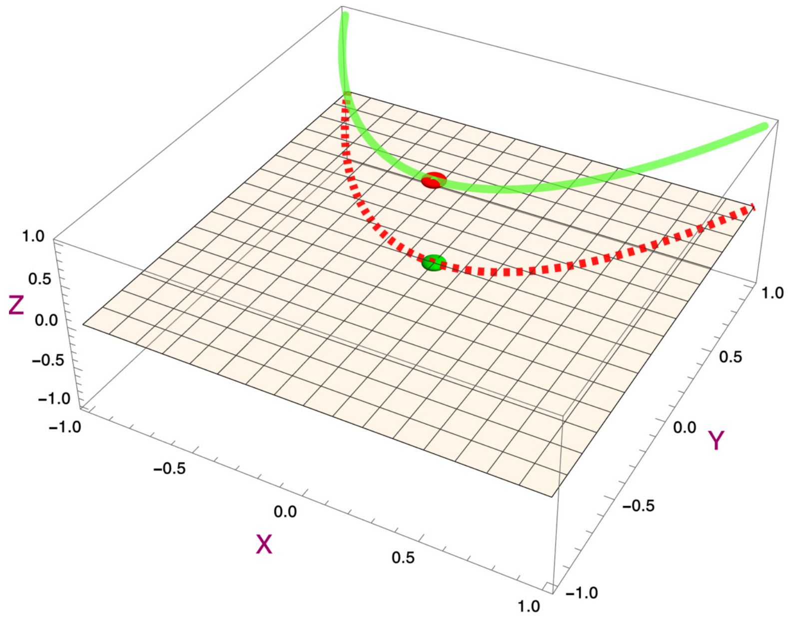

Figure 552. We see here a three dimensional figure of the graph of the function. In order to have a good interpretation of this function: the yellow plane except the points on the dashed red parabola belong to the graph of the function. The green points also belong to the graph of the function.

73.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

73.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

We remark that for every direction (u,v), there is always a neighbourhood of $(0,0)$ such that no points of that neighbourhood has a point of the dashed red parabola in it. So $f(0+hu, 0+hv) = 0$ for all $(0+hu, 0+hv)$ in that neighbourhood. Consult the figure for this observation.

So the directional derivatives do always exist.

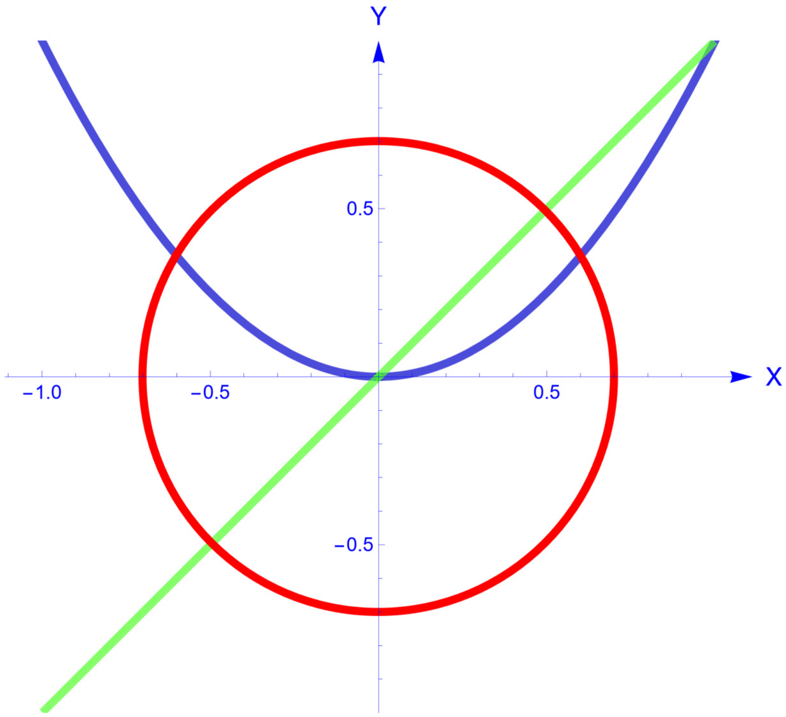

Figure 553. We see here a figure of the domain of the function. The green line is the picture of the points $(0+h\,u, 0+h\,v)$. The parabola is drawn in blue. We see that it is for every direction (u, v) always possible to draw a neighbourhood of $(0, 0)$ drawn in red so that the green line restricted to that neighbourhood has only one intersection point with the parabola and that point is $(0, 0)$.

73.4 Alternative proof of continuity (optional)

Irrelevant. The function is not continuous.

73.5 Differentiability

Irrelevant. The function is not continuous and consequently not differentiable.

73.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not continuous and consequently not differentiable.

73.7 Continuity of the partial derivatives

Irrelevant. The function is not continuous and consequently not differentiable.

73.8 Overview

$$f(x,y) = \begin{cases} 1 & \text{if } x^2 = y \text{ and } x \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

continuous	no
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	no
partials are continuous	irrelevant

Exercise 74.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} (x+1)^2 + (y+1)^2 - 2 & \text{if } x < y \text{ and } y < 2x \text{ and } x > 0, \\ 0 & \text{elsewhere.} \end{cases}$$

74.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| (x+1)^2 + (y+1)^2 - 2 - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$
\begin{aligned}
&\left|(x+1)^2+(y+1)^2-2\right| \\
&\quad \le x^2+2|x|+y^2+2|y| \\
&\quad \le \sqrt{x^2+y^2}^{\,2}+2\sqrt{x^2+y^2}+\sqrt{x^2+y^2}^{\,2}+2\sqrt{x^2+y^2} \\
&\quad \le 4\sqrt{x^2+y^2}+2\sqrt{x^2+y^2}^{\,2} \\
&\quad \le \sqrt{x^2+y^2}\left(4+2\sqrt{x^2+y^2}\right) \\
&\quad \le 6\sqrt{x^2+y^2}.
\end{aligned}
$$

We have restricted to the neighbourhood $\sqrt{x^2+y^2}<1$.

It is sufficient to take $\delta = \min\{1, \epsilon/6\}$. We can find a δ, so we conclude that the function is continuous.

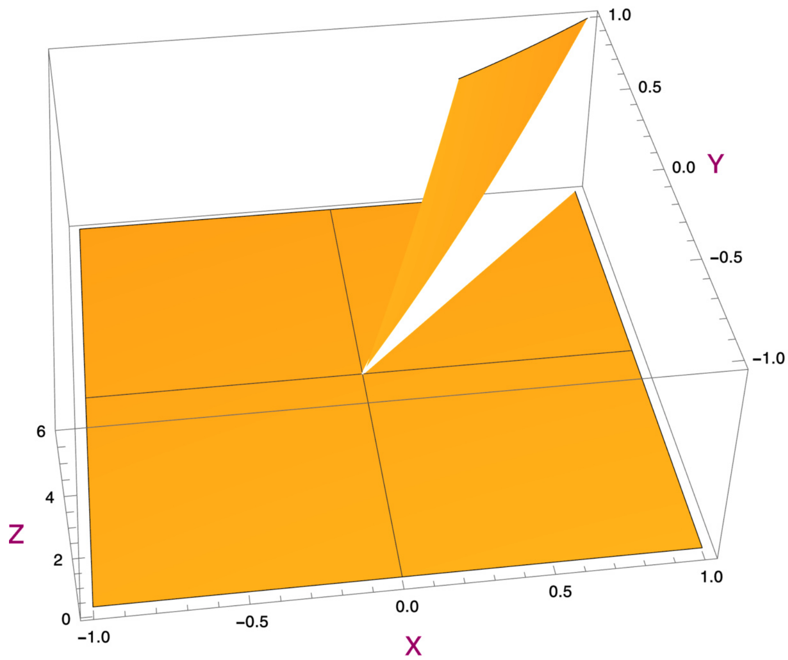

Figure 554. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

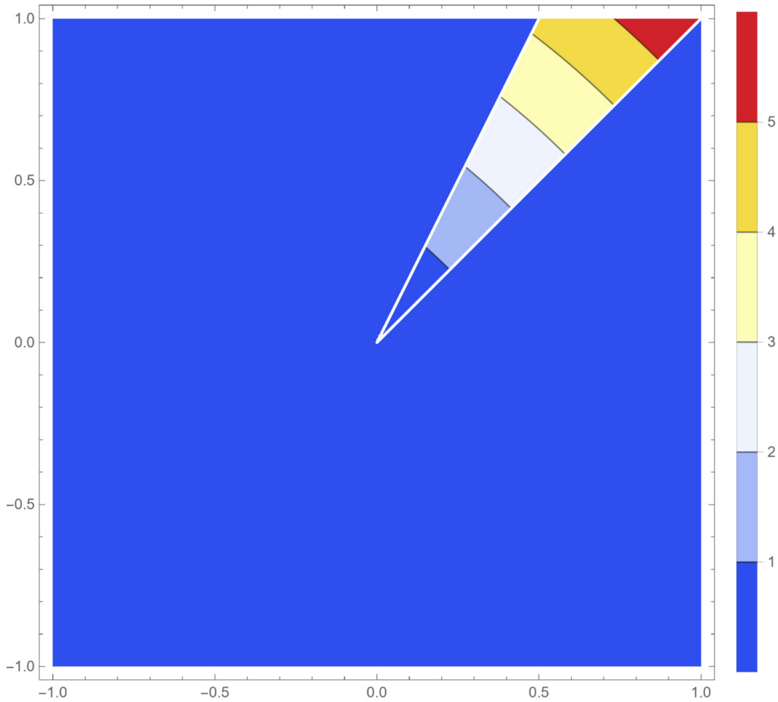

Figure 555. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0, 0)$.

74.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x, y) = \begin{cases} f(x, 0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = 0 & \text{if } y \neq 0;\\ 0 & \text{if } y = 0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

74.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2} = 1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

We will calculate the directional derivatives first for the normal vectors $(u,v) = (\cos(\alpha), \sin(\alpha))$ with $\alpha \in (\pi/4, \arctan(2))$. In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\ &= \lim_{h\to 0} \left(hu^2 + hv^2 + 2u + 2v\right) \\ &= 2u + 2v. \end{aligned}$$

The left limit is

$$\lim_{\substack{h\to 0 \\ <}} 0 = 0.$$

The right limit is

$$\lim_{\substack{h\to 0 \\ >}} \left(hu^2 + hv^2 + 2u + 2v\right) = 2u + 2v.$$

So in this case the directional limit does not exist. We have an analogue situation for the directional limits with direction $(u,v) = (\cos(\alpha), \sin(\alpha))$ where $\alpha \in (\pi + \pi/4, \pi + \arctan(2))$.

In all other directions, the directional limits exist.

So the directional derivatives do not always exist.

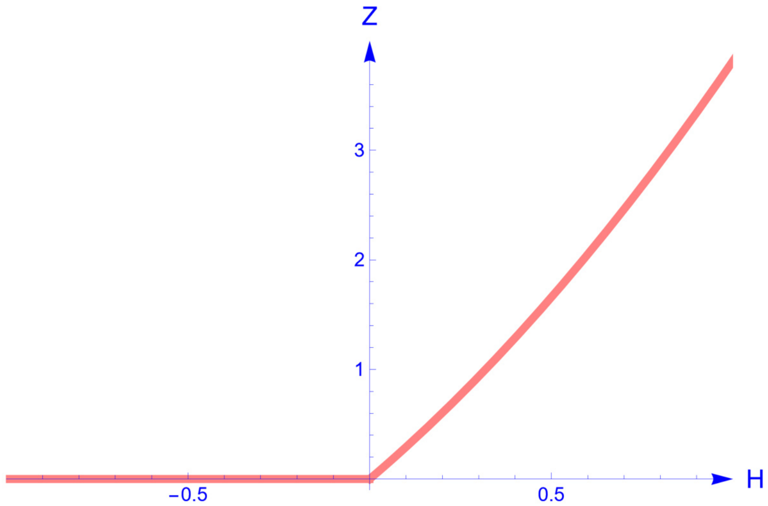

Figure 556. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = (\cos(50\pi/180), \sin(50\pi/180))$. We have plotted here the function $f(hu, hv)$. This curve is not differentiable.

74.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

The question is here whether the partial derivatives exist in a neighbourhood around $(0,0)$.

We take a point (a,a), $a \neq 0$, on the line with equation $y = x$. We draw the function in the direction of the Y-axis. We see that this function is not continuous. So it is certainly not differentiable and the partial derivative to Y does not exist in the point (a,a) where $a \neq 0$.

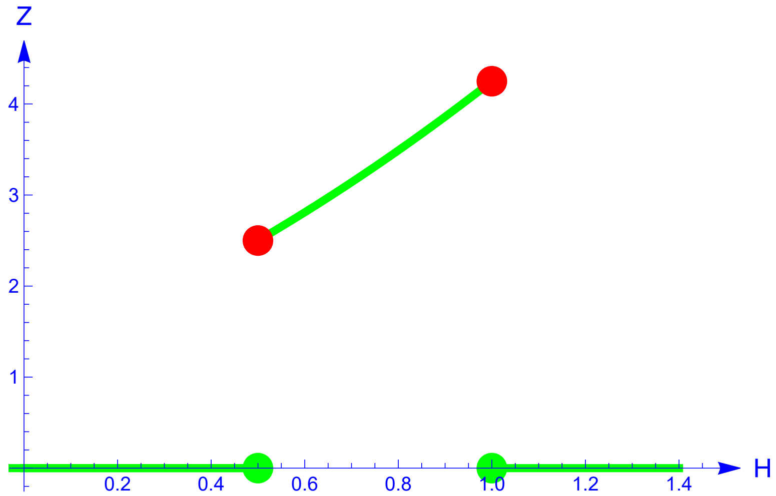

Figure 557. We see here a figure of the graph of the function $f(a,h)$. We have drawn the function here for the value $a = 1/2$ which is exemplary for the values of a close to 0. The function is not continuous. So the function is not differentiable.

Because some of the partial derivatives are not defined in any neighbourhood of $(0,0)$, we do not have an alternative proof for the continuity.

74.5 Differentiability

Because the directional derivatives do not all exist, the function cannot be differentiable.

74.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

74.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

74.8 Overview

$$f(x,y) = \begin{cases} (x+1)^2 + (y+1)^2 - 2 & \text{if } x < y \text{ and } y < 2x \text{ and } x > 0; \\ 0 & \text{elsewhere.} \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 75.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \lfloor x+y \rfloor.$$

75.1 Continuity

In order to avoid misunderstandings, please remark that these brackets are the brackets of the floor function.

We remark first that this floor function equals in a small neighbourhood of $(0,0)$, say $\sqrt{x^2+y^2} < 1/10$

$$f(x,y) = \begin{cases} -1 & \text{if } x+y < 0 \text{ and } \sqrt{x^2+y^2} < 1/10; \\ 0 & \text{if } x+y \geq 0 \text{ and } \sqrt{x^2+y^2} < 1/10. \end{cases}$$

It is technically sometimes easier to work with this form of the function definition in order to make some calculations.

We restrict the function to the continuous curves with equations $y = \lambda x$ with $\lambda > 0$. We observe then that

$$f\big|_{y=\lambda x}(x,y) = \begin{cases} f(x,\lambda x) = 0 & \text{if } x \geq 0 \text{ and } \sqrt{x^2+y^2} < 1/10; \\ -1 & \text{if } x < 0 \text{ and } \sqrt{x^2+y^2} < 1/10. \end{cases}$$

We see that these restricted functions have different limits. But if $f(x,y)$ is continuous, all these limit values should be $f(0,0) = 0$. So this function $f(x,y)$ is not continuous.

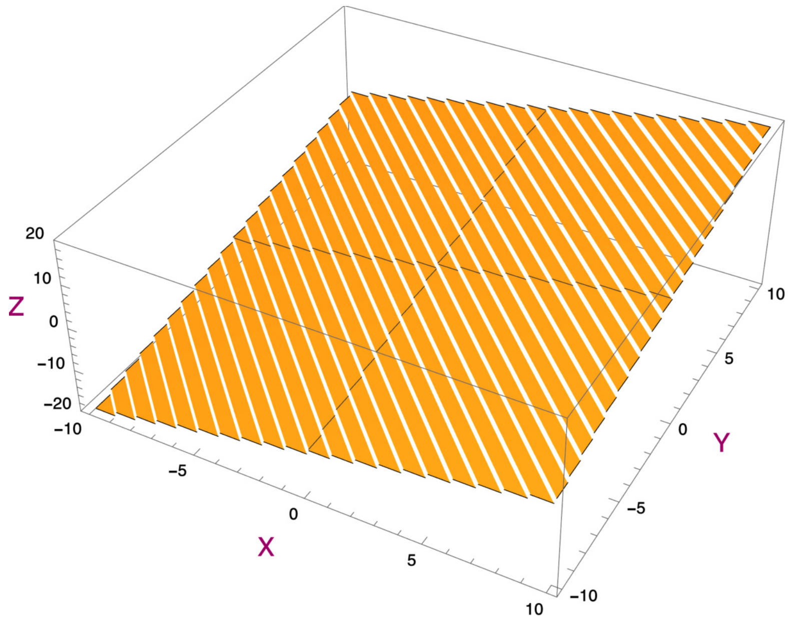

Figure 558. We see here a three dimensional figure of the graph of the function. This is a more global view upon the function. The discontinuities are in a manner of speaking almost build in. This does not seem to be a graph of a continuous function.

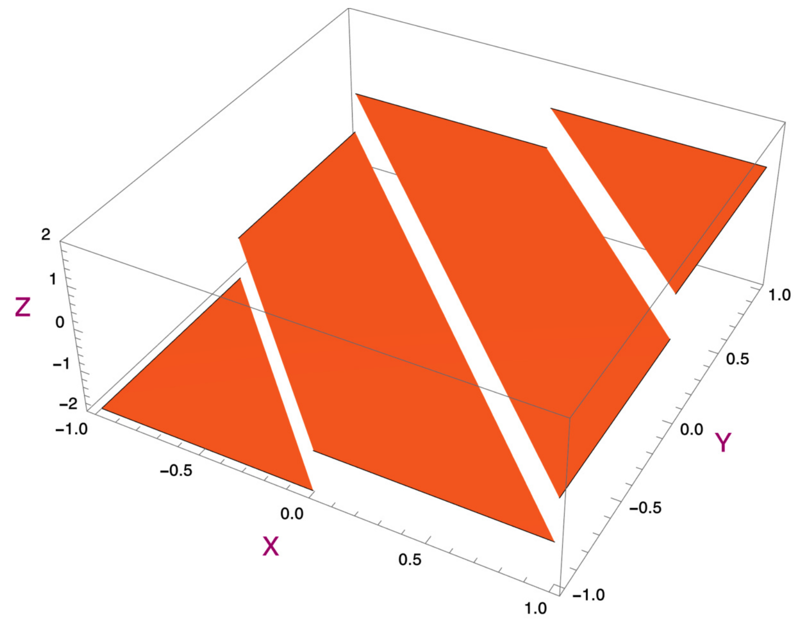

Figure 559. We see here a three dimensional figure of the graph of the function. This is a more local view upon the function. The discontinuities are in a manner of speaking almost build in. This does not seem to be a graph of a continuous function.

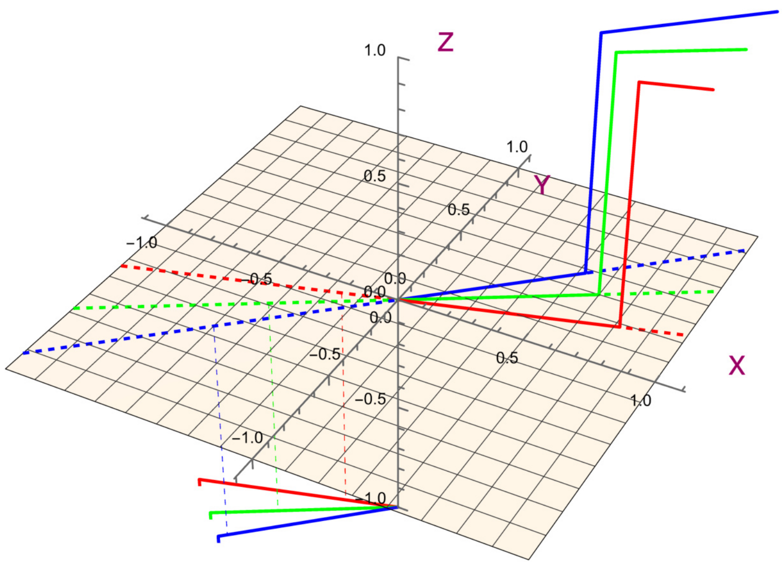

Figure 560. We have restricted the function here to $y = 3/10\,x$ and $y = 6/10\,x$ and $y = 9/10\,x$. We see in this figure clearly that the restrictions of the function to these lines are functions that have no limits in 0.

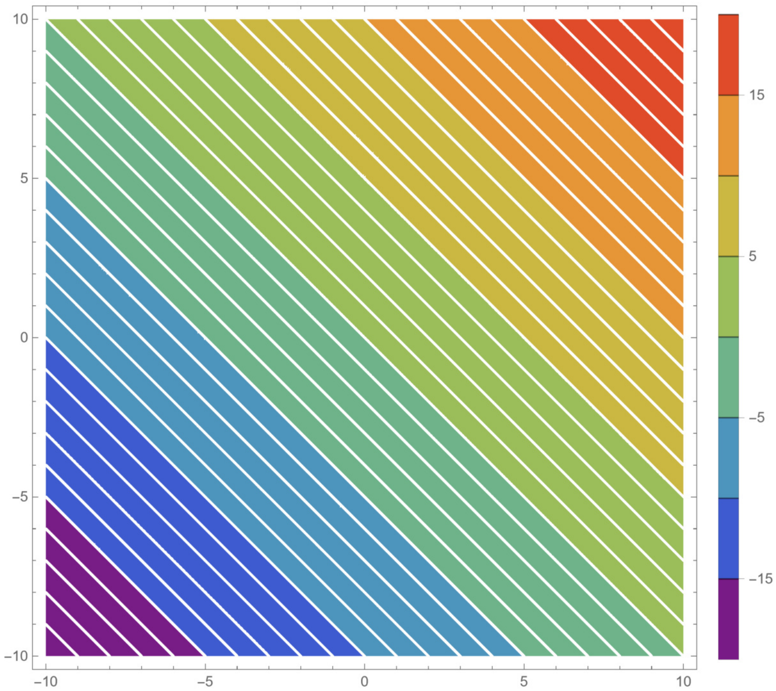

Figure 561. We see here a figure of the contour plot of the function. Level curves of very different levels approach $(0,0)$. This looks discontinuous indeed.

75.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = -1 & \text{if } x < 0 \text{ and } \sqrt{x^2+y^2} < 1/10; \\ 0 & \text{if } x \geq 0 \text{ and } \sqrt{x^2+y^2} < 1/10. \end{cases}$$

We conclude that this function is discontinuous and that the partial derivative does not exist.

Discussion of the partial derivative to y in (0,0).

We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y) = -1 & \text{if } y < 0 \text{ and } \sqrt{x^2+y^2} < 1/10; \\ 0 & \text{if } y \geq 0 \text{ and } \sqrt{x^2+y^2} < 1/10. \end{cases}$$

We conclude that this function is discontinuous and that the partial derivative does not exist.

We conclude that the partial derivative to x does not exist and that the partial derivative to y does not exist.

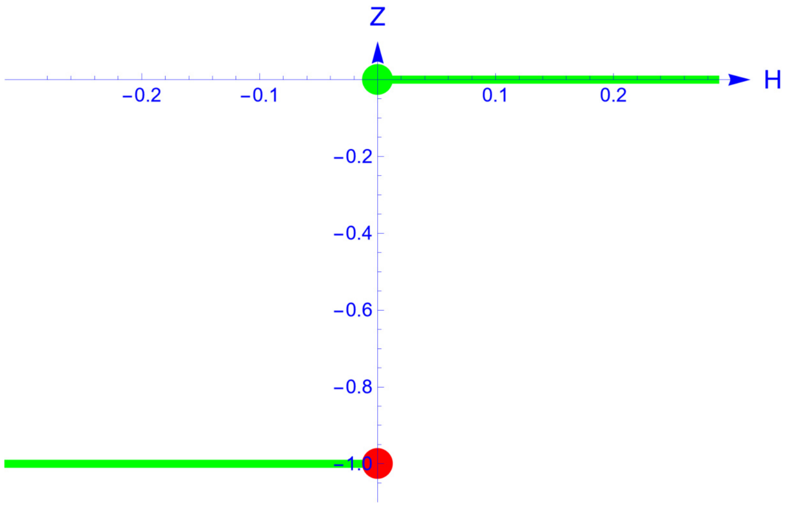

Figure 562. We see here a figure of the graph of the function $f(h,0)$ restricted to the horizontal X-axis through $(0,0)$. This function is not continuous and thus not differentiable.

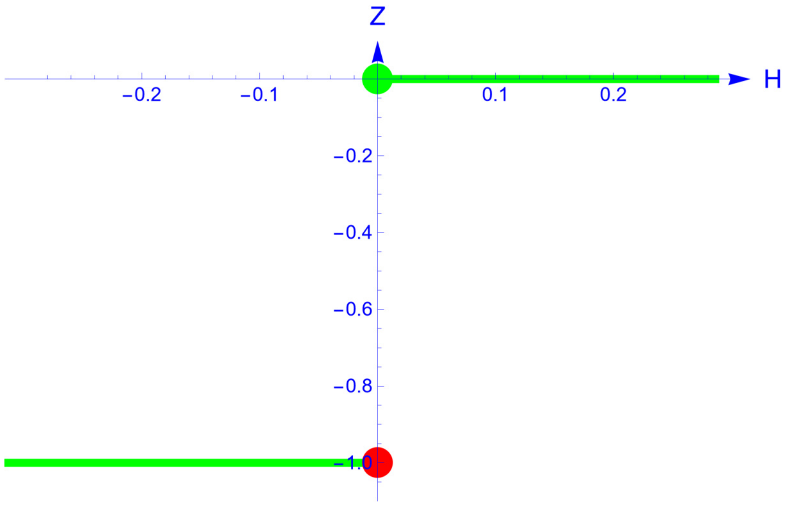

Figure 563. We see here a figure of the graph of the function restricted to the vertical Y-axis through $(0,0)$.

75.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0)=\lim_{h\to 0}\frac{f(0+h\,u,0+h\,v)-f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

We remark that the functions $f(0+h\,u,0+h\,v)$ are not continuous if $(u,v)\neq\left(-1/\sqrt{2},1/\sqrt{2}\right)$ and $(u,v)\neq\left(1/\sqrt{2},-1/\sqrt{2}\right)$ and are identically zero if $(u,v)=\left(-1/\sqrt{2},1/\sqrt{2}\right)$ or $(u,v)=\left(1/\sqrt{2},-1/\sqrt{2}\right)$. In these last cases, the functions are differentiable and the directional derivative exists.

So the directional derivatives do not always exist.

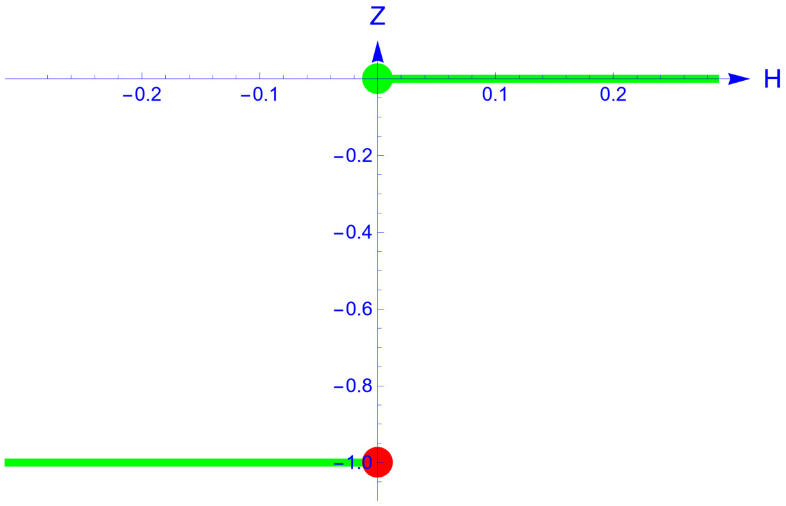

Figure 564. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$.

75.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

But we calculated that the function has no partial derivatives. So this criterion cannot be applied.

75.5 Differentiability

We have that the function is not continuous. So it is not differentiable.

75.6 Alternative proof of differentiability (optional)

Irrelevant. The function is not differentiable.

75.7 Continuity of the partial derivatives

Irrelevant. The function is not differentiable.

75.8 Overview

$$f(x, y) = \lfloor x + y \rfloor.$$

continuous	no
partial derivatives exist	no
all directional derivatives exist	no
differentiable	no
partials are continuous	irrelevant

Exercise 76.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{\cos(xy)-1}{x^2y^2} & \text{if } xy \neq 0, \\ \dfrac{-1}{2} & \text{if } xy = 0. \end{cases}$$

76.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y)-f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y)-(0,0)|| < \delta$ it follows that $|f(x,y)-f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y)-(0,0)|| = \sqrt{x^2+y^2} < \delta$, we have that

$$\left| \frac{\cos(xy)-1}{x^2y^2} + \frac{1}{2} \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

We remark also that by the Taylor expansion with remainder we have

$$\cos(\alpha) = 1 - \frac{\alpha^2}{2!} + \sin(\xi)\,\frac{\alpha^3}{3!}$$

where $0 \leq |\xi| \leq |\alpha|$.

$$
\begin{aligned}
\left|\frac{\cos(x\,y)-1}{x^2\,y^2}+\frac{1}{2}\right| &\le \left|\frac{1-\frac{x^2\,y^2}{2!}+\sin(\xi)\,\frac{x^3\,y^3}{3!}-1}{x^2\,y^2}+\frac{1}{2}\right| \\
&\le \left|\frac{-\frac{x^2\,y^2}{2}+\sin(\xi)\,\frac{x^3\,y^3}{3!}}{x^2\,y^2}+\frac{1}{2}\right| \\
&\le \left|-\frac{1}{2}+\sin(\xi)\,\frac{x\,y}{3!}+\frac{1}{2}\right| \\
&\le |\sin(\xi)|\,\frac{|x|\,|y|}{3!} \\
&\le \frac{|x|\,|y|}{3!} \\
&\le \frac{\sqrt{x^2+y^2}\,\sqrt{x^2+y^2}}{3!} \\
&\le \frac{\sqrt{x^2+y^2}^{\,2}}{3!}.
\end{aligned}
$$

It is sufficient to take $\delta = (3!\,\epsilon)^{1/2}$. We can find a δ, so we conclude that the function is continuous.

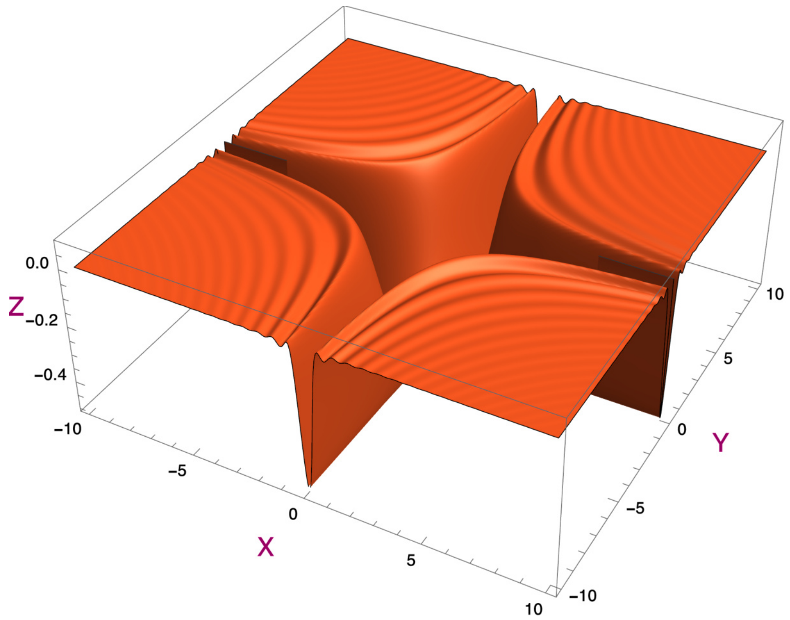

Figure 565. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

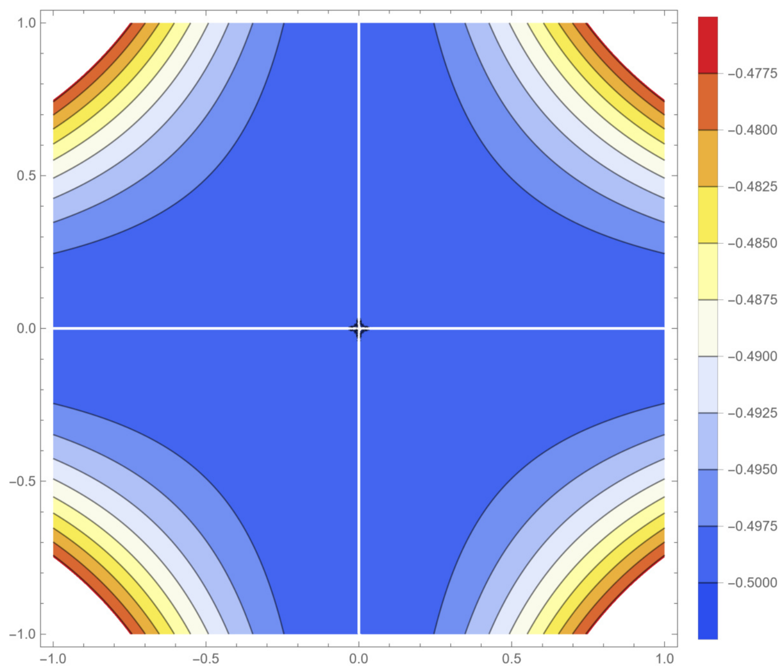

Figure 566. We see here a figure of the contour plot of the function. Only level curves of level around $-1/2$ come close to $(0,0)$.

76.2 Partial derivatives

Discussion of the partial derivative to x in (0,0).
We observe then that

$$f|_{y=0}(x,y) = f(x,0) = \frac{-1}{2}.$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative to y in (0,0).
We observe that

$$f|_{x=0}(x,y) = f(0,y) = \frac{-1}{2}.$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0) = 0.$$

76.3 Directional derivatives

Let (u, v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

In the case of our function we have to calculate the following limit

$$\begin{aligned}
D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{\cos(hu\,hv)-1}{(hu)^2\,(hv)^2} + \frac{1}{2}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{\cos(hu\,hv)-1}{(hu)^2\,(hv)^2} + \frac{1}{2}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1 - \frac{(hu)^2\,(hv)^2}{2!} + \sin(\xi)\,\frac{(hu)^3\,(hv)^3}{3!} - 1}{(hu)^2\,(hv)^2} + \frac{1}{2}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{-\frac{(hu)^2\,(hv)^2}{2} + \sin(\xi)\,\frac{(hu)^3\,(hv)^3}{3!}}{(hu)^2\,(hv)^2} + \frac{1}{2}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(-\frac{1}{2} + \sin(\xi)\,\frac{hu\,hv}{3!} + \frac{1}{2}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\,|\sin(\xi|)\,\frac{|hu|\,|hv|}{3!} \\
&= 0.
\end{aligned}$$

We have made use of the Taylor expansion with remainder.
So the directional derivatives do always exist.

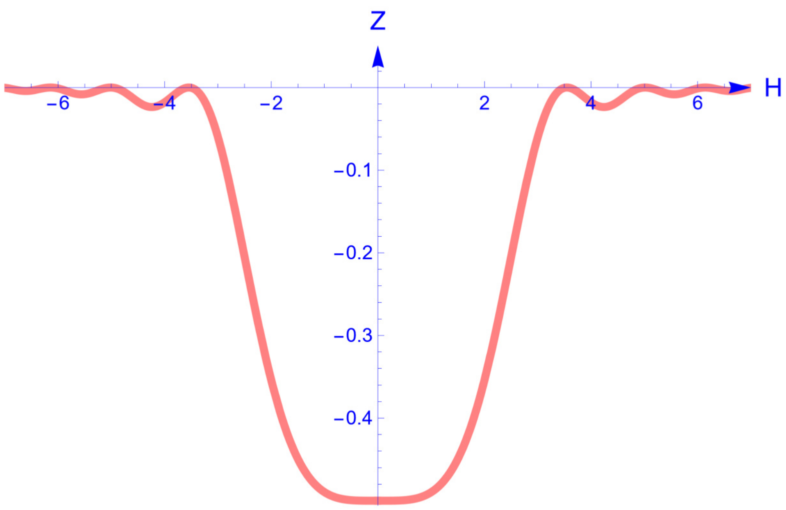

Figure 567. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$. We have drawn the graph of the function $f(hu, hv)$. This is certainly differentiable.

76.4 Alternative proof of continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

Let us first investigate if the function is partial differentiable in a neighbourhood of $(0,0)$. Let us investigate the partial derivatives in a point $(a,0)$ with $a \neq 0$. We are going to calculate the derivative in the Y-direction. We have $f(a,0) = -1/2$ and the derivative to the Y-direction is then

$$\begin{aligned}\frac{\partial f}{\partial y}(a,0) &= \lim_{h\to 0}\frac{f(a,h)-f(a,0)}{h}\\ &= \lim_{h\to 0}\frac{\frac{\cos(a\,h)-1}{a^2\,h^2}+\frac{1}{2}}{h}\\ &= \lim_{h\to 0}\frac{a^2\,h^2+2\cos(a\,h)-2}{2\,a^2\,h^3}\\ &= 0.\end{aligned}$$

We can use de l'Hospital for the last step in the calculation.

The function is in the X-direction constant, so this partial derivative to x also exists.

Because of the symmetry in the function definition, the reasoning is similar for points $(0,a)$.

Remember that we already calculated the partial derivatives in $(0,0)$. The partial derivative to x is:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases}\dfrac{-x\,y\,\sin(x\,y)-2\,\cos(x\,y)+2}{x^3\,y^2} & \text{if } x\,y\neq 0,\\ 0 & \text{if } x=0 \text{ or } y=0.\end{cases}$$

All calculations are similar for y because of the symmetry in the function definition.

We can consult figures of the absolute values of the first partial derivative at the end of this section.

We know by the theorem of Taylor with remainder that

$$\cos(\alpha) = 1-\frac{\alpha^2}{2}+\sin(\xi)\,\frac{\alpha^3}{6}$$

with $0\le|\xi|\le|\alpha|$
and

$$\sin(\alpha) = \alpha - \frac{\alpha^3}{6} + \sin(\xi)\,\frac{\alpha^4}{24}$$

with $0 \le |\xi| \le |\alpha|$.

So we have the following equality

$$\begin{aligned}
&\frac{-x\,y\,\sin(x\,y) - 2\,\cos(x\,y) + 2}{x^3\,y^2} \\
&= \frac{2 - 2\left(1 - \frac{(x\,y)^2}{2} + \frac{1}{6}\,\sin(\xi_1)\,(x\,y)^3\right) - x\,y\left((x\,y) - \frac{(x\,y)^3}{6} + \frac{1}{24}\sin(\xi_2)(x\,y)^4\right)}{x^3\,y^2} \\
&= \frac{1}{24}\left(-x^2 y^3 \sin(\xi_2) + 4xy^2 - 8y\sin(\xi_1)\right).
\end{aligned}$$

Let us try to prove that $\left|\frac{\partial f}{\partial x}\right|$ is bounded.

$$\begin{aligned}
&\left|\frac{1}{24}\left(-x^2\,y^3\,\sin(\xi_2) + 4\,x\,y^2 - 8\,y\,\sin(\xi_1)\right)\right| \\
&\le \frac{1}{24}\left(x^2\,|y|^3\,|\sin(\xi_2)| + 4\,|x|\,y^2 + 8\,|y|\,|\sin(\xi_1)|\right) \\
&\le \frac{1}{24}\left(\sqrt{x^2+y^2}^{\,2}\,\sqrt{x^2+y^2}^{\,3} + 4\sqrt{x^2+y^2}\sqrt{x^2+y^2}^{\,2} + 8\sqrt{x^2+y^2}\right) \\
&\le \frac{1}{24}\left(\sqrt{x^2+y^2}^{\,5} + 4\sqrt{x^2+y^2}^{\,3} + 8\sqrt{x^2+y^2}\right) \\
&\le \frac{13}{24}.
\end{aligned}$$

We have chosen here the restriction to the neighbourhood defined by $\sqrt{x^2+y^2} < 1$ for the last step.

We can reason similarly for the derivative to y because of the symmetry in the function definition.

Because the two partial derivatives are bounded in a neighbourhood of $(0, 0)$, we have an alternative proof for the continuity.

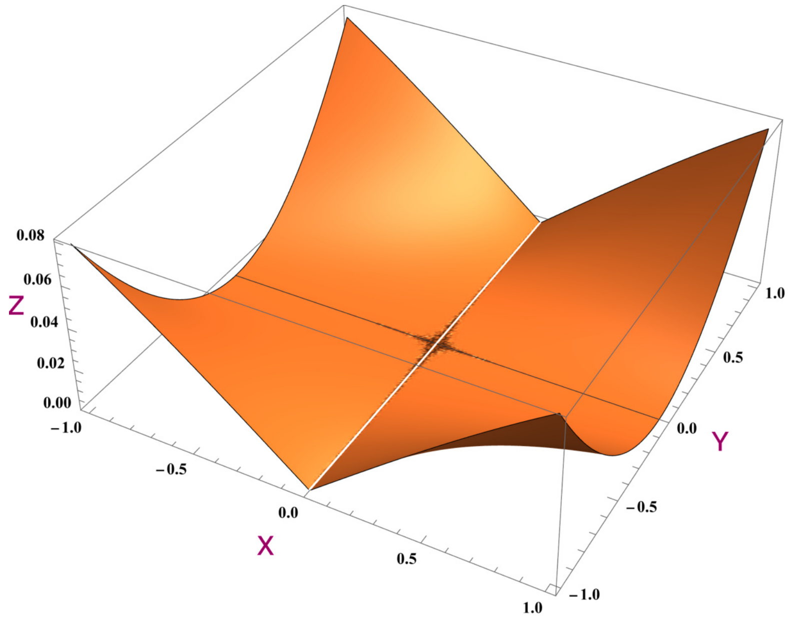

Figure 568. We see here the absolute value of the first partial derivative $\left|\frac{\partial f}{\partial x}\right|$. We can observe the boundedness from this picture.

76.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks based on the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue our reasoning as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

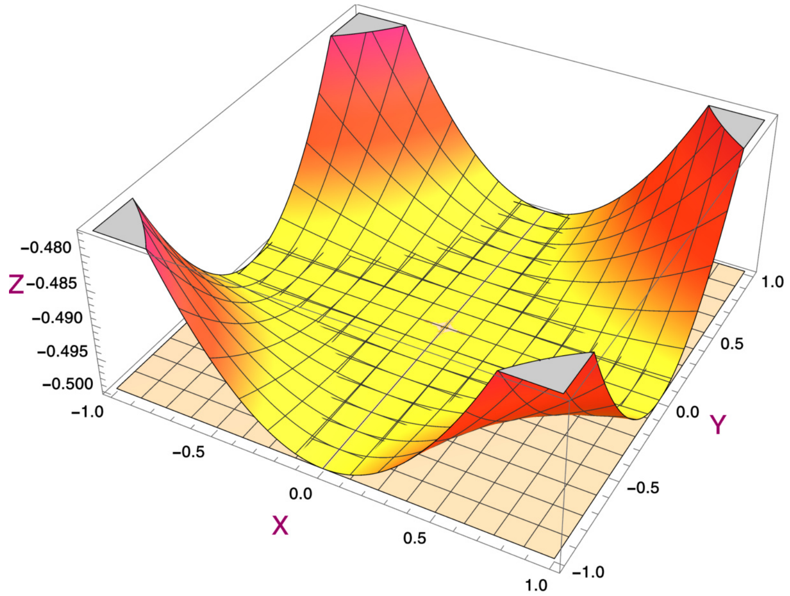

Figure 569. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function very nicely.

We perform now our second visual check. We can take a look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the following vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

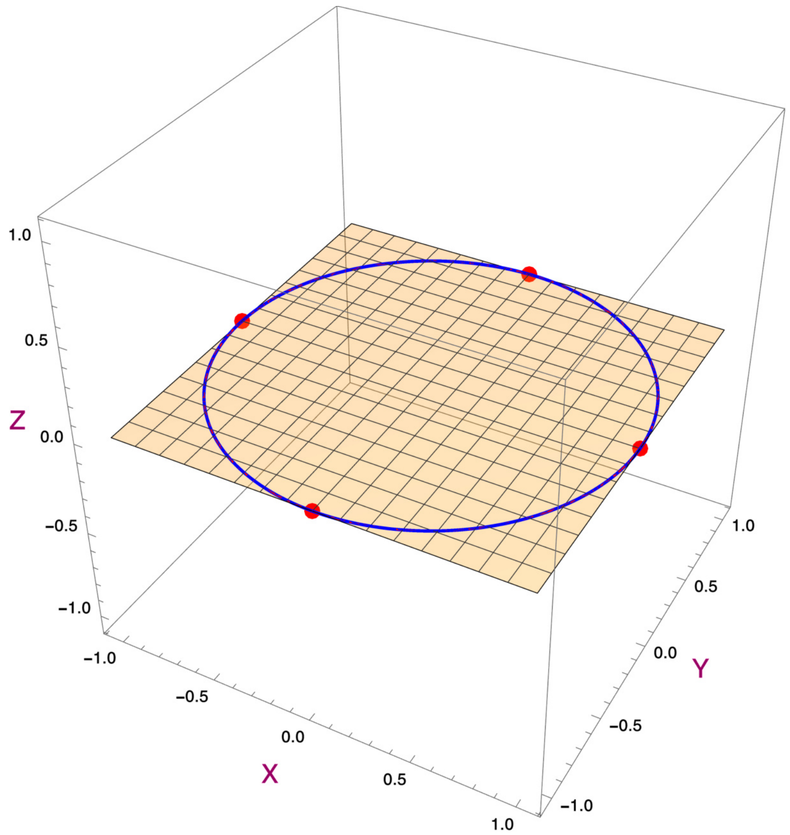

Figure 570. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points.

We are going to define the quotient for this function.

We define the quotient function, notation $q(h, k)$ in $(0, 0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(h,k)$ and not the differential quotient.

If

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x,y)$ is by definition differentiable in $(0,0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{\dfrac{\cos(h\,k)-1}{h^2\,k^2} + \dfrac{1}{2}}{\sqrt{h^2+k^2}} & \text{if } h\,k \neq 0; \\ 0 & \text{if } h\,k = 0. \end{cases}$$

By using the Taylor expansion with remainder we find

$$q(h,k) = \begin{cases} \dfrac{\dfrac{\frac{1}{6}h^3\,k^3\sin(\xi) - \frac{h^2\,k^2}{2}}{h^2k^2} + \dfrac{1}{2}}{\sqrt{h^2+k^2}} & \text{if } h\,k \neq 0; \\ 0 & \text{if } h\,k = 0. \end{cases}$$

is continuous in $(0,0)$.
We can rewrite this as

$$q(h,k) = \begin{cases} \dfrac{h\,k\sin(\xi)}{6\sqrt{h^2+k^2}} & \text{if } h\,k \neq 0; \\ 0 & \text{if } h\,k = 0. \end{cases}$$

Discussion of the continuity of *q(h,k)* in (0,0).

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h,k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h,k) - (0,0)\| < \delta$ it follows that $|q(h,k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h,k) - (0,0)\| = \sqrt{h^2 + k^2} < \delta$, we have that

$$\left| \frac{\frac{\cos(h\,k)-1}{h^2\,k^2} + \frac{1}{2}}{\sqrt{h^2 + k^2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h,k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h,k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}
\left| \frac{\frac{\cos(h\,k)-1}{h^2\,k^2} + \frac{1}{2}}{\sqrt{h^2 + k^2}} \right| &\leq \left| \frac{h\,k\,\sin(\xi)}{6\sqrt{h^2 + k^2}} \right| \\
&\leq \frac{|h|\,|k|\,|\sin(\xi)|}{6\sqrt{h^2 + k^2}} \\
&\leq \frac{\sqrt{h^2 + k^2}\sqrt{h^2 + k^2}}{6\sqrt{h^2 + k^2}} \\
&\leq 1/6\sqrt{h^2 + k^2}.
\end{aligned}$$

It is sufficient to take $\delta = 6\,\epsilon$. We can find a δ, so we conclude that the function $q(h,k)$ is continuous. Thus the function $f(x,y)$ is differentiable.

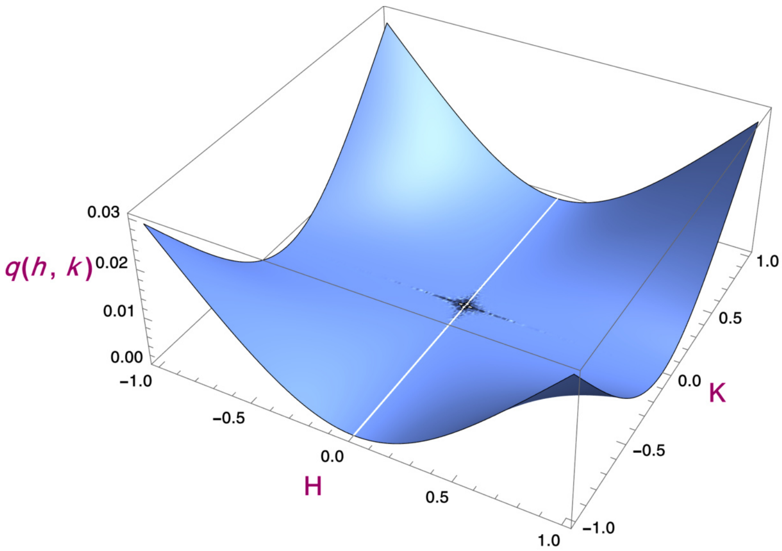

Figure 571. We see here a three dimensional figure of the graph of the function $q(h, k)$. This looks like a continuous function.

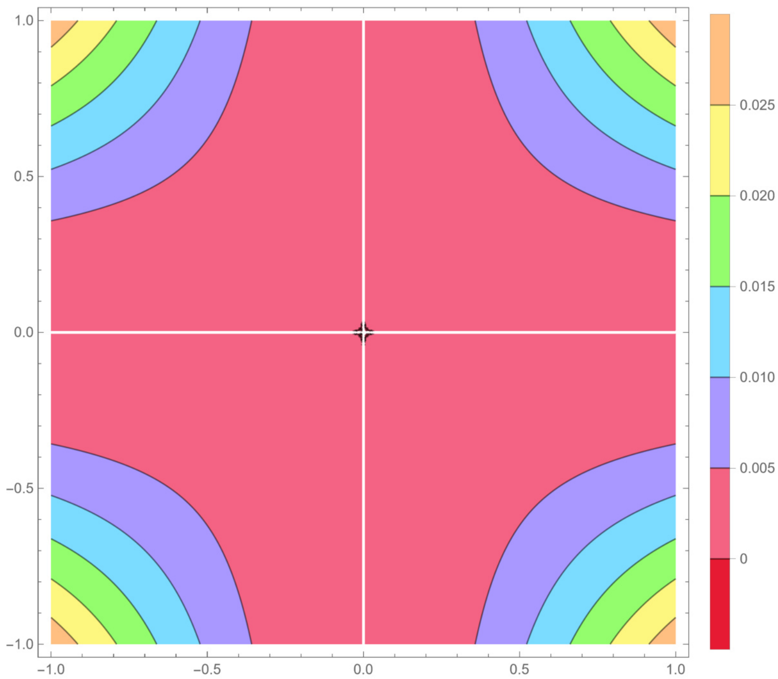

Figure 572. We see here a figure of the contour plot of the function $q(h, k)$. Only level curves of level around 0 come close to $(0, 0)$.

76.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

We have calculated all directional derivatives. We have seen that the vectors $(u, v, D_{(u,v)}(0,0))$ are nicely coplanar and that they satisfy the formula

$$D_{(u,v)}(0,0) = \frac{\partial f}{\partial x}(0,0)\, u + \frac{\partial f}{\partial y}(0,0)\, v.$$

So we have now almost an intuitive tangent plane.

So we wonder if we could use an alternative proof of differentiability without the nuisance of calculating the continuity of the quotient function $q(h,k)$. It turns out that we only have to prove now that the function $f(x,y)$ is locally Lipschitz continuous in $(0,0)$. We cite here the criterion that we will use.

A function is differentiable in (a,b) if it satisfies the three following conditions

1. All the directional derivatives of the function exist.
2. The directional derivatives have the following form: $D_{(u,v)}(a,b) = \nabla f(a,b) \cdot (u,v) = \frac{\partial f}{\partial x}(a,b)\, u + \frac{\partial f}{\partial y}(a,b)\, v$.
3. The function is locally Lipschitz continuous in (a,b). This means that there exists at least one neighbourhood of (a,b) and a number $K > 0$ such that for all (x_1, y_1) and (x_2, y_2) in the neighbourhood, we have $|f(x_1,y_1) - f(x_2,y_2)| < K\, ||(x_1,y_1) - (x_2,y_2)||$. Remark that the same K must be valid for all (x_1,y_1) and (x_2,y_2).

Remark. It is to be expected that a very strong continuity condition must be satisfied. The Lipschitz local continuity is indeed a very strong condition but this is not unexpected because the differentiable function f must be "locally flat" and thus "locally linear" which is partly expressed by this Lipschitz continuity condition.

Let us try to prove the Lipschitz condition. We are going to use a well known method.

$$
\begin{aligned}
&|f(x_1,y_1) - f(x_2,y_2)| \\
&\qquad = |f(x_1,y_1) - f(x_1,y_2) + f(x_1,y_2) - f(x_2,y_2)| \\
&\qquad \le |f(x_1,y_1) - f(x_1,y_2)| + |f(x_1,y_2) - f(x_2,y_2)|.
\end{aligned}
$$

We focus now on the first term $|f(x_1,y_1) - f(x_1,y_2)|$. We fix now x_1 and look upon $f(x_1,y)$ as a function in one variable y. So it springs to mind that we can estimate this by using Lagrange's intermediate value theorem. So we have

$$
|f(x_1,y_1) - f(x_1,y_2)| = \left|\frac{\partial f}{\partial y}(x_1,\xi)\right| \, |y_1 - y_2|
$$

where ξ is a number lying in the open interval (y_1,y_2) if e.g. $y_1 \le y_2$. Remark that we already have proven that the partial derivatives are bounded in a neighbourhood, we can take that bound found for $\frac{\partial f}{\partial y}$, say M_2 and that neighbourhood in this case also. So $\left|\frac{\partial f}{\partial y}(x_1,\xi)\right| \le M_2$. We work in a completely analogous way for the second term $|f(x_1,y_2) - f(x_2,y_2)|$. We have in that case $\left|\frac{\partial f}{\partial x}(\xi,y_2)\right| \le M_1$ where ξ is now a number in the open interval (x_1,x_2) if e.g. $x_1 \le x_2$.

We take up our inequality again

$$
\begin{aligned}
&|f(x_1,y_1) - f(x_2,y_2)| \\
&\qquad \le |f(x_1,y_1) - f(x_1,y_2) + f(x_1,y_2) - f(x_2,y_2)| \\
&\qquad \le |f(x_1,y_1) - f(x_1,y_2)| + |f(x_1,y_2) - f(x_2,y_2)| \\
&\qquad \le M_2\,|(y_1 - y_2)| + M_1\,|(x_1 - x_2)| \\
&\qquad \le M_2\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} + M_1\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \\
&\qquad \le (M_1 + M_2)\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}.
\end{aligned}
$$

So this function is locally Lipschitz continuous. We have an alternative proof for the differentiability.

76.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

We know that the partial derivative to x exists and is equal to

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{-x\,y\,\sin(x\,y) - 2\,\cos(x\,y) + 2}{x^3\,y^2} & \text{if } x\,y \neq 0, \\ 0 & \text{if } x = 0 \text{ of } y = 0. \end{cases}$$

We want to see if it is continuous or not.
Discussion of the continuity of the first partial derivative in $(0,0)$.
We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove the inequality $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if the inequality $||(x,y) - (0,0)|| < \delta$ holds, it follows that $|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left|\frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0)\right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.
We ask the reader to read section 4 of this exercise again. We are going to use those reasoning and notations but change only the inequalities at the end of that calculation.

$$\begin{aligned}
&\left|\frac{\partial f}{\partial x}\right| \\
&\quad \le \left|\frac{1}{24}\left(-x^2\,y^3\,\sin(\xi_2) + 4xy^2 - 8\,y\,\sin(\xi_1)\right)\right| \\
&\quad \le \frac{1}{24}\left(x^2\,|y|^3\,|\sin(\xi_2)| + 4\,|x|\,y^2 + 8\,|y|\,|\sin(\xi_1)|\right) \\
&\quad \le \frac{1}{24}\left(\sqrt{x^2+y^2}^2\sqrt{x^2+y^2}^3 + 4\sqrt{x^2+y^2}\sqrt{x^2+y^2}^2 + 8\sqrt{x^2+y^2}\right) \\
&\quad \le \frac{1}{24}\left(\sqrt{x^2+y^2}^5 + 4\sqrt{x^2+y^2}^3 + 8\sqrt{x^2+y^2}\right) \\
&\quad \le \frac{1}{24}\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2}^4 + 4\sqrt{x^2+y^2}^2 + 8\right) \\
&\quad \le \frac{13}{24}\sqrt{x^2+y^2}.
\end{aligned}$$

We have chosen here the restriction to the neighbourhood defined by $\sqrt{x^2+y^2} < 1$.

It is sufficient to take $\delta = \min\{24/13\,\epsilon, 1\}$. We can find a δ, so we conclude that the function is continuous.

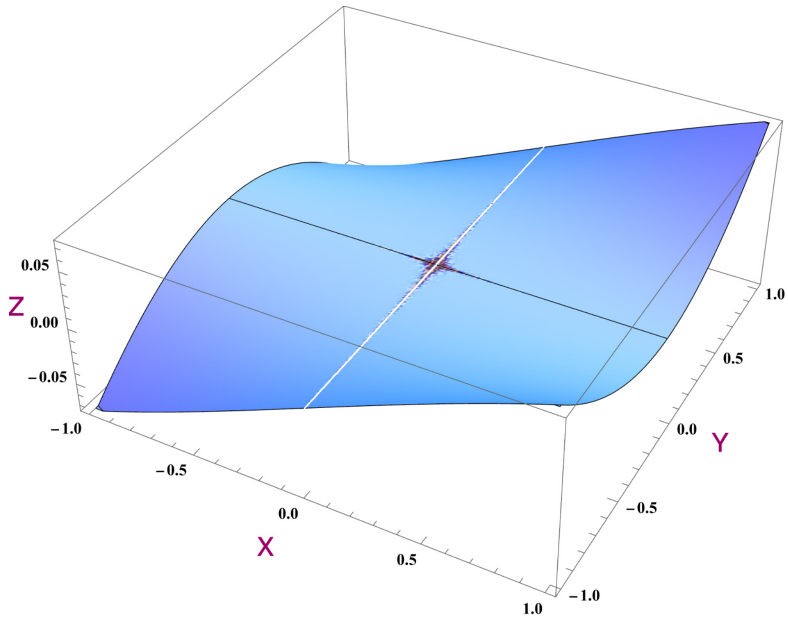

Figure 573. We see here a three dimensional figure of the graph of the first partial derivative $\frac{\partial f}{\partial x}(x, y)$. This looks like a continuous function.

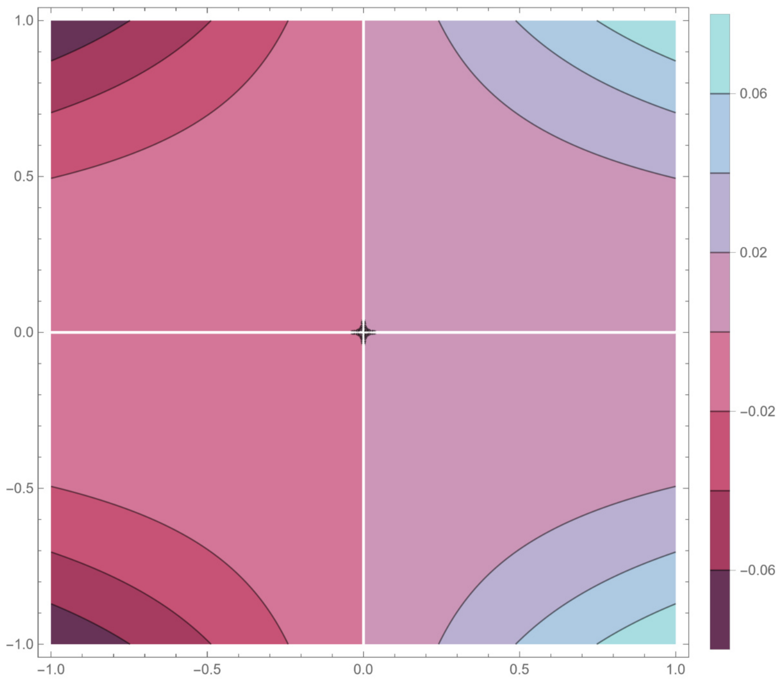

Figure 574. We see here a figure of the contour plot of the $\frac{\partial f}{\partial x}(x, y)$. Only level curves of level around 0 come close to $(0, 0)$.

76.8 Overview

$$f(x,y) = \begin{cases} \dfrac{\cos(xy) - 1}{x^2 y^2} & \text{if } xy \neq 0, \\ \dfrac{-1}{2} & \text{if } xy = 0. \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	yes

Exercise 77.

Discuss the

1. continuity,
2. partial derivatives,
3. directional derivatives,
4. differentiability

of the following function in $(0,0)$.

$$f(x,y) = \begin{cases} x^2 + y^2 & \text{if } x < y \text{ and } y < 2x \text{ and } x > 0, \\ 0 & \text{elsewhere.} \end{cases}$$

77.1 Continuity

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|f(x,y) - f(0,0)| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $||(x,y) - (0,0)|| < \delta$ it follows that $|f(x,y) - f(0,0)| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $||(x,y) - (0,0)|| = \sqrt{x^2 + y^2} < \delta$, we have that

$$\left| x^2 + y^2 - 0 \right| < \epsilon.$$

We are looking for a function $\ell(x,y)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(x,y)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned} \left| x^2 + y^2 - 0 \right| &\leq |x^2 + y^2| \\ &\leq \sqrt{x^2 + y^2}^2 + \sqrt{x^2 + y^2}^2 \\ &\leq 2\sqrt{x^2 + y^2}^2. \end{aligned}$$

It is sufficient to take $\delta = (\epsilon/2)^{1/2}$. We can find a δ, so we conclude that the function is continuous.

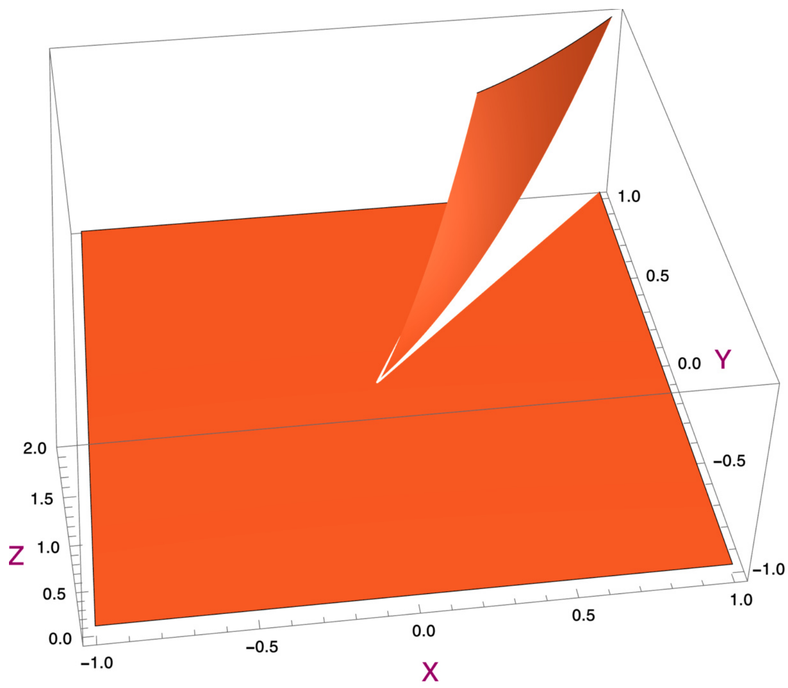

Figure 575. We see here a three dimensional figure of the graph of the function. This looks like a continuous function.

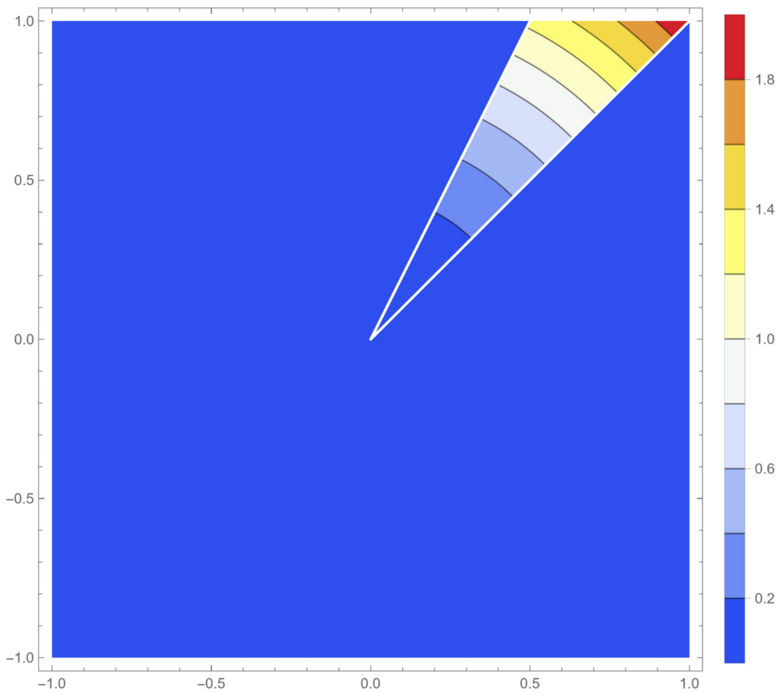

Figure 576. We see here a figure of the contour plot of the function. Only level curves of level around 0 come close to $(0,0)$.

77.2 Partial derivatives

Discussion of the partial derivative of x in $(0,0)$.

We observe then that

$$f|_{y=0}(x,y) = \begin{cases} f(x,0) = 0 & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to x does exist.

Discussion of the partial derivative of y in $(0,0)$.

We observe that

$$f|_{x=0}(x,y) = \begin{cases} f(0,y)=0 & \text{if } y\neq 0;\\ 0 & \text{if } y=0.\end{cases}$$

So

$$\begin{aligned}\frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}\\ &= \lim_{h\to 0} 0\\ &= 0.\end{aligned}$$

So the partial derivative to y does exist.
We conclude that

$$\frac{\partial f}{\partial x}(0,0)=0 \quad \text{and} \quad \frac{\partial f}{\partial y}(0,0)=0.$$

77.3 Directional derivatives

Let (u,v) be a normalised direction vector. So it has length 1, this means that $\sqrt{u^2+v^2}=1$. Then the directional derivative in $(0,0)$ in the direction (u,v), notation $D_{(u,v)}(0,0)$ is by definition

$$D_{(u,v)}(0,0) = \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h}.$$

The classical partial derivatives are of course also directional derivatives in the directions $(1,0)$ and $(0,1)$.

We will calculate the directional derivatives first for the normal vectors $(u,v) = (\cos(\alpha), \sin(\alpha))$ with $\alpha \in (\pi/4, \arctan(2))$. In the case of our function we have to calculate the following limit

$$\begin{aligned} D_{(u,v)}(0,0) &= \lim_{h\to 0} \frac{f(0+hu, 0+hv) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{(hu)^2 + (hv)^2}{h} \\ &= \lim_{h\to 0} hu^2 + hv^2 \\ &= 0. \end{aligned}$$

Let us calculate the right limit.

$$\lim_{\substack{h\to 0 \\ >}} \frac{(hu)^2 + (hv)^2}{h} = 0.$$

The left limit is evidently 0. So the directional limit does exist.

In all other directions, the directional derivative is 0.

So the directional derivatives do always exist.

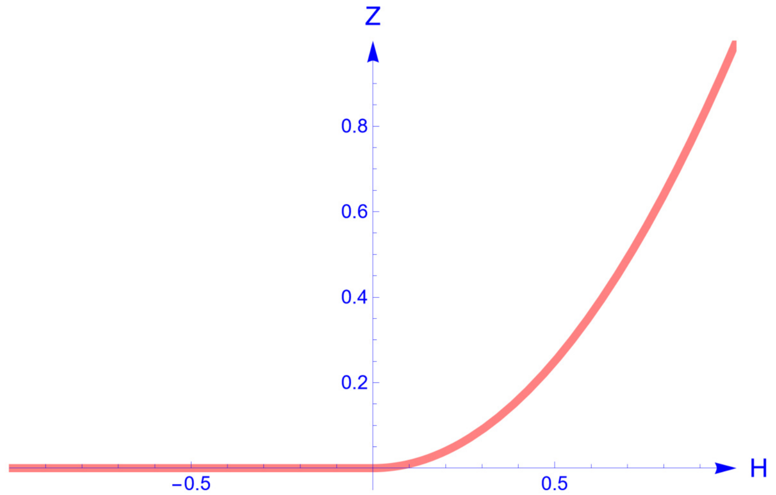

Figure 577. We see here a figure of the graph of the function restricted to the line through $(0,0)$ with direction $(u,v) = (\cos(50\pi/180), \sin(50\pi/180))$. We have plotted here the function $f(hu,hv)$. We see that this function is differentiable.

77.4 Alternative proof of Continuity (optional)

This item is not suited for a first reading. The following criterion for continuity has not been treated in some calculus courses. In that case, skip this and carry on reading the section about differentiability.

If the partial derivatives exist and are bounded in a neighbourhood of $(0,0)$, then this implies that the function is continuous. Because we have calculated the partial derivatives, this gives us an easy opportunity to see if we can have a supplementary proof of the continuity.

The question is here whether the partial derivatives exist in a neighbourhood around $(0,0)$.

We take a point (a,a), $a \neq 0$, on the line with equation $y = x$. We draw the function in the direction of the Y-axis. We see that this function is not continuous. So it is certainly not differentiable and the partial derivative to Y does not exist in the point (a,a) where $a \neq 0$.

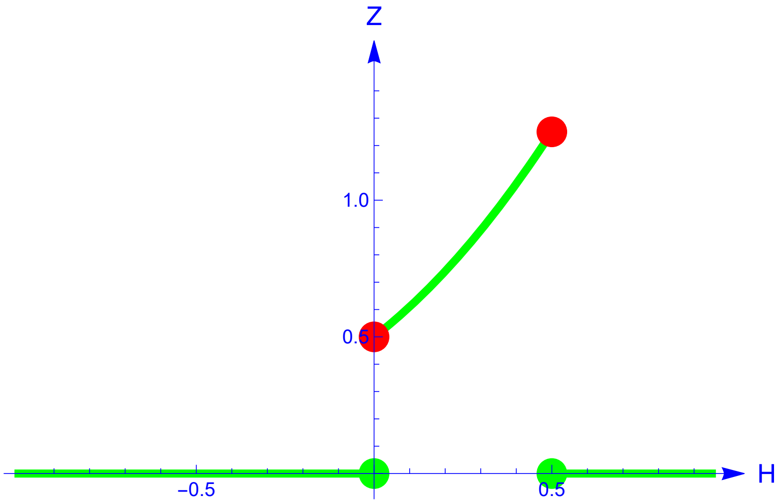

Figure 578. We see here a figure of the graph of the function $f(a, a+h)$. We have drawn the function here for the value $a = 1/2$ which is exemplary for the values of a close to 0. The function is not continuous. So the function is not differentiable. The partial derivatives do not exist everywhere in any neighbourhood of $(0, 0)$.

We cannot use this criterion for proving the continuity.

77.5 Differentiability

The function is continuous and all directional derivatives exist, so there is a possibility that this function is differentiable.

Some preliminary visual tests.

Before setting out to do an investigation of the differentiability, let us take the liberty to do a few visual checks of the calculations that we have performed until now. Maybe these figures can make us doubtful about the differentiability. Because we do not rely on purely visual proofs, we will then continue as if we did not perform these visual checks.

The partial derivatives exist and all other directional derivatives exist. The function is also continuous. So it can be useful to show the function and its candidate tangent plane in $(0, 0)$ in order to see what is going on.

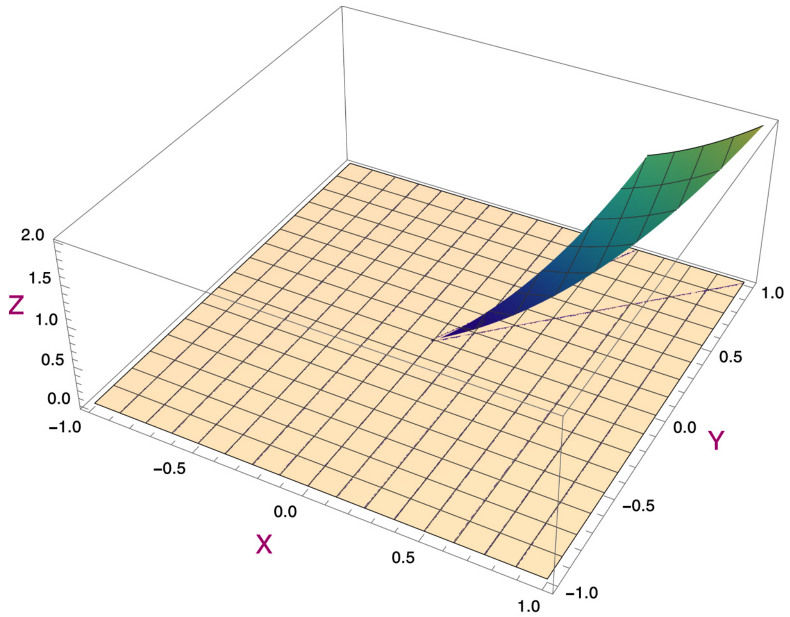

Figure 579. We see here a three dimensional figure of the graph of the function $\frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\,y$, which is graphically the candidate tangent plane and the function $f(x,y)$. We see here that the candidate tangent plane fits the function very nicely.

We can look at it in another way. We have calculated all the directional derivatives and we know that if the function is differentiable, then the vectors $(u, v, D_{(u,v)}(0,0))$ must lie in one plane, which is the tangent plane if the function is differentiable. So let us visually check that these vectors are coplanar.

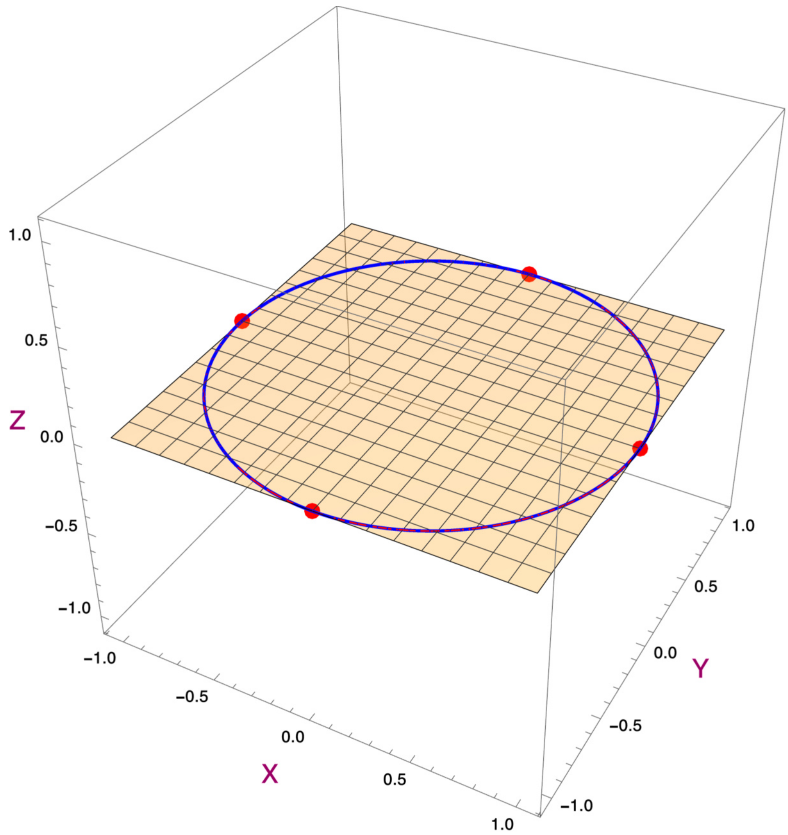

Figure 580. We see here a three dimensional figure of the graph of the function and the position of the vectors $(u, v, D_{(u,v)}(0,0))$ which must sweep out an ellipse in the candidate tangent plane. If they do not, then there is no tangent plane and the function cannot be differentiable. We see the red circle of the unit vectors (u, v) in the X-Y plane. We see also in the blue circle the vectors $(u, v, D_{(u,v)}(0,0))$. It is possible that the red circle is completely covered by the blue circle and then it is made invisible. Four points on the blue circle are indicated by large red points. We see here that the vectors $(u, v, D_{(\mathbf{u},\mathbf{v})}(0,0))$ sweep out a nice ellipse in the candidate tangent plane.

Discussion of the continuity of the quotient.

We define the quotient function, notation $q(h, k)$ in $(0,0)$ as

$$q(h,k) = \frac{f(h,k) - f(0,0) - \left(\frac{\partial f}{\partial x}(0,0)\,h + \frac{\partial f}{\partial y}(0,0)\,k\right)}{\|(h,k)\|}.$$

Remark that this quotient is the two variable equivalent of the one variable quotient

$$q(h) = \frac{f(h) - f(0) - f'(0)\,h}{h}.$$

So please do not confuse this with $\frac{f(h)-f(0)}{h}$, which is commonly called the differential quotient! To avoid any misunderstandings we call our q from now on the quotient, notation $q(x, y)$ and not the differential quotient.

If the

$$\lim_{(h,k)\to(0,0)} q(h,k) = 0,$$

then the function $f(x, y)$ is by definition differentiable in $(0, 0)$. So we have to prove that the function

$$q(h,k) = \begin{cases} \dfrac{h^2+k^2}{\sqrt{h^2+k^2}} & \text{if } h < k \text{ and } k < 2\,h \text{ and } h > 0; \\ 0 & \text{elsewhere.} \end{cases}$$

is continuous in $(0, 0)$.

Discussion of the continuity in of $q(h, k)$ in $(0, 0)$.

We investigate continuity with an ϵ-δ approach.

We take an arbitrary $\epsilon > 0$ and we have to prove that $|q(h, k) - 0| < \epsilon$. The problem is now to find a $\delta > 0$ such that if $\|(h, k) - (0, 0)\| < \delta$ it follows that $|q(h, k) - 0| < \epsilon$ is valid.

When applying our function definition, we have then the following statements. Try to find a δ such that if $\|(h, k) - (0, 0)\| = \sqrt{h^2 + k^2} < \delta$, we have that

$$\left| \frac{h^2+k^2}{\sqrt{h^2+k^2}} - 0 \right| < \epsilon.$$

We are looking for a function $\ell(h, k)$ that is larger then or equal to the left hand side of this last inequality. This function $\ell(h, k)$ has to have the property that it can be made smaller then ϵ by carefully manipulating the value δ. This is sufficient for our continuity proof.

$$\begin{aligned}\left|\frac{h^2+k^2}{\sqrt{h^2+k^2}}-0\right| &\leq \frac{h^2+k^2}{\sqrt{h^2+k^2}} \\ &\leq \frac{\sqrt{h^2+k^2}^2}{\sqrt{h^2+k^2}} \\ &\leq \sqrt{h^2+k^2}.\end{aligned}$$

It is sufficient to take $\delta = \epsilon$. We can find a δ, so we conclude that the function $q(h, k)$ is continuous.

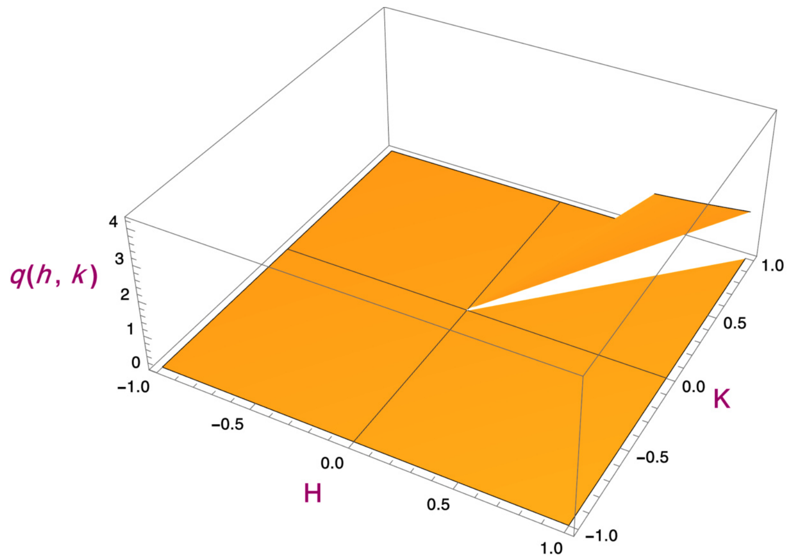

Figure 581. We see here a three dimensional figure of the graph of the function $q(h, k)$. This looks like a continuous function.

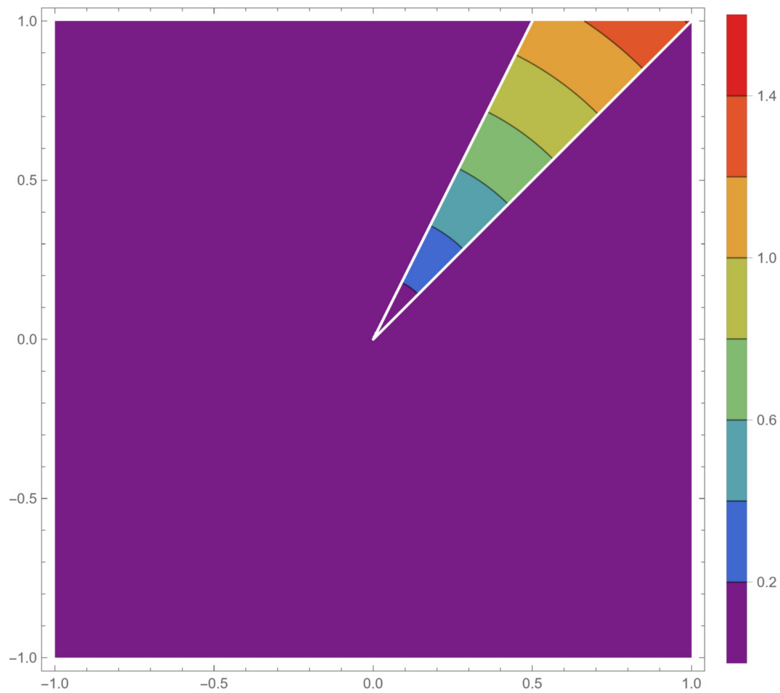

Figure 582. We see here a figure of the contour plot of the function $q(h,k)$. Only level curves of level around 0 come close to $(0,0)$.

77.6 Alternative proof of differentiability (optional)

This is not suited for a first reading or if the student is not acquainted with Lipschitz continuity. Skip this section then and continue with the continuity of the first partial derivatives.

This section is **irrelevant** for this exercise, because the function is not continuous in any neighbourhood of $(0,0)$.

77.7 Continuity of the partial derivatives

We are looking for an alternative proof for the differentiability.

If both the partial derivative derivatives are continuous, then we have an alternative proof of the existence of the derivative. The condition that both of the partial derivatives are continuous is in fact too strong in the sense that it is not equivalent with differentiability only. But this criterion is in fact used by many instructors and textbooks, so it is interesting to take a look at it.

We have seen that the partial derivatives do not all exist in any neighbourhood of $(0,0)$.

77.8 Overview

$$f(x,y) = \begin{cases} x^2 + y^2 & \text{if } x < y \text{ and } y < 2x \text{ and } x > 0, \\ 0 & \text{elsewhere.} \end{cases}$$

continuous	yes
partial derivatives exist	yes
all directional derivatives exist	yes
differentiable	yes
partials are continuous	no

Appendix

Alternative proof of continuity

If the function f has bounded partial derivatives in a neighbourhood of $(0, 0)$, then it is continuous. See Rudin, principles of mathematical analysis, third edition, p. 239 exercise 7.

Lipschitz continuity

Definition of Lipschitz continuity

A function is Lipschitz continuous in a set $S \subseteq \mathbf{R}$ if there exists a positive real number L such that for all x_1 and x_2 in S we have that

$$|f(x_1) - f(x_2)| \le L\,|x_1 - x_2|.$$

Remark.
The positive number L is called a Lipschitz constant for f in S. Many values for L are possible. In most applications it is irrelevant which value L has. Only the existence of a L usually matters. This is also the case in these exercises. It is possible that no L satisfies the inequality. Then we say that f is not Lipschitz continuous.

Example and visualisation of Lipschitz continuity

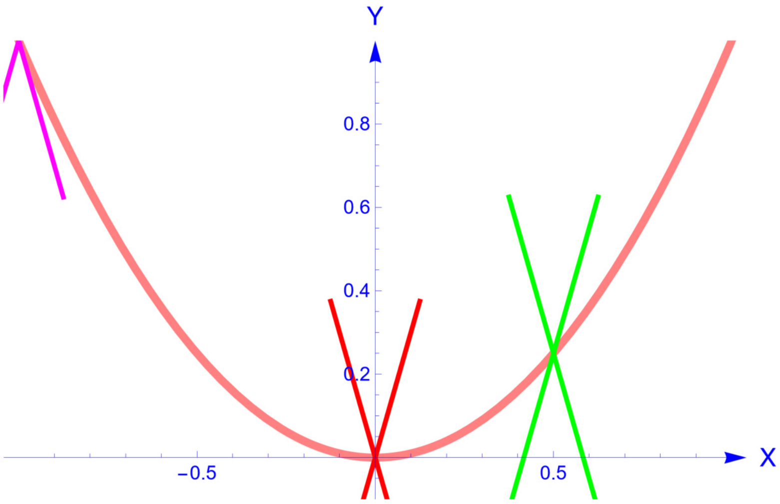

Figure 583. We consider here the function $f(x) = x^2$ on the interval $S = [-1, 1]$. We have taken $L = 3$. We drew a red cone or degenerate hyperbola consisting of the lines $y = -Lx$ and $y = Lx$ in red. We can translate this cone to any point on the graph of the function. An example of this is the green cone. No cone may have points of the graph of the function in its interior, we mean by that the region containing the vertical line. We see in this figure that no cone has points of the function in its interior. This means that the function f is Lipschitz continuous in S with Lipschitz constant $L = 3$.

We consider now the the same function $f(x) = x^2$ on the interval $[-1, 1]$. We try to see if this function is Lipschitz continuous with constant $L = 1/3$.

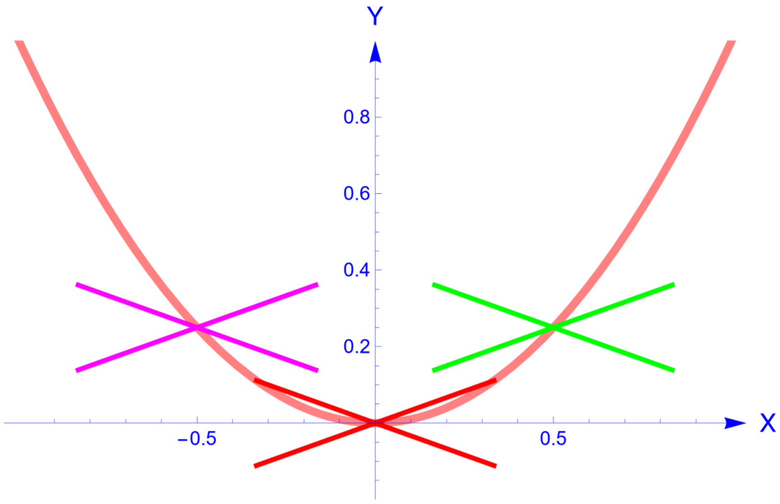

Figure 584. We consider here the function $f(x) = x^2$ on the interval $S = [-1, 1]$. We have taken $L = 1/3$. We drew a red cone or degenerate hyperbola consisting of the lines $y = -Lx$ and $y = Lx$ in red. We can translate this cone to any point on the graph of the function. An example of this is the green cone. No cone may have points of the graph of the function in its interior, we mean by that the region containing the vertical line. We see in this figure that at least one cone has points of the function in its interior. This means that the function f is not Lipschitz continuous in S with Lipschitz constant $L = 1/3$.

Example of non Lipschitz continuity

The following figure is an example of a function that is not Lipschitz continuous. There does not exist a real number L such that $|f(x_1) - f(x_2)| \leq L\,|x_1 - x_2|$ in S. We take the function $f(x) = 1/x$ on $S = \mathrm{R}_0^+$.

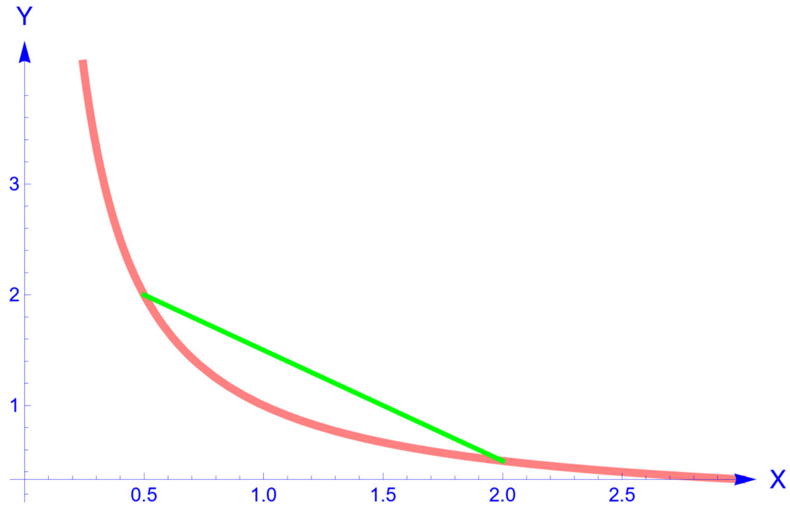

Figure 585. We see here the graph of the function $f(x) = 1/x$ on the set $S = R_0^+$. We have drawn the green line piece with vertices $(x_1, f(x_1))$ and $(x_2, f(x_2))$. The absolute value of the slope of this line piece is $|(f(x_1) - f(x_2))/(x_1 - x_2)|$. This value is not bounded on the set S. It is easy to see that the value becomes arbitrarily large if we move either x_1 or x_2 or both close to 0. A small calculation shows indeed $|(f(x_1) - f(x_2))/(x_1 - x_2)| = |(1/x_1 - 1/x_2)/(x_1 - x_2)| = |-1/(x_1 x_2)|$. This value is unbounded in any neighbourhood of 0.

Here is another example of a function that is not Lipschitz continuous. We take the function $f(x) = \sqrt{x}$ in the set $S = \mathbf{R}^+$.

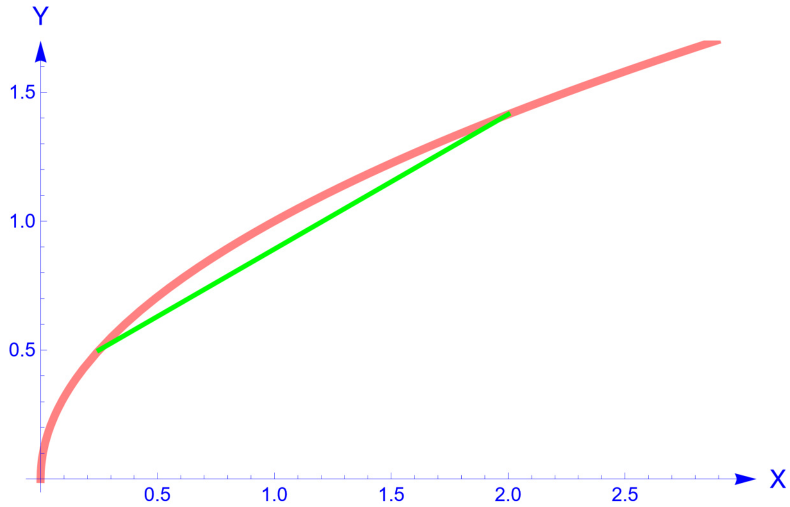

Figure 586. We see here the graph of the function $f(x) = \sqrt{x}$ on the set $S = \mathbf{R}_0^+$. We have drawn the green line piece with vertices $(x_1, f(x_1))$ and $(x_2, f(x_2))$. The absolute value of the slope of this line piece is $|(f(x_1) - f(x_2))/(x_1 - x_2)|$. This value is not bounded on the set S. It is easy to see that the value becomes arbitrarily large if we take $x_1 = 0$ and move x_2 close to 0. A small calculation shows indeed $|(f(0) - f(x_2))/(0 - x_2)| = |(0 - \sqrt{x_2})/(0 - x_2)| = |1/\sqrt{x_2}|$. This value is unbounded in any neighbourhood of 0.

The previous examples show that it is possible to interpret the meaning of the existence of Lipschitz continuity as the fact that the slopes $(f(x_1) - f(x_2))/(x_1 - x_2)$ of line pieces supported by the function must not increase too fast. This means also that the slopes of the tangents must be bounded. We can see that again in another example.

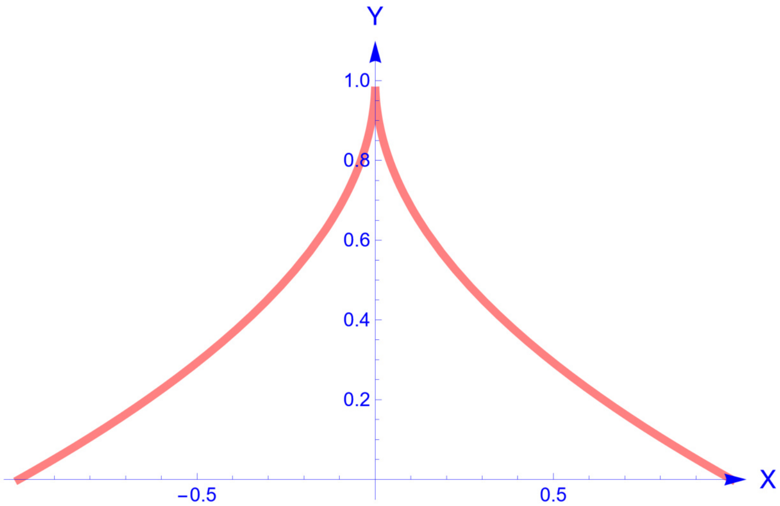

Figure 587. We have drawn the function $f(x) = 1 - \sqrt{-x}$ if $x < 0$ and $f(x) = 1 - \sqrt{x}$ if $x \geq 0$. The graph of this function has a vertical tangent in $x = 0$. The slopes $f(x_1) - f(x_2)$ are unbounded in any neighbourhood of $x = 0$. This function cannot be Lipschitz on the interval $[-1, 1]$.

Lipschitz continuity in several variables

It is clear that Lipschitz continuity can also be defined in $\mathbf{R}^2$. We have then that the following inequality must be satisfied.

$$|f(x_1, y_1) - f(x_2, y_2)| \leq L\,||(x_1, y_1) - (x_2, y_2)||.$$

The rest of the definition holds word by word by analogy.

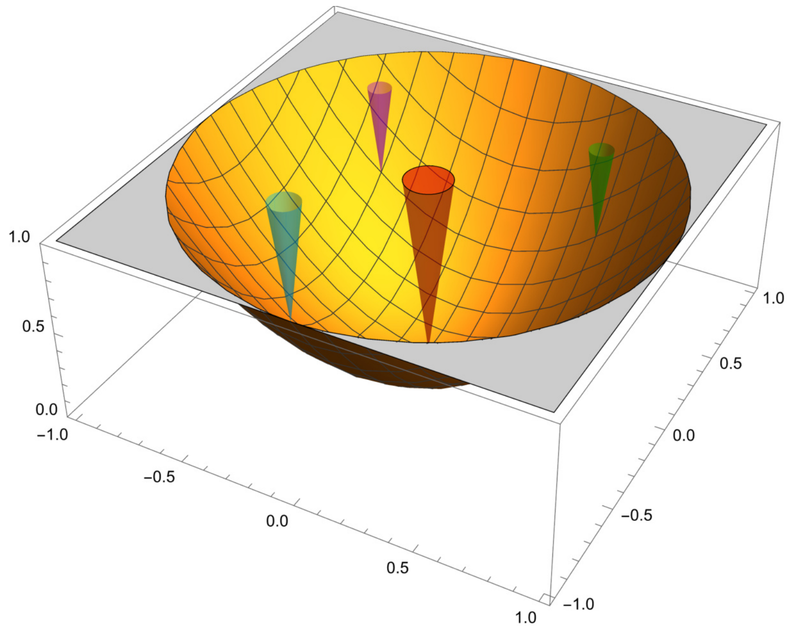

Figure 588. We see here the plot of the function $f(x, y) = x^2 + y^2$ on the set $S = [-1, 1] \times [-1, 1]$. Lipschitz continuity with Lipschitz constant L on a set $S \subseteq \mathbf{R}^2$ can be interpreted graphically as follows. Take a cone with equation $L^2 z^2 = (x^2 + y^2)$. Move this cone to any point on the surface by translating the origin to that point. See the figure. If the interior of these cones do not contain points of the surface, then the function f is Lipschitz continuous with Lipschitz constant L on the set S.

We end by remarking that Lipschitz continuity implies continuity in every point. We have used this fact frequently in the text. Much more can be said. We refer the reader to *www.wikipedia.org*.

Alternative proof of differentiability

Suppose we have done some research on the continuity and the directional derivatives. These investigations do almost the job of proving differentiability if there is continuity and if the directional derivatives exist and are linear in a certain way. Suppose we do not want to investigate differentiability by using the definition. We are looking for a tiny piece of further evidence that can show differentiability. It turns out that having also Lipschitz continuity in a neighbourhood of (a, b) would do the job of proving differentiability. This is a possible alternative way that is applicable in some circumstances. Remark that the following is an implication and not an equivalence.

Lemma
Suppose $U \subseteq \mathbf{R}^2$ is an open set containing the point (a, b). Let $f : U \to \mathbf{R} : (x, y) \mapsto f(x, y)$. Suppose also that

1. the function is locally Lipschitz continuous in (a, b);
2. the partial first order derivatives in $(0, 0)$, $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ exist;
3. for every unit direction vector (u, v), we have that the directional derivative is of the form $D_{(u,v)} f(a, b) = \frac{\partial f}{\partial x}(0, 0)\, u + \frac{\partial f}{\partial y}(0, 0)\, v$,

then the function is differentiable in (a, b).

Proof.
Choose an arbitrary $\epsilon > 0$. We have to prove that there exists a δ such that if $||(x, y) - (a, b)|| < \delta$, we have that

$$\frac{\left| f(x, y) - f(a, b) - \left(\frac{\partial f}{\partial x}(0, 0)\, h_1 + \frac{\partial f}{\partial y}(0, 0)\, h_2\right) \right|}{\|(h_1, h_2)\|} < \epsilon.$$

By subtracting $f(a, b) + \left(\frac{\partial f}{\partial x}(0, 0)\,(x - a) + \frac{\partial f}{\partial y}(0, 0)\,(y - b)\right)$ of $f(x, y)$, we can assume that $f(a, b) = 0$ and that $\frac{\partial f}{\partial x}(0, 0) = 0$ and $\frac{\partial f}{\partial y}(0, 0) = 0$. Now let L be a Lipschitz constant that is valid in at least one neighbourhood of (a, b).

We cover the unit circle B around (a, b) with open disks with radius $\epsilon/(2L)$ with their centre points on B. We take an finite sub cover of this covering which is possible by the compactness of the unit circle. We denote the centre points of this finite sub cover by (u_i, v_i), $i \in \{1, \ldots, n\}$.

We investigate now the n functions

$$g_i(t) = \frac{f\,((a, b) + t\,(u_i, v_i))}{t}.$$

We assume that t is taken small enough so that these functions are defined. Because all directional derivatives exist, we have for every everyone of our n functions $g_i(t)$ continuity in $t = 0$. So for all $\epsilon/(2L)$, we have a δ_i such that if $|t| < \delta_i$, then $|g_i(t)| < \epsilon/(2L)$. So by the definition of the $g_i(t)$

$$|f\,((a, b) + t(u_i, v_i))\,| < |t|\,\epsilon/(2L) \text{ if } |t| < \delta_i.$$

Let now δ be defined as follows

$$\delta = \min_{i \in \{1,\ldots,n\}} \delta_i.$$

We have to prove for differentiability that for all ϵ, there exists a δ such that for all $|t| < \delta$ it holds for all unit vectors (u, v) then

$$|f((a,b) + t(u,v))| < |t|\epsilon.$$

We take any unit vector (u, v). We know by the construction of the sub cover that there exists at least one $i \in \min\{1,\ldots,n\}$ such that $|(u_i, v_i) - (u, v)| < \epsilon/(2L)$. Then by the Lipschitz continuity, suppose the Lipschitz constant is L,

$$\begin{aligned} &|f((a,b) + t(u,v)) - f((a,b) + t(u_i,v_i))| \\ &\quad \le L|t|\,\|(u,v) - (u_i,v_i)\| \\ &\quad < L|t|\epsilon/(2L) \\ &\quad < |t|\epsilon/2. \end{aligned}$$

Now we use the triangle inequality

$$\begin{aligned} &|f((a,b) + t(u,v))| \\ &\quad \le |f((a,b) + t(u_i,v_i))| + |f((a,b) + t(u,v)) - (f((a,b) + t(u_i,v_i)))| \\ &\quad < |t|\epsilon/2 + |t|\epsilon/2 \\ &\quad = |t|\epsilon. \end{aligned}$$

Remark. We adapted this proof from a sketch made in the website *www.stackexchange.com* by an anonymous author. We thank prof. Leoni from Carnegie Mellon university for pointing this out to us.

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