Exercise notes on

the Jordan normal form and the Jordan chains

by Dirk Bollaerts

May 2024

Preface & Content

Preface

We constructed these exercises in four different periods when we had to provide students with assistance in solving Jordan normal form problems. These students encountered the Jordan normal form in a course on differential equations, and some of them also encountered it when they had to solve recursion equations.

The intention was to help students quickly handle Jordan normal form problems so that they could promptly tackle problems in those other branches of mathematics. This text reflects this objective. If one is seeking a deeper theoretical understanding of the theory behind the Jordan normal form, then we can only advise to use other websites and theoretical texts both on the web and in algebra books. There are many algebra courses on the web that develop this theory exquisitely and extensively. The Jordan normal form theory offers various proofs with distinct flavours. We can work within pure matrix theory, explore it through linear maps, or delve into it from another perspective by studying modules over principal ideal domains. All these approaches are readily available online. This text is however not suited for that purpose.

The students with which we worked had a very good theoretical and practical background in general vector space theory. Linear independence, generating vectors, basis, linear maps, matrices and eigenvectors were very well known. And last but not least everyone of them could solve very fluently systems of linear equations working against the clock.

Jordan theory usually occupies an awkward position in a typical linear algebra course. It often comes at the end of the chapters of the general theory of eigenvectors and their application to symmetric matrices. Consequently, it is frequently covered by lack of time with only brief explanations of the key concepts. This results in exercises being frequently skipped, leaving students with limited practical knowledge in this area.

This is a typical how-to text with no pretence of a theoretical approach. We designed the exercises assuming that the theory behind Jordan decomposition is already known in skeleton or that the reader has some familiarity with it. If there are doubts about the theory, we would readily recommend consulting the excellent Wikipedia article on the theorem.

We worked with students as follows. Every group of students could choose their favourite size of matrices, and we constructed these exercises to suit their flavours and needs. Every student was offered two exercises on nilpotent matrices and two exercises on general matrices. We kept one additional matrix for demonstration purposes. We strongly advise current readers to follow this strategy. One could select arbitrarily two exercises from part 2 on nilpotent matrices and similarly choose two exercises from part 3 on general matrices. This should be sufficient; doing more could be a bit of overkill.

This text is not intended to be read linearly from top to bottom. We expect readers instead to immediately focus on the type and size of matrices they are currently dealing with in homework or exercises. Consequently, there is some repetition in the exercises. We originally planned to include only six matrices in this text but abandoned that idea because it was unclear which ones to omit and we wanted also to have Jordan matrices with different normal form structure, and readers often have strong preferences and their motivation is often based on their current particular needs in their studies.

Great care is taken to ensure that each exercise can be solved manually with ease. This led us to keep the eigenvalues small to facilitate the calculation of certain matrix powers. We do not use consequently a wide range of values for the eigenvalues; they are frequently -1, 1, or 0. However, this should not give the wrong impression that eigenvalues are almost always small in this context. We experimented with eigenvalues like $\lambda = 3$ and $\lambda = 4$, but for matrices of a substantial size (e.g., 7×7), this often results in large numbers when calculating higher powers of related matrices. We wanted to avoid this to ensure that the exercises always remain manageable for manual computation.

We made an effort to provide sufficient intermediate results to facilitate the repetition of calculations if necessary. We hope that it will be possible to check intermediate calculations in order to make exercise work as fluent as possible. This also makes it easier for readers who are allowed to experiment with a calculator. We worked with students who had a solid understanding of the fundamentals of linear algebra, assuming they could handle essential parts of the calculations easily. It would have been impractical to solve systems of equations step by step in this text, as it would have significantly increased the volume of this text. We include intermediate results so that readers can check their solutions at their convenience as much as it is possible. Additionally, many instructors follow a similar practice of starting with intermediate results, allowing students to fill in the gaps to avoid overwhelming them with extensive calculation work. Readers can consult their instructors to determine which intermediate results will be provided in tests or homework and adapt to their instructor's preferences.

Some instructors may insist that the vector spaces are defined over either **R** or **C**. This does not affect the solutions presented in this text, which remain valid as long as we are working in characteristic zero fields. The situation differs for fields with characteristics other than zero. One crucial point to note is that when selecting basis vectors, larger fields offer a broader choice for these basis vectors. The matrices P representing the change of basis can be defined over these larger fields, such as **R** or **C**. We consistently work however with the field **Q** and restrict ourselves to basis change matrices P over this field. This restriction was chosen to simplify manual calculations.

If the instructor insists on using another field **K** of numbers, in most cases it will be **C**, then our calculations and solutions remain invariably correct if that field is of characteristic 0. We remark that if one works over larger fields then **Q**, one is allowed to choose generating vectors for the Jordan chains that are vectors in **K**. The matrix P in the answers can be chosen in the larger field. Our choice remains valid however and will be always easier to work with. We give a concrete example. Suppose we have to choose a vector in $\mathbf{K}^3$ of the form (a, a, a) with $a \neq 0$, then we will choose $(1, 1, 1)$ but if we work in **C**, one is also allowed to choose $(2 + 3\,i, 2 + 3\,i, 2 + 3\,i)$. Let us summarise. Suppose we work in a larger field then **Q** of characteristic 0 then our answers are also valid in that field. But more answers are theoretically allowed. If one works in a **finite** field then it is not true that one can use our answer entirely. Though we can reuse many of our calculations we will see that we sometimes used fractions with small denominators. Depending on our characteristic, we have to be careful there. And when one encounters matrices from which the kernel must be calculated, then the rank of that matrix can be different from our rank because some elements of that matrix can be zero in the finite field but are not zero in a field of characteristic 0.

There are many solutions for every exercise. This is an inherent disadvantage when working with matrices. Matrices almost never reflect the fundamental mathematics because they are completely dependent on making arbitrary choices when choosing bases. Other choices of a basis give different matrices which are equally valid. This is also the case in Jordan theory where the essence of the theory is reflected by the invariants of certain vector spaces related to a linear map f. The choice of a basis is in this theory dependent on choosing the generating vector of Jordan chains.

There are often uncertainties about verifying linear independence when constructing Jordan chains to build the basis. We have endeavoured to clarify precisely what needs to be checked and which independencies are theoretically implied automatically. An explanation of the procedure is provided in the appendix, so readers with doubts should refer to it. We have summarized it in the appendix, and it is advisable to consult it.

Some readers may wish to experiment with these matrices using calculators. We have listed all matrices at the end of the document, and presented them without any additional typographic formatting, making it easy to copy and paste them into the text editor of your choice. One can adjust if necessary the delimiters using the text editor's replace function.

The solutions of these exercises are not unique. There are many ways in which a basis can be chosen. The same holds for choosing a generating vector of Jordan chains. We would urge the reader to explicitly check their results as we did in our solutions. Notice that the resulting Jordan normal form is unique. So it is perfectly possible to check the end result.

This text is provided to the reader as-is. After digitising our handwritten notes, there was no external proofreading. While we made an effort to check everything and the final results, we acknowledge that the law of Murphy often plays a strong role in mathematics and that is especially true in exercises of this nature. The culprit is frequently the copy and paste procedure, which can be both a blessing and a source of errors. We would appreciate it if users could point out any errors they encounter.

Content

List of Exercises

We will use the notation $J_n(\lambda)$ for an elementary Jordan block. The number n indicates the size of the elementary Jordan block, the number λ indicates the eigenvalue associated to the elementary Jordan block. Note that some exercises are deliberately skipped in this list.

Exercise 1. Jordan structure: $(\mathbf{4} \times \mathbf{4})$; $(\mathbf{J_4}(\mathbf{0}))$.

$$B = \begin{pmatrix} 1 & -2 & 3 & 1 \\ 4 & -5 & 7 & 2 \\ 3 & -3 & 4 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Exercise 2. $(\mathbf{4} \times \mathbf{4})$; $(\mathbf{J_3}(\mathbf{0}), \mathbf{J_1}(\mathbf{0}))$.

$$B = \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

Exercise 3. $(\mathbf{4} \times \mathbf{4})$; $(\mathbf{J_2}(\mathbf{0}), \mathbf{J_2}(\mathbf{0}))$.

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{pmatrix}.$$

Exercise 4. $(\mathbf{7} \times \mathbf{7})$; $(\mathbf{J_3}(\mathbf{0}), \mathbf{J_3}(\mathbf{0}), \mathbf{J_1}(\mathbf{0}))$.

$$B = \begin{pmatrix} -2 & 1 & 0 & -2 & 1 & 3 & 2 \\ 1 & -1 & 0 & 1 & -1 & -2 & -1 \\ 1 & -1 & 0 & 1 & -1 & -2 & -1 \\ 0 & -1 & 0 & 2 & 2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 2 & 2 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 1 & 5 & 5 & -1 \end{pmatrix}.$$

Exercise 5. $(\mathbf{7} \times \mathbf{7})$; $(\mathbf{J_3}(\mathbf{0}), \mathbf{J_2}(\mathbf{0}), \mathbf{J_2}(\mathbf{0}))$.

$$B = \begin{pmatrix} 3 & 3 & -4 & 2 & 1 & 1 & -1 \\ -3 & -3 & 3 & -2 & 0 & 0 & 1 \\ -1 & -1 & 1 & -1 & 0 & 0 & 1 \\ -1 & -1 & 3 & -1 & -2 & -2 & 1 \\ -5 & -5 & 6 & -4 & 0 & 0 & 3 \\ 4 & 4 & -5 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}.$$

Exercise 6. $(\mathbf{7}\times\mathbf{7})$; $(\mathbf{J_2(0)}, \mathbf{J_2(0)}, \mathbf{J_2(0)}, \mathbf{J_1(0)})$.

$$B = \begin{pmatrix} 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -4 & 3 & -1 & -1 \\ 0 & -2 & -1 & -5 & 4 & -1 & -1 \\ 1 & -1 & -2 & -5 & 4 & -1 & -1 \\ 1 & -2 & -3 & -9 & 7 & -2 & -2 \\ -1 & -1 & 2 & 3 & -2 & 1 & 1 \\ 0 & 2 & -1 & -1 & 0 & -1 & -1 \end{pmatrix}.$$

Exercise 7. $(\mathbf{6}\times\mathbf{6})$; $(\mathbf{J_5(0)}, \mathbf{J_1(0)})$.

$$B = \begin{pmatrix} 1 & 2 & -1 & 0 & 2 & 1 \\ -1 & -1 & 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 2 & 1 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Exercise 8. $(\mathbf{6}\times\mathbf{6})$; $(\mathbf{J_4(0)}, \mathbf{J_2(0)})$.

$$B = \begin{pmatrix} 2 & -2 & -1 & 2 & 1 & 1 \\ 4 & -5 & -3 & 5 & 1 & 2 \\ -1 & 2 & 1 & -3 & 1 & 0 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 3 & -4 & -2 & 2 & 0 & 1 \end{pmatrix}.$$

Exercise 9. $(\mathbf{6}\times\mathbf{6})$; $(\mathbf{J_3(0)}, \mathbf{J_3(0)})$.

$$B = \begin{pmatrix} 3 & 3 & -1 & 2 & 2 & 1 \\ -5 & -5 & 1 & -3 & -3 & -1 \\ -12 & -10 & 3 & -8 & -5 & -3 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -4 & -4 & 1 & -3 & -2 & -1 \end{pmatrix}.$$

Exercise 10. **(5×5); ($J_4(0), J_1(0)$).**

$$B = \begin{pmatrix} 0 & 2 & -1 & -4 & -2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & -2 & 0 \end{pmatrix}.$$

Exercise 11. **(5×5); ($J_3(0), J_2(0)$).**

$$B = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 & 1 \\ 3 & -1 & 1 & 3 & 2 \\ -2 & 1 & -1 & -2 & -1 \\ 2 & -1 & 0 & 1 & 1 \end{pmatrix}.$$

Exercise 12. **(5×5); ($J_2(0), J_2(0), J_1(0)$).**

$$B = \begin{pmatrix} 1 & 0 & 1 & -1 & -2 \\ -1 & 0 & -1 & 1 & 2 \\ 1 & -1 & 0 & 0 & -2 \\ 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 & -2 \end{pmatrix}.$$

Exercise 13. **(3×3); ($J_2(0), J_1(0)$).**

$$B = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Exercise 14. **(3×3); ($J_3(0)$).**

$$B = \begin{pmatrix} -2 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & -1 & 2 \end{pmatrix}.$$

Exercise 15. **(2×2); ($J_2(0)$).**

$$B = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Exercise 16. **(8×8); ($J_4(0), J_2(0), J_2(0)$).**

$$B = \begin{pmatrix} 10 & -1 & 9 & -11 & 6 & 3 & -5 & -9 \\ 20 & 0 & 19 & -20 & 16 & 4 & -15 & -19 \\ -5 & 0 & -5 & 5 & -4 & -1 & 4 & 5 \\ 6 & -1 & 5 & -7 & 3 & 2 & -2 & -5 \\ -3 & 0 & -3 & 3 & -2 & -1 & 2 & 3 \\ -23 & 1 & -21 & 24 & -15 & -6 & 13 & 20 \\ 21 & -1 & 19 & -22 & 15 & 5 & -13 & -19 \\ -25 & 1 & -23 & 26 & -18 & -6 & 16 & 23 \end{pmatrix}.$$

Exercise 17. $(\mathbf{3} \times \mathbf{3})$; $(\mathbf{J}_3(\mathbf{0}))$.

$$B = \begin{pmatrix} 3 & 2 & -1 \\ -3 & -2 & 1 \\ 2 & 1 & -1 \end{pmatrix}.$$

Exercise 18. $(\mathbf{5} \times \mathbf{5})$; $(\mathbf{J}_3(\mathbf{0}), \mathbf{J}_2(\mathbf{0}))$.

$$B = \begin{pmatrix} 2 & 1 & 1 & -2 & -1 \\ -2 & -1 & 0 & 1 & 2 \\ -3 & -1 & -1 & 2 & 2 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

Exercise 19. $(\mathbf{4} \times \mathbf{4})$; $(\mathbf{J}_4(\mathbf{0}))$.

$$B = \begin{pmatrix} -1 & 1 & -1 & 0 \\ 7 & -4 & 9 & 3 \\ 3 & -2 & 4 & 1 \\ 3 & -2 & 3 & 1 \end{pmatrix}.$$

Exercise 23. $(\mathbf{7} \times \mathbf{7})$; $(\mathbf{J}_2(\mathbf{1}), \mathbf{J}_2(\mathbf{1}), \mathbf{J}_3(-\mathbf{1}))$.

$$B = \begin{pmatrix} 0 & 2 & 0 & -2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 4 & 3 & -1 & 0 \\ 1 & 0 & 0 & -4 & -4 & 2 & 0 \\ -1 & 1 & 1 & -3 & -2 & 0 & 1 \\ -1 & 1 & -2 & -2 & -1 & 0 & -1 \end{pmatrix}.$$

Exercise 24. $(\mathbf{7} \times \mathbf{7})$; $(\mathbf{J}_3(\mathbf{1}), \mathbf{J}_2(\mathbf{1}), \mathbf{J}_2(\mathbf{2}))$.

$$B = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 1 & 1 & -1 & -2 \\ -2 & -2 & 3 & 1 & -1 & -3 & -1 \\ 1 & 1 & -1 & 1 & 1 & 2 & 1 \\ -1 & 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Exercise 25. $(\mathbf{7 \times 7})$; $(\mathbf{J_2(2), J_2(2), J_1(2), J_2(1)})$.

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & -2 & 1 & 2 & 1 \\ -3 & -1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 1 & -2 & -1 \\ 4 & 1 & -1 & 1 & -1 & -1 & -1 \\ -15 & -4 & 2 & -5 & 4 & 10 & 6 \end{pmatrix}.$$

Exercise 26. $(\mathbf{7 \times 7})$; $(\mathbf{J_4(1), J_1(1), J_2(-1)})$.

$$B = \begin{pmatrix} -2 & 1 & -2 & -1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 & 0 & -1 & 1 \\ 2 & 0 & 2 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 2 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Exercise 27. $(\mathbf{6 \times 6})$; $(\mathbf{J_3(-1), J_1(-1), J_2(1)})$.

$$B = \begin{pmatrix} -1 & -2 & 3 & 2 & -4 & 2 \\ 0 & -1 & -4 & -3 & -2 & -3 \\ 0 & 0 & 2 & 2 & 1 & 2 \\ 0 & 4 & -2 & -4 & 4 & -3 \\ 0 & 0 & 3 & 2 & 0 & 2 \\ 0 & -8 & 0 & 3 & -10 & 2 \end{pmatrix}.$$

Exercise 28. $(\mathbf{6 \times 6})$; $(\mathbf{J_2(-1), J_2(-1), J_2(1)})$.

$$B = \begin{pmatrix} -4 & 5 & 0 & 3 & 8 & 6 \\ -8 & 5 & 2 & -1 & 15 & 8 \\ -6 & 8 & -1 & 3 & 14 & 9 \\ 7 & -3 & -2 & 1 & -11 & -6 \\ 9 & -5 & -2 & 2 & -16 & -8 \\ -13 & 9 & 2 & 0 & 23 & 13 \end{pmatrix}.$$

Exercise 29. $(\mathbf{6} \times \mathbf{6})$; $(\mathbf{J_3(-1)}, \mathbf{J_1(-1)}, \mathbf{J_2(1)})$.

$$B = \begin{pmatrix} -7 & 7 & 1 & 2 & 13 & 8 \\ -8 & 5 & 2 & -1 & 15 & 8 \\ -5 & 7 & -1 & 3 & 12 & 8 \\ 10 & -5 & -3 & 2 & -16 & -8 \\ 9 & -5 & -2 & 2 & -16 & -8 \\ -16 & 11 & 3 & -1 & 28 & 15 \end{pmatrix}.$$

Exercise 30. $(\mathbf{6} \times \mathbf{6})$; $(\mathbf{J_2(-1)}, \mathbf{J_2(-1)}, \mathbf{J_2(1)})$.

$$B = \begin{pmatrix} -4 & 5 & 0 & 3 & 8 & 6 \\ -8 & 5 & 2 & -1 & 15 & 8 \\ -6 & 8 & -1 & 3 & 14 & 9 \\ 7 & -3 & -2 & 1 & -11 & -6 \\ 9 & -5 & -2 & 2 & -16 & -8 \\ -13 & 9 & 2 & 0 & 23 & 13 \end{pmatrix}.$$

Exercise 31. $(\mathbf{5} \times \mathbf{5})$; $(\mathbf{J_3(-1)}, \mathbf{J_2(2)})$.

$$B = \begin{pmatrix} -3 & 0 & -5 & 4 & 1 \\ -4 & 2 & -4 & -1 & -1 \\ 2 & 1 & 3 & -5 & 1 \\ 0 & -1 & 1 & 1 & -2 \\ 3 & -2 & 5 & 0 & -2 \end{pmatrix}.$$

Exercise 32. $(\mathbf{4} \times \mathbf{4})$; $(\mathbf{J_2(-2)}, \mathbf{J_2(2)})$.

$$B = \begin{pmatrix} 10 & 5 & 3 & 4 \\ -12 & -7 & -3 & -4 \\ -8 & -5 & -1 & -4 \\ -5 & -1 & -4 & -2 \end{pmatrix}.$$

Exercise 33. $(\mathbf{3} \times \mathbf{3})$; $(\mathbf{J_3(-2)})$.

$$B = \begin{pmatrix} -4 & 0 & 1 \\ -3 & -3 & 2 \\ -5 & -1 & 1 \end{pmatrix}.$$

Exercise 34. $(\mathbf{3 \times 3})$; $(\mathbf{J_2(3), J_1(-2)})$.

$$B = \begin{pmatrix} 10 & 1 & 5 \\ -14 & 1 & -10 \\ -12 & -1 & -7 \end{pmatrix}.$$

Part 1
Notation

1. Elementary Jordan matrices or block matrices.

A matrix of the form

$$\begin{pmatrix} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \end{pmatrix}$$

is called an elementary Jordan matrix or an elementary Jordan block with eigenvalue λ.

The eigenvalue can be any number of the field **K** in which we work. In the exercises that follow we will always work in the field **Q**.

The eigenvalue is on all the diagonal cells of the matrix and the superdiagonal consists of 1's. Several sizes are possible but the matrices have to be square matrices.

The colouring of the matrix is done to emphasise the position of the elementary Jordan blocks in the matrix, see the following section. The reader may drop this colouring mentally. We believe however that the colouring makes the structure of Jordan matrices much clearer.

A 1×1 block is also allowed. It has then the eigenvalue on the diagonal and there is no superdiagonal in that case.

We will use the notation $J_n(\lambda)$ for an elementary Jordan block. The number n indicates the size of the elementary Jordan block, the number λ indicates the eigenvalue associated to the elementary Jordan block.

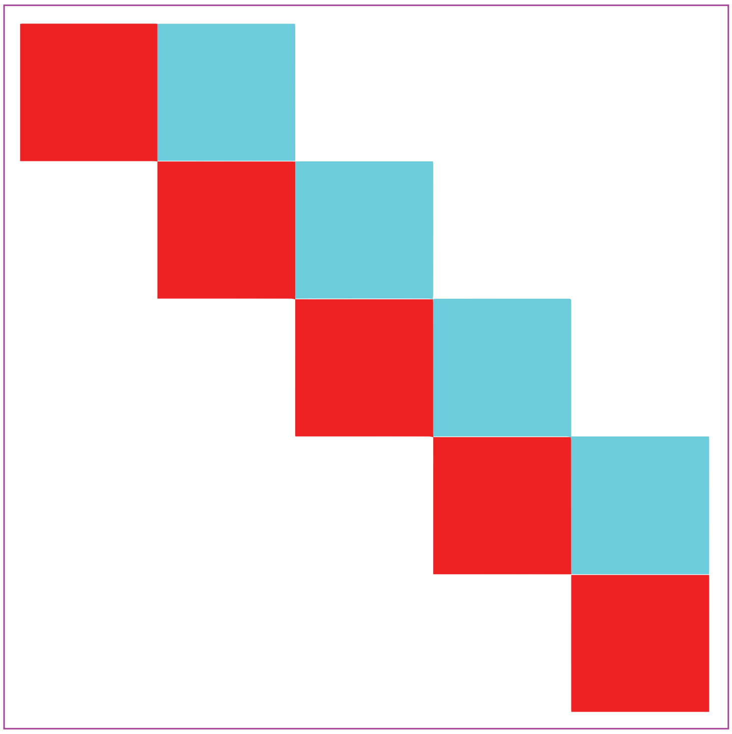


Figure 1. This is a figure representing an elementary Jordan block. It is the graphical representation of a 5×5 elementary Jordan block. Its diagonal is represented by the red squares. The red cells contain the eigenvalue λ. The superdiagonal is represented by the blue squares. The blue cells all contain the number 1. All cells in the white area contain the number 0.

2. Large Jordan matrices or block matrices of one eigenvalue.

A matrix of the following form is a more general Jordan matrix. All these matrices have elementary Jordan block matrices on their diagonal. These matrices have all one and the same eigenvalue. The following matrix is a large block Jordan matrix with only one eigenvalue $\lambda = 3$. If all elementary blocks have the same eigenvalue, one calls this type of matrix sometimes a large Jordan block with eigenvalue λ.

$$\begin{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \end{pmatrix}.$$

We see that the matrix consists of two elementary Jordan blocks on the diagonal of sizes 3×3 and 2×2. Other sizes or other numbers of Jordan blocks are also allowed. We remark the eigenvalue 3 appearing on the diagonals and the superdiagonals consisting of the number 1 in each of the elementary Jordan blocks. Other eigenvalues are also allowed.

We have done the colouring and additional coloured brackets to emphasise the position of the elementary Jordan blocks.

Sometimes one uses in the literature the abbreviation

$$A = \begin{pmatrix} J_3(\lambda = 0) & 0 \\ 0 & J_2(\lambda = 0) \end{pmatrix}.$$

This is a shorthand for the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We will try to make the structure of the matrix more clear and colour the positions of the elementary Jordan blocks

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

3. General Jordan matrices.

We look now at a Jordan matrix of the most general form. These are matrices consisting of elementary Jordan submatrices on the diagonal with possibly more then one eigenvalue. An example of this most general form is the following.

$$\begin{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix}.$$

This matrix consists of one elementary Jordan block of size 3×3 with eigenvalue $\lambda = 1$, one elementary Jordan block of size 2×2 with eigenvalue $\lambda = 1$ and one elementary Jordan block of size 2×2 with eigenvalue $\lambda = 2$.

In the literature one uses a shorthand for this

$$A = \begin{pmatrix} J_3(\lambda = 1) & 0 & 0 \\ 0 & J_2(\lambda = 1) & 0 \\ 0 & 0 & J_2(\lambda = 2) \end{pmatrix}.$$

In this notation, all cells are matrix blocks.

We remark that k elementary Jordan blocks on the diagonal are allowed, and that the formats of these square matrices can be of any size n_i. So in general this is

$$A = \begin{pmatrix} J_{n_1}(\lambda_1) & & & \\ & J_{n_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{n_k}(\lambda_k) \end{pmatrix}.$$

It is understood that in this notation empty cells have to be filled with blocks of 0's. The eigenvalues λ_i do not have to be different.

4. Notation for vectors.

We will write vectors in $\mathbf{K}^n$ with the shortcut $\mathbf{v}$ in bold letters. We write the coordinates in two different styles. In full text, we will of course write e.g. $\mathbf{v} = (1, 2, 3)$. In mathematical context we will write, if A is a matrix, $A\,\mathbf{v}$ where $\mathbf{v}$ is then obviously written in column form. We believe that this convention cannot lead to confusion. The explicit transposition notation leads to possible notational overkill.

5. Linear transformations.

We call a linear map from a vector space V to the same vector space V a linear transformation or a linear operator. We work exclusively in the coordinate space relative to a basis. In that context we will even say that the matrix A representing the linear map relative to a basis will map a vector to another vector. We believe that this language cannot hurt the reasoning though it is not optimal language from the theoretical point of view.

6. Theorem of Jordan.

Theorem of Jordan. Let $\mathbf{K}$ be a field. Let $B \in \mathbf{K}^{n\times n}$. Assume that the characteristic polynomial of B is a product of first degree polynomials defined over $\mathbf{K}$. Then there exists a Jordan matrix A and an invertible matrix $P \in \mathbf{K}^{n\times n}$ such that $A = P^{-1}\,B\,P$. This matrix is unique upon the ordering of the elementary Jordan blocks.

Remark. Many readers will work in the field of complex numbers $\mathbf{C}$. If this is the case then we can drop the assumption that the characteristic polynomial is a product of first degree polynomials. This condition is satisfied for every polynomial defined over the field $\mathbf{C}$.

7. Nilpotency of matrices, Jordan chains and height of vectors.

A matrix B is called a nilpotent matrix if there is some natural number n_0 such $B^{n_0} = 0$. The smallest n_0 for which this is the case is called the height of nilpotency of that matrix.

Let $\mathbf{v}$ be in a vector space V. Let A be a linear transformation on that space that is nilpotent. Consider the set

$$\{A^{n_0-1}\,\mathbf{v}, A^{n_0-2}\,\mathbf{v}, A^{n_0-3}\,\mathbf{v}, \ldots, A^2\,\mathbf{v}, A\,\mathbf{v}, \mathbf{v}\}.$$

Suppose now that no vector in this set is a zero vector. Suppose also that $A^{n_0}\,\mathbf{v} = \mathbf{0}$. Then we call the set a Jordan chain of length n_0 with respect to the nilpotent operator. Because it is clear in every exercise which operator is meant, we call it shortly a Jordan chain without mentioning the nilpotent operator A.

We follow the tradition that we write the Jordan chain in reversed order. The order seems at first glance to be rather strange. But it will be clear in the exercises why this is done. It makes mistakes in constructing a certain matrix almost impossible.

The vector $\mathbf{v}$ is called the **generating** vector of the chain. It is also called sometimes the **leading** vector of the chain. Remark that this vector is traditionally written at the back of the chain. The word “generating” has two meanings. We can say that a vector is “generating for a Jordan chain” and also that a vector “generates a vector space”. There are obviously two different meanings for that word in these sentences. But we use this term in this text almost always in the first meaning.

The **height** of a nonzero vector is equal to the length of the Jordan chain generated by that vector which is in that case considered a finite chain. It is thus the smallest natural number i for which $A^i\,\mathbf{v} = \mathbf{0}$. The height of a vector may not be confused with the height of nilpotency of a matrix.

8. Vector spaces.

We will invariably work in the vector spaces $\mathbf{K^n}$. This is also the general case by the isomorphism theorem of finite dimensional spaces. We will use the traditional notation $\mathbf{e_i}$ for the canonical basis vectors of $\mathbf{K^n}$.

9. About the span of vectors.

We have in this text taken the convention that the notation **span** is always used in the context of taking a minimal span. This means that the vectors in a span are always carefully chosen to be also linearly independent in these notes. The consequence is that the vectors in every span are a **basis** for the subspace they span. We do not mention this fact explicitly every time it is used.

10. About the name Jordan.

Camille Jordan was a French mathematician and is the discoverer of the technique with which we will work in this text. He is sometimes confused with two other German mathematicians. The first one is Wilhelm Jordan who is famous for the Gauss-Jordan row reducing algorithm. The second one is Pascual Jordan who is famous for the Jordan algebras. Note that the French name and the German name have a different pronunciation of the letter “J”. In French the “J” is pronounced in the same way as in the English name “Jordan”. In German the “J” is pronounced like the “y” in the English word “you”.

Part 2
Nilpotent matrices

1 exercise. (4×4); $(\mathbf{J_4(0)})$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is a matrix in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 1 & -2 & 3 & 1 \\ 4 & -5 & 7 & 2 \\ 3 & -3 & 4 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial $p_{\text{C-H}}(\lambda)$.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_4| = \lambda^4.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 4. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field $\mathbf{K}$. It is thus possible to put the matrix B in Jordan normal form over the field $\mathbf{K}$.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 2. The solution will be completely independent from this section.

We want to investigate the endomorphism A associated with this Jordan matrix. We compute also the powers of A.

$$A = \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right), \qquad A^2 = \left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right),$$

$$A^3 = \left(\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right), \qquad A^4 = \left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right).$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the fourth power of A. The matrix itself is already an elementary Jordan block. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 4, is called the height of nilpotency of the matrix A.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A) = \operatorname{span}\{\mathbf{e}_1\}; \\ \ker(A^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}; \\ \ker(A^3) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}; \\ \ker(A^4) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following information table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	1	$1 = \dim(\ker(A))$
$\ker(A^2)$	2	$1 = \dim(\ker(A^2)) - \dim(\ker(A))$
$\ker(A^3)$	3	$1 = \dim(\ker(A^3)) - \dim(\ker(A^2))$
$\ker(A^4)$	4	$1 = \dim(\ker(A^4)) - \dim(\ker(A^3))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.
2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.
3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.
4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.
5. The “remaining dim” give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the fourth power onwards.

$$\ker(A) \subsetneq \ker(A^2) \subsetneq \ker(A^3) \subsetneq \ker(A^4) = \ker(A^5) = \cdots .$$

This chain of inclusions stabilises from the fourth power onwards.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 + n_3 + n_4 = 1 = \dim(\ker(A)), \\ \qquad n_2 + n_3 + n_4 = 1 = \dim(\ker(A^2)) - \dim(\ker(A)), \\ \qquad\qquad n_3 + n_4 = 1 = \dim(\ker(A^3)) - \dim(\ker(A^2)), \\ \qquad\qquad\qquad n_4 = 1 = \dim(\ker(A^4)) - \dim(\ker(A^3)). \end{cases}$$

Solving this system, we have $n_1 = 0$, $n_2 = 0$, $n_3 = 0$, $n_4 = 1$.

A consequence from this fact is that the numbers in the last column are descending.

We remark by looking at the matrices A^i that we have the following mappings

$$\begin{cases} A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A\,\mathbf{e}_3 = \mathbf{e}_2, \\ A\,\mathbf{e}_2 = \mathbf{e}_1, \\ A\,\mathbf{e}_1 = \mathbf{0}, \end{cases}$$

or

$$\begin{cases} A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A^2\,\mathbf{e}_4 = \mathbf{e}_2, \\ A^3\,\mathbf{e}_4 = \mathbf{e}_1, \\ A^4\,\mathbf{e}_4 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = A^3\,\mathbf{e}_4, \mathbf{e}_2 = A^2\,\mathbf{e}_4, \mathbf{e}_3 = A\,\mathbf{e}_4, \mathbf{e}_4\}.$$

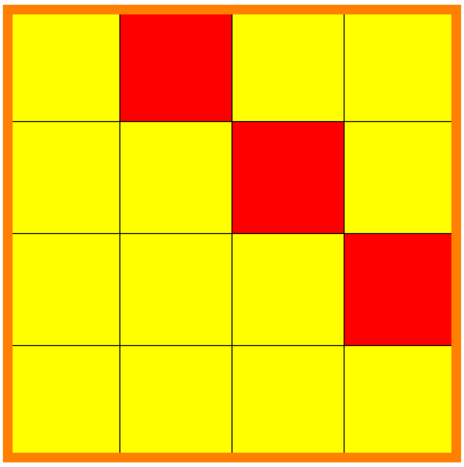
After we have found the first Jordan chain of length 4, we have then the following table.

Keeping track of chains and dimensions.

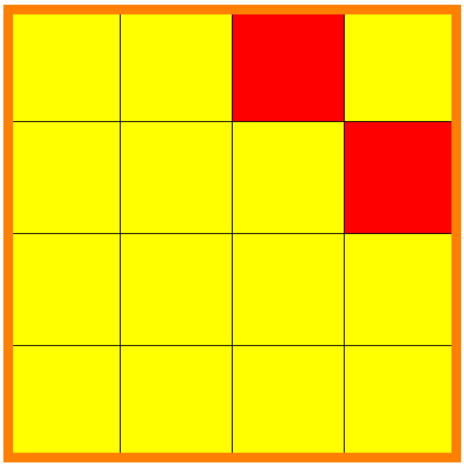
	dim	chain 1	remaining dim
$\ker(A)$	1	$\mathbf{e_1}$	0
$\ker(A^2)$	2	$\mathbf{e_2}$	0
$\ker(A^3)$	3	$\mathbf{e_3}$	0
$\ker(A^4)$	4	$\mathbf{e_4}$	0

Because the last column consists now of 0's, we are done with looking for Jordan chains. We have found at this point a basis of V consisting entirely of vectors in Jordan chains.

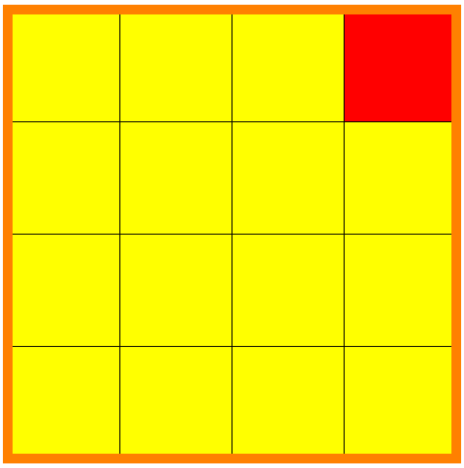
Let us express our findings in a more visual way that emphasises the dimensions and positions of the elementary Jordan blocks in the matrix.



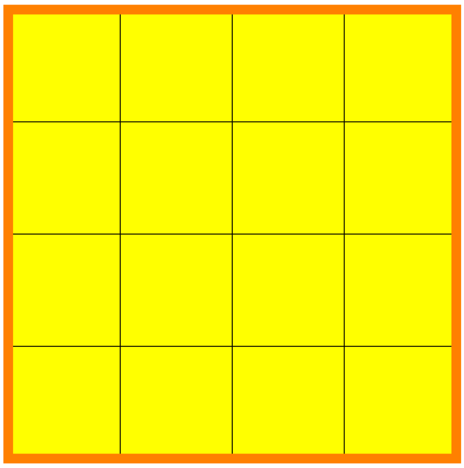
(a) Visualisation of A.



(b) Visualisation of A^2.



(c) Visualisation of A^3.



(d) Visualisation of A^4.

Figure 2

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up

into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

Let us calculate the powers of the matrix B.

$$B = \begin{pmatrix} 1 & -2 & 3 & 1 \\ 4 & -5 & 7 & 2 \\ 3 & -3 & 4 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}; \qquad B^2 = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 3 & -4 & 5 & 1 \\ 2 & -3 & 4 & 1 \\ -1 & 2 & -3 & -1 \end{pmatrix};$$

$$B^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{pmatrix}; \qquad B^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} 1 & -2 & 3 & 1 \\ 4 & -5 & 7 & 2 \\ 3 & -3 & 4 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} z_1 - 2z_2 + 3z_3 + z_4 = 0, \\ 4z_1 - 5z_2 + 7z_3 + 2z_4 = 0, \\ 3z_1 - 3z_2 + 4z_3 + z_4 = 0, \\ -z_1 = 0. \end{cases}$$

This system can be immediately solved and this gives us the solution set or space

$$\ker(B) = \{(0, r_2, r_2, -r_2) \mid r_2 \in \mathbf{K}\} = \text{span}\{(0, 1, 1, -1)\}.$$

Kernel of B^2.
We calculate the kernel of B^2 and we have to solve the matrix equation

$$B^2 = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 3 & -4 & 5 & 1 \\ 2 & -3 & 4 & 1 \\ -1 & 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} z_1 - z_2 + z_3 = 0, \\ 3z_1 - 4z_2 + 5z_3 + z_4 = 0, \\ 2z_1 - 3z_2 + 4z_3 + z_4 = 0, \\ -z_1 + 2z_2 - 3z_3 - z_4 = 0. \end{cases}$$

This system can be immediately solved and this gives us the solution set or space

$$\begin{aligned} \ker(B^2) &= \{(r_1, r_2, -r_1 + r_2, 2r_1 - r_2) \mid r_1, r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, 0, -1, 2), (0, 1, 1, -1)\}. \end{aligned}$$

Kernel of B^3.
We calculate the kernel of B^3 and we have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} z_1 - z_2 + z_3 = 0, \\ z_1 - z_2 + z_3 = 0, \\ -z_1 + z_2 - z_3 = 0. \end{cases}$$

This system can be immediately solved and this gives us the solutions set

$$\begin{aligned} \ker(B^3) &= \{(r_1, r_2, -r_1 + r_2, r_4) \mid r_1, r_2, r_4 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, 0, -1, 0), (0, 1, 1, 0), (0, 0, 0, 1)\}. \end{aligned}$$

Kernel of B^4.
We want to calculate the kernel of B^4 and we observe first that

$$B^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So the kernel is $\mathbf{K}^4$.

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
ker(B)	1	$1 = \dim(\ker(B))$
ker(B^2)	2	$1 = \dim(\ker(B^2)) - \dim(\ker(B))$
ker(B^3)	3	$1 = \dim(\ker(B^3)) - \dim(\ker(B^2))$
ker(B^4)	4	$1 = \dim(\ker(B^4)) - \dim(\ker(B^3))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.
2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.
3. In the first row, we have $\dim(\ker(B))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.
4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.
5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column is 1 and this gives the information that there will be a Jordan chain of length 4. The first number 1 in the last column is the dimension of the kernel of B. After we have calculated this chain, the last column will be $\{0, 0, 0, 0\}$. There are no linearly independent vectors left to be found. We have indeed at that moment already 4 linearly independent vectors which form a base for this vector space.

4. Calculation of the Jordan chains.

The first Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ satisfying

$$\begin{cases} B\,\mathbf{w}_1 = \mathbf{0}, \\ B\,\mathbf{w}_2 = \mathbf{w}_1, \\ B\,\mathbf{w}_3 = \mathbf{w}_2, \\ B\,\mathbf{w}_4 = \mathbf{w}_3 \end{cases}$$

or

$$\begin{cases} B^4\,\mathbf{w}_4 = \mathbf{0}, \\ B^3\,\mathbf{w}_4 = \mathbf{w}_1, \\ B^2\,\mathbf{w}_4 = \mathbf{w}_2, \\ B\,\mathbf{w}_4 = \mathbf{w}_3 \end{cases}$$

where $\mathbf{w}_4$ is in the vector space $\ker(B^4)$ but not in $\ker(B^3)$.

We look for a starting $\mathbf{w}_4$. We see from the information table that there is a generating vector $\mathbf{w}_4$ for a chain of length 4.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^4)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^4)$.

2. The generating vector may not be in the $\ker(B^3)$ because the length of the chain must be exactly 4. So it has to be independent from all vectors in $\ker(B^3)$. It is sufficient that it is linearly independent from a basis of $\ker(B^3)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 4. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^4 together with the vectors in $\ker(B^3)$ and also the vectors, if any, of exactly height 4 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that the kernel $\ker(B^4)$ is generated by the following set of vectors.

$$\ker(B^4) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}.$$

So the generating vector has the following generic form

$$a\,\mathbf{e}_1 + b\,\mathbf{e}_2 + c\,\mathbf{e}_3 + d\,\mathbf{e}_4 = (a, b, c, d).$$

The kernel of B^3.
We remember that the kernel $\ker(B^3)$ is

$$\ker(B^3) = \operatorname{span}\{(1, 0, -1, 0), (0, 1, 1, 0), (0, 0, 0, 1)\}.$$

Vectors chosen in previous Jordan chains.
We have previously not chosen any vector in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{pmatrix}.$$

If we impose the condition $a - b + c \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We conclude that if we impose that $a - b + c \neq 0$, then these vectors are certainly linearly independent.

We can choose $a = 1$, $b = 0$, $c = 0$, $d = 0$ and we have the valid generating vector

$$\mathbf{w_4} = (1, 0, 0, 0).$$

We calculate $B(1,0,0,0)$, $B^2(1,0,0,0)$, $B^3(1,0,0,0)$ and we know that the set of vectors

$$\{\mathbf{w_1} = B^3\,\mathbf{w_4}, \mathbf{w_2} = B^2\,\mathbf{w_4}, \mathbf{w_3} = B\,\mathbf{w_4}, \mathbf{w_4}\}$$

is a Jordan chain of 4 linearly independent vectors and because $B^4\,\mathbf{w_4} = \mathbf{0}$ we know that the length of the chain is exactly 4.

We calculate $\mathbf{w_3}$.

$$\mathbf{w_3} = B\,\mathbf{w_4} = \begin{pmatrix} 1 & -2 & 3 & 1 \\ 4 & -5 & 7 & 2 \\ 3 & -3 & 4 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \\ -1 \end{pmatrix}$$

and

$$\mathbf{w_2} = B^2\,\mathbf{w_4} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 3 & -4 & 5 & 1 \\ 2 & -3 & 4 & 1 \\ -1 & 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ -1 \end{pmatrix}$$

and

$$\mathbf{w_1} = B^3\,\mathbf{w_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}.$$

We have now found the Jordan chain

$$\{\mathbf{w_1} = (0, 1, 1, -1), \mathbf{w_2} = (1, 3, 2, -1), \mathbf{w_3} = (1, 4, 3, -1), \mathbf{w_4} = (1, 0, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	1	$\mathbf{w_1} = (0, 1, 1, -1)$	0
$\ker(B^2)$	2	$\mathbf{w_2} = (1, 3, 2, -1)$	0
$\ker(B^3)$	3	$\mathbf{w_3} = (1, 4, 3, -1)$	0
$\ker(B^4)$	4	$\mathbf{w_4} = (1, 0, 0, 0)$	0

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 3 & 4 & 0 \\ 1 & 2 & 3 & 0 \\ -1 & -1 & -1 & 0 \end{pmatrix}.$$

$$A = P^{-1} B P$$

$$= \begin{pmatrix} 0 & -1 & 1 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & -1 & 2 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 & 1 \\ 4 & -5 & 7 & 2 \\ 3 & -3 & 4 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 3 & 4 & 0 \\ 1 & 2 & 3 & 0 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right).$$

Remark that the matrix A is itself an elementary Jordan block.

2 exercise. (4×4); $(\mathbf{J_3(0), J_1(0)})$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is a matrix in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial $p_{\text{C-H}}(\lambda)$.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_4| = \lambda^4.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 4. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field $\mathbf{K}$. It is thus possible to put the matrix B in Jordan normal form over the field $\mathbf{K}$.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 2. The solution will be completely independent from this section.

We will investigate here the matrices A^i.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We show the matrix in a way that emphasises the positions of the elementary Jordan blocks.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & (\mathbf{0}) \end{pmatrix}.$$

We see that we have a matrix consisting of an elementary Jordan three block and an elementary Jordan one block.

We compute also the powers of A.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & (\mathbf{0}) \end{pmatrix}; \qquad A^2 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & (\mathbf{0}) \end{pmatrix};$$

$$A^3 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & (\mathbf{0}) \end{pmatrix}.$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the third power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 3, is called the height of nilpotency of the matrix A.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_4\}, \\ \ker(A^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4\}, \\ \ker(A^3) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	2	$2 = \dim(\ker(A))$
$\ker(A^2)$	3	$1 = \dim(\ker(A^2)) - \dim(\ker(A))$
$\ker(A^3)$	4	$1 = \dim(\ker(A^3)) - \dim(\ker(A^2))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.
2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.
3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.
4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.
5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the third power onwards.

$$\ker(A) \subsetneq \ker(A^2) \subsetneq \ker(A^3) = \ker(A^4) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 + n_3 = 2 = \dim(\ker(A)), \\ \qquad n_2 + n_3 = 1 = \dim(\ker(A^2)) - \dim(\ker(A)), \\ \qquad\qquad n_3 = 1 = \dim(\ker(A^3)) - \dim(\ker(A^2)). \end{cases}$$

Solving this system, we have $n_1 = 1$, $n_2 = 0$, $n_3 = 1$.

A consequence from this fact is that the numbers in the last column are descending.

We remark by looking at the matrices A^i that the vector $\mathbf{e}_3$ satisfies

$$\begin{cases} A\,\mathbf{e}_3 = \mathbf{e}_2, \\ A\,\mathbf{e}_2 = \mathbf{e}_1, \\ A\,\mathbf{e}_1 = \mathbf{0} \end{cases}$$

or

$$\begin{cases} A\,\mathbf{e}_3 = \mathbf{e}_2, \\ A^2\,\mathbf{e}_3 = \mathbf{e}_1, \\ A^3\,\mathbf{e}_3 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain of three linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = A^2\,\mathbf{e}_3, \mathbf{e}_2 = A\,\mathbf{e}_3, \mathbf{e}_3\}.$$

After we have found the first Jordan chain of length 3, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
ker(A)	2	$\mathbf{e}_1$	1
ker(A^2)	3	$\mathbf{e}_2$	0
ker(A^3)	4	$\mathbf{e}_3$	0

We look at the last column and see now that we have a second Jordan chain. It has length 1. We have the chain

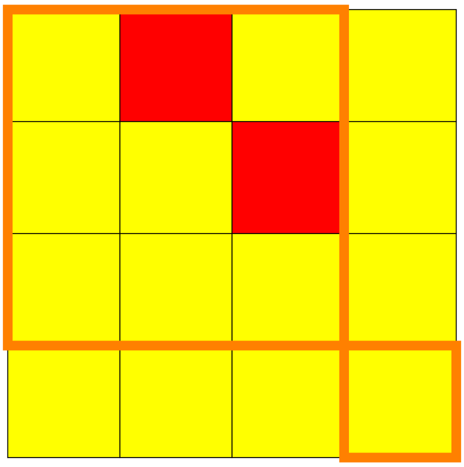
$$\{\mathbf{e}_4\}.$$

Keeping track of chains and dimensions.

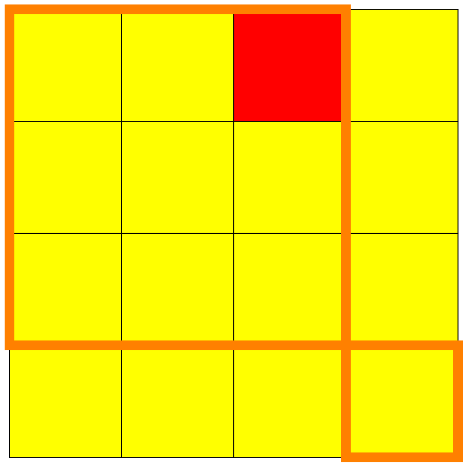
	dim	chain 1	chain 2	rem dim
ker(A)	2	$\mathbf{e}_1$	$\mathbf{e}_4$	0
ker(A^2)	3	$\mathbf{e}_2$		0
ker(A^3)	4	$\mathbf{e}_3$		0

The last column consists entirely out of 0's and we have found a basis for the vector space consisting of vectors in Jordan chains.

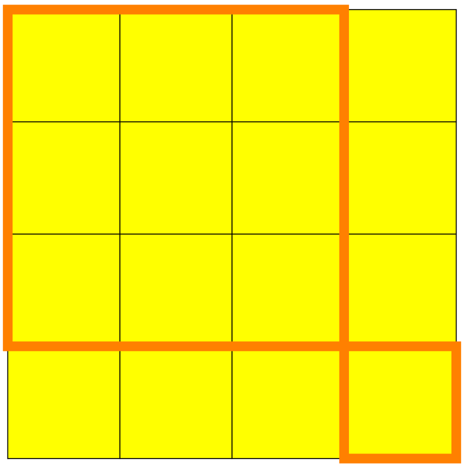
Let us express our findings in a way that emphasises the position of the elementary Jordan blocks and their relation with the powers of the matrix A.



(a) Visualisation of A.



(b) Visualisation of A^2.



(c) Visualisation of A^3.

Figure 3

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up

into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

$$B = \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}; \qquad B^2 = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$B^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} z_1 + z_2 - 2z_3 = 0, \\ z_1 \qquad - \ z_3 = 0, \\ z_1 \qquad - \ z_3 = 0, \\ \qquad z_2 - \ z_3 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\ker(B) = \{(r_1, r_1, r_1, r_4) \mid r_1, r_4 \in \mathbf{K}\} = \operatorname{span}\{(1,1,1,0), (0,0,0,1)\}.$$

Kernel of B²$^{}$.
We calculate the kernel of B^2 and we have to solve the matrix equation

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} z_2 - z_3 = 0, \\ z_2 - z_3 = 0, \\ z_2 - z_3 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B^2) &= \{(r_1, r_2, r_2, r_4) \mid r_1, r_2, r_4 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0), (0,1,1,0), (0,0,0,1)\}. \end{aligned}$$

Kernel of B³.
We calculate the kernel of B^3 and we have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The solutions set is the space $\mathbf{K}^4$.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	2	2
$\ker(B^2)$	3	1
$\ker(B^3)$	4	1

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have dim(ker(B)). This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

4. Calculation of the Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ satisfying

$$\begin{cases} B\,\mathbf{w}_1 = \mathbf{0}, \\ B\,\mathbf{w}_2 = \mathbf{w}_1, \\ B\,\mathbf{w}_3 = \mathbf{w}_2 \end{cases}$$

or

$$\begin{cases} B^3\,\mathbf{w}_3 = \mathbf{0}, \\ B^2\,\mathbf{w}_3 = \mathbf{w}_1, \\ B\,\mathbf{w}_3 = \mathbf{w}_2. \end{cases}$$

The first Jordan chain.
We look for a generating vector $\mathbf{w}_3$. We see from the information table that there is a generating vector $\mathbf{w}_3$ for a chain of length 3.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel ker(B^3). So we can describe such a vector as a linear combination of vectors of a basis of the subspace ker(B^3).

2. The generating vector may not be in the $\ker(B^2)$ because the length of the chain must be exactly 3. So it has to be independent from all vectors in $\ker(B^2)$. It is sufficient that it is linearly independent from a basis of $\ker(B^2)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 3. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^3 together with the vectors in $\ker(B^2)$ and also the vectors, if any, of exactly height 3 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that the kernel $\ker(B^3)$ is generated by the following set of vectors.

$$\ker(B^3) = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}.$$

So the generating vector has the following generic form

$$a\,\mathbf{e}_1 + b\,\mathbf{e}_2 + c\,\mathbf{e}_3 + d\,\mathbf{e}_4 = (a, b, c, d).$$

The kernel of $\ker(B^2)$**.**
We remember that the kernel $\ker(B^2)$ is

$$\ker(B^2) = \text{span}\{(1, 0, 0, 0), (0, 1, 1, 0), (0, 0, 0, 1)\}.$$

Vectors chosen in previous Jordan chains.
We have previously not chosen any vector in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{pmatrix}.$$

If we impose the condition $b - c \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We conclude that if we impose that $b - c \neq 0$, then these vectors are certainly linearly independent.

We can choose $a = 0$, $b = 1$, $c = 0$ and $d = 0$.
We have the valid generating vector

$$\mathbf{w_3} = (0, 1, 0, 0).$$

We calculate $\mathbf{w_2}$.

$$\mathbf{w_2} = B\,\mathbf{w_3} = \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and

$$\mathbf{w_1} = B^2\,\mathbf{w_3} = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

We have found the first Jordan chain.

$$\{\mathbf{w_1} = (1, 1, 1, 0), \mathbf{w_2} = (1, 0, 0, 1), \mathbf{w_3} = (0, 1, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	2	$\mathbf{w_1}$	1
$\ker(B^2)$	3	$\mathbf{w_2}$	0
$\ker(B^3)$	4	$\mathbf{w_3}$	0

with

$$\mathbf{w_1} = (1,1,1,0)$$
$$\mathbf{w_2} = (1,0,0,1)$$
$$\mathbf{w_3} = (0,1,0,0)$$

We know from the information table that we have one Jordan chain with length 1 left.

The second Jordan chain.
We look for a generating vector $\mathbf{w_4}$ of height 1.

The last column in the information table tells us that there is still a chain of length 1 left to be found. This is an eigenvector $\mathbf{w_4}$. We know that we have in our previous Jordan chain found an eigenvector $\mathbf{w_1} = (1,1,1,0)$. So we have to be careful when choosing another eigenvector.

The vector must be an eigenvector. It must be in the space

$$\ker(B) = \text{span}\{(1,1,1,0),(0,0,0,1)\}.$$

The vector must be of the generic form

$$a\,(1,1,1,0) + b\,(0,0,0,1) = (a,a,a,b).$$

We have at this point chosen in $\ker(B)$ already the following vector $\mathbf{w_1}$ of height 1.

$$\mathbf{w_1} = (1,1,1,0).$$

of height exactly height 1.

We have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 \\ a & a & a & b \end{pmatrix}.$$

We row reduce this matrix H and find then if we impose the condition that $b \neq 0$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $b \neq 0$.

So we can choose $a = 0$, $b = 1$.

We have then generating vector

$$\mathbf{w_4} = (0, 0, 0, 1).$$

This vector forms a Jordan chain on its own. We have now found the second Jordan chain

$$\{\mathbf{w_4} = (0, 0, 0, 1)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	rem dim
$\ker(B)$	2	$\mathbf{w_1}$	$\mathbf{w_4}$	0
$\ker(B^2)$	3	$\mathbf{w_2}$		0
$\ker(B^3)$	4	$\mathbf{w_3}$		0

with

$$\mathbf{w_1} = (1,1,1,0)$$

$$\mathbf{w_2} = (1,0,0,1)$$

$$\mathbf{w_3} = (0,1,0,0)$$

$$\mathbf{w_4} = (0,0,0,1)$$

The last column tells us that there is nothing left to be found. We have now a basis of vectors consisting of independent Jordan chains.

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

$$\begin{aligned} A &= P^{-1} B P \\ &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & \quad 0 & \quad 0 \end{matrix} & (\mathbf{0}) \end{pmatrix}. \end{aligned}$$

3 exercise. (4×4); $(\mathrm{J}_2(0), \mathrm{J}_2(0))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is a matrix in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial $p_{\text{C-H}}(\lambda)$.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_4| = \lambda^4.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 4. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field $\mathbf{K}$. It is thus possible to put the matrix B in Jordan normal form over the field $\mathbf{K}$.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 2. The solution will be completely independent from this section.

We will investigate here the matrices A^i.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We show the matrix in a way that emphasises the positions of the elementary Jordan blocks.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}.$$

We see that we have a matrix consisting of an elementary Jordan three block and an elementary Jordan one block.

We compute also the powers of A.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}; \qquad A^2 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}.$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the second power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 2, is called the height of nilpotency of the matrix A.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_3\}, \\ \ker(A^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	2	$2 = \dim(\ker(A))$
$\ker(A^2)$	4	$2 = \dim(\ker(A^2)) - \dim(\ker(A))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.
2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.
3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.
4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.
5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A) \subsetneq \ker(A^2) = \ker(A^3) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 = 2 = \dim(\ker(A)), \\ \quad\quad\; n_2 = 2 = \dim(\ker(A^2)) - \dim(\ker(A)). \end{cases}$$

Solving this system, we have $n_1 = 0$, $n_2 = 2$.

A consequence from this fact is that the numbers in the last column are descending.

We remark by looking at the matrices A^i that we have

$$\begin{cases} A\,\mathbf{e}_2 = \mathbf{e}_1, \\ A\,\mathbf{e}_1 = \mathbf{0} \end{cases}$$

and

$$\begin{cases} A\,\mathbf{e}_2 = \mathbf{e}_1, \\ A^2\,\mathbf{e}_2 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain of two linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = A\,\mathbf{e}_2, \mathbf{e}_2\}.$$

After we have found the first Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A)$	2	$\mathbf{e}_1$	1
$\ker(A^2)$	4	$\mathbf{e}_2$	1

We look at the last column and see now that we have still one Jordan chain of length 2.

We remark by looking at the matrices A^i that

$$\begin{cases} A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A\,\mathbf{e}_3 = \mathbf{0} \end{cases}$$

and

$$\begin{cases} A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A^2\,\mathbf{e}_4 = \mathbf{0}. \end{cases}$$

One sees that we have a second Jordan chain. It has two linearly independent vectors. We write a Jordan chain in reverse order.

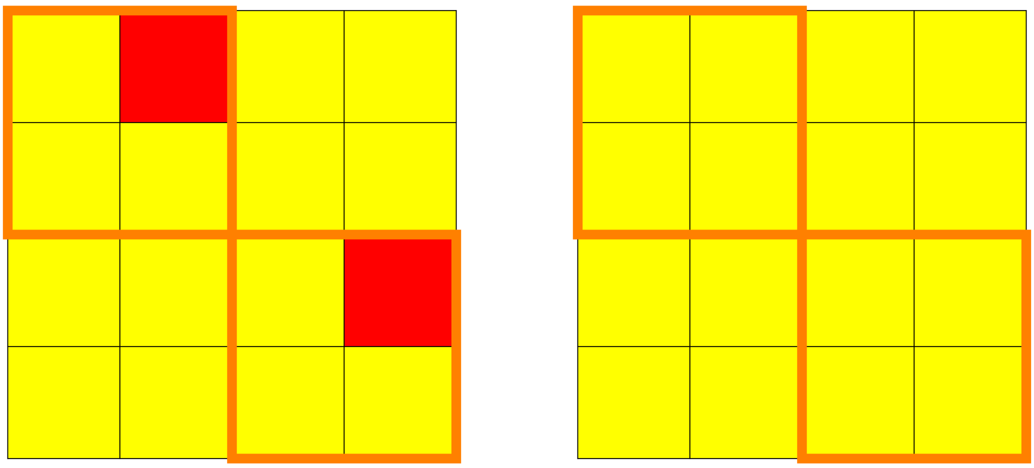
$$\{\mathbf{e}_3 = A\,\mathbf{e}_4, \mathbf{e}_4\}.$$

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	rem dim
$\ker(A)$	2	$\mathbf{e}_1$	$\mathbf{e}_3$	0
$\ker(A^2)$	4	$\mathbf{e}_2$	$\mathbf{e}_4$	0

The last column consists entirely out of 0's and we have found a basis for the vector space consisting of vectors in Jordan chains.

Let us express our findings in a way that emphasises the position of the elementary Jordan blocks and their relation with the powers of the matrix A.



(a) Visualisation of A. (b) Visualisation of A^2.

Figure 4

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{pmatrix}; \qquad B^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} z_1 + z_3 & = 0, \\ z_3 - z_4 & = 0, \\ -z_1 - z_3 & = 0, \\ -z_1 - z_3 & = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\ker(B) = \{(r_1, r_2, -r_1, -r_1) \mid r_1, r_2 \in \mathbf{K}\} = \text{span}\{(1,0,-1,-1),(0,1,0,0)\}.$$

Kernel of $\mathbf{B^2}$.
We calculate the kernel of B^2. This is the space

$$\ker(B^2) = \mathbf{K}^4.$$

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	2	2
$\ker(B^2)$	4	2

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.
2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.
3. In the first row, we have $\dim(\ker(B))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.
4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

4. Calculation of the Jordan chains.

We see from the information table that there is a generating vector $\mathbf{w}_2$ for a chain of length 2.

We look for a linearly independent set of vectors $\{\mathbf{w}_1, \mathbf{w}_2\}$ satisfying

$$\begin{cases} B\,\mathbf{w}_1 = \mathbf{0}, \\ B\,\mathbf{w}_2 = \mathbf{w}_1, \end{cases}$$

or

$$\begin{cases} B^2\,\mathbf{w}_2 = \mathbf{0}, \\ B\,\mathbf{w}_2 = \mathbf{w}_1. \end{cases}$$

The first Jordan chain.
We look for a generating vector $\mathbf{w}_2$.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^2)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B)$.

2. The generating vector may not be in the $\ker(B)$ because the length of the chain must be exactly 2. So it has to be independent from all vectors in $\ker(B)$. It is sufficient that it is linearly independent from a basis of $\ker(B)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 2. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^2 together with the vectors in $\ker(B)$ and also the vectors, if any, of exactly height 2 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that the kernel $\ker(B^2)$ is generated by the following set of vectors.

$$\ker(B^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\} = \mathbf{K}^4.$$

So the generating vector has the following generic form

$$a\,\mathbf{e}_1 + b\,\mathbf{e}_2 + c\,\mathbf{e}_3 + d\,\mathbf{e}_4 = (a, b, c, d).$$

The kernel of $\ker(B)$.
We remember that the kernel $\ker(B)$ is

$$\ker(B) = \operatorname{span}\{(1, 0, -1, -1), (0, 1, 0, 0)\}.$$

Vectors chosen in previous Jordan chains.
We have previously not chosen any vector in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ a & b & c & d \end{pmatrix}.$$

If we impose the condition $a + c \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \frac{d-c}{a+c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{a+d}{a+c} \end{pmatrix}.$$

We conclude that if we impose that $a + c \neq 0$, then these vectors are certainly linearly independent.

We can choose $a = 1$, $b = 0$, $c = 0$ and $d = 0$.
We have the valid generating vector

$$\mathbf{w}_2 = (1, 0, 0, 0).$$

We calculate $\mathbf{w}_1$.

$$\mathbf{w_1} = B\,\mathbf{w_2} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}.$$

We have now a Jordan chain

$$\{\mathbf{w_1} = (1, 0, -1, -1), \mathbf{w_2} = (1, 0, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	2	$\mathbf{w_1}$	1
$\ker(B^2)$	4	$\mathbf{w_2}$	1

with

$\mathbf{w_1} = (1, 0, -1, -1)$
$\mathbf{w_2} = (1, 0, 0, 0)$

We know from the information table that we have one Jordan chain with length 2 left.

The second Jordan chain.
We look for a generating vector $\mathbf{w_4}$ of height 2.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^2)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B)$.

2. The generating vector may not be in the $\ker(B)$ because the length of the chain must be exactly 2. So it has to be independent from all vectors in $\ker(B)$. It is sufficient that it is linearly independent from a basis of $\ker(B)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 2. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^2 together with the vectors in $\ker(B)$ and also the vectors, if any, of exactly height 2 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that the kernel $\ker(B^2)$ is generated by the following set of vectors.

$$\ker(B^2) = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}.$$

So the generating vector has the following generic form

$$a\,\mathbf{e}_1 + b\,\mathbf{e}_2 + c\,\mathbf{e}_3 + d\,\mathbf{e}_4 = (a, b, c, d).$$

The kernel of $\ker(B)$**.**
We remember that the kernel $\ker(B)$ is

$$\ker(B) = \text{span}\{(1, 0, -1, -1), (0, 1, 0, 0)\}.$$

Vectors chosen in previous Jordan chains.
We have previously chosen a vector $\mathbf{w}_2 = (1, 0, 0, 0)$ of exactly height 2 in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ a & b & c & d \end{pmatrix}.$$

If we impose the condition $c - d \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We conclude that if we impose that $c - d \neq 0$, then these vectors are certainly linearly independent.

So we can choose $a = 0$, $b = 0$, $c = 0$ and $d = 1$.

We have then generating vector

$$\mathbf{w}_4 = (0, 0, 0, 1).$$

We calculate $\mathbf{w}_3$.

$$\mathbf{w}_3 = B\,\mathbf{w}_4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

We have now a second Jordan chain. It has length 2.

$$\{\mathbf{w}_3 = (0, -1, 0, 0), \mathbf{w}_4 = (0, 0, 0, 1)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	rem dim
ker(B)	2	$\mathbf{w}_1$	$\mathbf{w}_3$	0
ker(B^2)	3	$\mathbf{w}_2$	$\mathbf{w}_4$	0

with

$$\mathbf{w_1} = (1, 0, -1, -1)$$

$$\mathbf{w_2} = (1, 0, 0, 0)$$

$$\mathbf{w_3} = (0, -1, 0, 0)$$

$$\mathbf{w_4} = (0, 0, 0, 1)$$

The last column tells us that there is nothing left to be found. We have now a basis of vectors consisting of independent Jordan chains.

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{aligned} A &= P^{-1} B P \\ &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}. \end{aligned}$$

4 exercise. (7×7); $(J_3(0), J_3(0), J_1(0))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is a matrix in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} -2 & 1 & 0 & -2 & 1 & 3 & 2 \\ 1 & -1 & 0 & 1 & -1 & -2 & -1 \\ 1 & -1 & 0 & 1 & -1 & -2 & -1 \\ 0 & -1 & 0 & 2 & 2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 2 & 2 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 1 & 5 & 5 & -1 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial $p_{\text{C-H}}(\lambda)$.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_7| = -\lambda^7.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 7. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field $\mathbf{K}$. It is thus possible to put the matrix B in Jordan normal form over the field $\mathbf{K}$.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 2. The solution will be completely independent from this section.

We want to investigate that endomorphism A. We compute the powers of A.

We can show these matrices as follows.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & (\mathbf{0}) \end{matrix} \end{pmatrix},$$

$$A^2 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & (\mathbf{0}) \end{matrix} \end{pmatrix},$$

$$A^3 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & (\mathbf{0}) \end{matrix} \end{pmatrix}.$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the third power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 3, is called the height of nilpotency of the matrix A.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\ker(A) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_7\};$$
$$\ker(A^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_7\};$$
$$\ker(A^3) = \mathbf{K}^7.$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	3	$3 = \dim(\ker(A))$
$\ker(A^2)$	5	$2 = \dim(\ker(A^2)) - \dim(\ker(A))$
$\ker(A^3)$	7	$2 = \dim(\ker(A^3)) - \dim(\ker(A^2))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The “remaining dim” give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the third power onwards.

$$\ker(A) \subsetneq \ker(A^2) \subsetneq \ker(A^3) = \ker(A^4) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 + n_3 = 3 = \dim(\ker(A)), \\ \qquad n_2 + n_3 = 2 = \dim(\ker(A^2)) - \dim(\ker(A)), \\ \qquad\qquad n_3 = 2 = \dim(\ker(A^3)) - \dim(\ker(A^2)). \end{cases}$$

Solving this system, we have $n_1 = 1$, $n_2 = 0$, $n_3 = 2$.

A consequence from this fact is that the numbers in the last column are descending.

We remark by looking at the matrices A^i that the vector $\mathbf{e}_3$ satisfies

$$\begin{cases} A\,\mathbf{e}_3 = \mathbf{e}_2, \\ A^2\,\mathbf{e}_3 = \mathbf{e}_1, \\ A^3\,\mathbf{e}_3 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain with length three of three linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = A^2\,\mathbf{e}_3, \mathbf{e}_2 = A\,\mathbf{e}_3, \mathbf{e}_3\}.$$

After we have found the first Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A)$	3	$\mathbf{e}_1$	2
$\ker(A^2)$	5	$\mathbf{e}_2$	1
$\ker(A^3)$	7	$\mathbf{e}_3$	1

We remark by looking at the matrices A^i that the vector $\mathbf{e}_6$ satisfies

$$\begin{cases} A\,\mathbf{e}_6 = \mathbf{e}_5, \\ A^2\,\mathbf{e}_6 = \mathbf{e}_4, \\ A^3\,\mathbf{e}_6 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain with length three of three linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_4 = A^2\,\mathbf{e}_6, \mathbf{e}_5 = A\,\mathbf{e}_6, \mathbf{e}_6\}.$$

We have found our second Jordan chain and we have the following information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	rem dim
$\ker(A)$	3	$\mathbf{e}_1$	$\mathbf{e}_4$	1
$\ker(A^2)$	5	$\mathbf{e}_2$	$\mathbf{e}_5$	0
$\ker(A^3)$	7	$\mathbf{e}_3$	$\mathbf{e}_6$	0

Because the last column consists now of at least one not 0, we are **not** done with looking for Jordan chains. We see that we find another Jordan chain with length 1. We are looking for a classical eigenvector. We find a third chain. It has length 1.

$$\{\mathbf{e}_7\}.$$

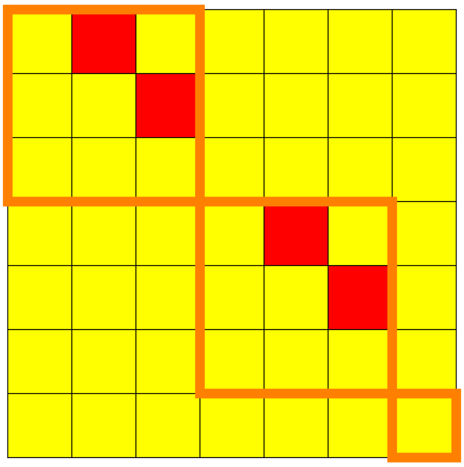
Let us note this in the new information table.

Keeping track of chains and dimensions.

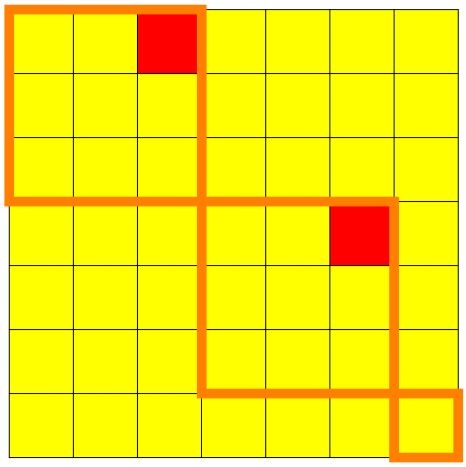
	dim	chain 1	chain 2	chain 3	rem dim
ker(A)	3	$\mathbf{e}_1$	$\mathbf{e}_4$	$\mathbf{e}_7$	0
ker(A^2)	5	$\mathbf{e}_2$	$\mathbf{e}_5$		0
ker(A^3)	7	$\mathbf{e}_3$	$\mathbf{e}_6$		0

Now all cells in the last column are zero. This means that we are done looking for a basis of vectors containing vectors from Jordan chains.

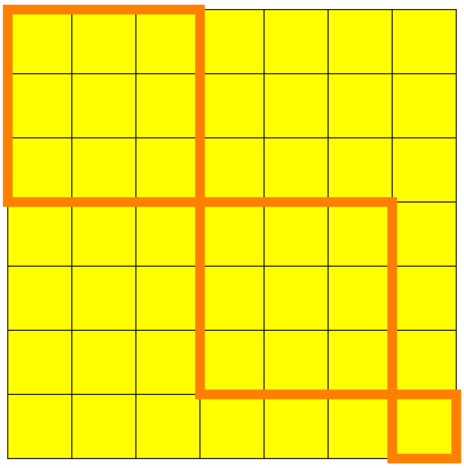
Let us express our findings in a way that emphasises the position of the elementary Jordan blocks and their relation with the powers of the matrix A.



(a) Visualisation of A.



(b) Visualisation of A^2.



(c) Visualisation of A^2.

Figure 5

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up

into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

Let us calculate the powers of the matrix B.

$$B = \begin{pmatrix} -2 & 1 & 0 & -2 & 1 & 3 & 2 \\ 1 & -1 & 0 & 1 & -1 & -2 & -1 \\ 1 & -1 & 0 & 1 & -1 & -2 & -1 \\ 0 & -1 & 0 & 2 & 2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 2 & 2 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 1 & 5 & 5 & -1 \end{pmatrix}; \quad B^2 = \begin{pmatrix} 0 & 2 & 0 & 0 & 2 & 2 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 \end{pmatrix};$$

$$B^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} -2 & 1 & 0 & -2 & 1 & 3 & 2 \\ 1 & -1 & 0 & 1 & -1 & -2 & -1 \\ 1 & -1 & 0 & 1 & -1 & -2 & -1 \\ 0 & -1 & 0 & 2 & 2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 2 & 2 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 1 & 5 & 5 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} -2z_1 + z_2 - 2z_4 + z_5 + 3z_6 + 2z_7 = 0, \\ z_1 - z_2 + z_4 - z_5 - 2z_6 - z_7 = 0, \\ z_1 - z_2 + z_4 - z_5 - 2z_6 - z_7 = 0, \\ -z_2 + 2z_4 + 2z_5 + z_6 - 2z_7 = 0, \\ +z_2 + z_3 + 2z_5 + 2z_6 = 0, \\ -z_1 - z_3 - z_4 - z_5 + z_7 = 0, \\ -z_1 + z_2 + z_3 + z_4 + 5z_5 + 5z_6 - z_7 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned}&\ker(B)\\&\quad=\{(r_1,r_2,r_2,r_4,-2r_1-4r_2,2r_1+3r_2,-r_1-3r_2+r_4)\mid r_1,r_2,r_4\in\mathbf{K}\}\\&\quad=\operatorname{span}\{(1,0,0,0,-2,2,-1),(0,1,1,0,-4,3,-3),(0,0,0,1,0,0,1)\}.\end{aligned}$$

Kernel of $\mathbf{B^2}$.
We calculate the kernel of B^2 and we have to solve the matrix equation

$$\begin{pmatrix}0&2&0&0&2&2&0\\0&-1&0&0&-1&-1&0\\0&-1&0&0&-1&-1&0\\0&-1&-1&0&-2&-2&0\\0&0&0&0&0&0&0\\0&1&0&0&1&1&0\\0&0&-1&0&-1&-1&0\end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\\z_4\\z_5\\z_6\\z_7\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases}2z_2+2z_5+2z_6=0,\\-z_2-z_5-z_6=0,\\-z_2-z_5-z_6=0,\\-z_2-z_3-2z_5-2z_6=0,\\z_2+z_5+z_6=0,\\-z_3-z_5-z_6=0.\end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned}&\ker(B^2)\\&\quad=\{(r_1,r_2,r_2,r_4,r_5,-r_2-r_5,r_7)\mid r_1,r_2,r_4,r_5,r_7\in\mathbf{K}\}\\&\quad=\operatorname{span}\{(1,0,0,0,0,0,0),(0,1,1,0,0,-1,0),(0,0,0,1,0,0,0),\\&\qquad(0,0,0,0,1,-1,0),(0,0,0,0,0,0,1)\}.\end{aligned}$$

Kernel of $\mathbf{B^3}$.
We have of course also $\ker(B^3)=\mathbf{K}^7$.

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	3	$3 = \dim(\ker(B))$
$\ker(B^2)$	5	$2 = \dim(\ker(B^2)) - \dim(\ker(B))$
$\ker(B^3)$	7	$2 = \dim(\ker(B^3)) - \dim(\ker(B^2))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(B))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column is 2 and this says that there will be two Jordan chains of length 3. The first number 1 in the last column is 3 and this gives the information that there will be three elementary Jordan chains. After we have calculated the chain with length 3, the last column will be from top to bottom $\{2, 1, 1\}$. There are still linearly independent vectors left to be found. We see that we can calculate again a Jordan chain with length 3. We have at this moment 6 linearly independent vectors. The last column will then be from top to bottom $\{1, 0, 0\}$. We calculate then the remaining eigenvector which is a chain on its own.

Please consult the tables in subsection 2 of this exercise for a full view of the situation. Then the procedure will stop because all entries in the last column are 0's. We will have at that stage a basis for V consisting entirely out of linearly independent vectors in Jordan chains.

4. Calculation of the Jordan chains.

The first Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ satisfying

$$\begin{cases} B^3\,\mathbf{w}_3 = \mathbf{0}, \\ B^2\,\mathbf{w}_3 = \mathbf{w}_1, \\ B\,\mathbf{w}_3 = \mathbf{w}_2, \end{cases}$$

where $\mathbf{w}_3$ is in the vector space $\ker(B^3)$ but not in $\ker(B^2)$ in order to guarantee that this is a chain of length 3.

We look for a starting $\mathbf{w}_3$. We see from the information table that there is a generating vector $\mathbf{w}_3$ for a chain of length 3.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^3)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^3)$.

2. The generating vector may not be in the $\ker(B^2)$ because the length of the chain must be exactly 3. So it has to be independent from all vectors in $\ker(B^2)$. It is sufficient that it is linearly independent from a basis of $\ker(B^2)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 3. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^3 together with the vectors in $\ker(B^2)$ and also the vectors, if any, of exactly height 3 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that the kernel $\ker(B^3)$ is generated by the following set of vectors.

$$\ker(B^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\} = \mathbf{K}^7.$$

So the generating vector has the following generic form

$$(a, b, c, d, e, f, g).$$

The kernel of B^2.
We remember that the kernel $\ker(B^2)$ is

$$\ker(B^2) = \operatorname{span}\{(1,0,0,0,0,0,0),(0,1,1,0,0,-1,0),(0,0,0,1,0,0,0),\\ (0,0,0,0,1,-1,0),(0,0,0,0,0,0,1)\}.$$

Vectors chosen in previous Jordan chains.
We have previously not chosen a vector in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ a & b & c & d & e & f & g \end{pmatrix}.$$

If we impose the condition $b - c \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{c+e+f}{b-c} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{-b-e-f}{b-c} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $b - c \neq 0$. So we can choose $a = 0$, $b = 1$, $c = 0$, $d = 0$, $e = 0$, $f = 0$, $g = 0$.

So we have the valid generating vector

$$\mathbf{w_3} = (0, 1, 0, 0, 0, 0, 0).$$

We calculate $B\,(0, 1, 0, 0, 0, 0, 0)$ and $B^2\,(0, 1, 0, 0, 0, 0, 0)$ and we know that the vectors

$$\{\mathbf{w_1} = B^2\,\mathbf{w_3}, \mathbf{w_2} = B\,\mathbf{w_3}, \mathbf{w_3}\}$$

are a Jordan chain of 3 linearly independent vectors and because $B^3\,\mathbf{w_3} = \mathbf{0}$ we know that the length of the chain is exactly 3.

We calculate $\mathbf{w_2}$.

$$\mathbf{w_2} = B\,\mathbf{w_3} = \begin{pmatrix} -2 & 1 & 0 & -2 & 1 & 3 & 2 \\ 1 & -1 & 0 & 1 & -1 & -2 & -1 \\ 1 & -1 & 0 & 1 & -1 & -2 & -1 \\ 0 & -1 & 0 & 2 & 2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 2 & 2 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 1 & 5 & 5 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

We find $\mathbf{w_2} = (1, -1, -1, -1, 1, 0, 1)$.

We calculate now $\mathbf{w_1}$.

$$\mathbf{w_1} = B^2\,\mathbf{w_3} = \begin{pmatrix} 0 & 2 & 0 & 0 & 2 & 2 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

We find $\mathbf{w_1} = (2, -1, -1, -1, 0, 1, 0)$.

These vectors $\mathbf{w_3}$, $\mathbf{w_2}$ and $\mathbf{w_1}$ can be checked to be linearly independent though this is not necessary. They are automatically linearly independent, see the appendix.

We have now found our first Jordan chain. It has length 3.

$$\{\mathbf{w_1} = (2, -1, -1, -1, 0, 1, 0), \mathbf{w_2} = (1, -1, -1, -1, 1, 0, 1), \\ \mathbf{w_3} = (0, 1, 0, 0, 0, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	3	$\mathbf{w_1}$	2
$\ker(B^2)$	5	$\mathbf{w_2}$	1
$\ker(B^3)$	7	$\mathbf{w_3}$	1

with

$$\mathbf{w_1} = (2, -1, -1, -1, 0, 1, 0)$$
$$\mathbf{w_2} = (1, -1, -1, -1, 1, 0, 1)$$
$$\mathbf{w_3} = (0, 1, 0, 0, 0, 0, 0)$$

We see that we have one Jordan chain with length 3 left.

The second Jordan chain.
The last information table tells us that we have to find a generating vector $\mathbf{w_6} \in \ker(B^3)$ satisfying

1. It has to be in the kernel $\ker(B^3)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^3)$.
2. The generating vector may not be in the $\ker(B^2)$ because the length of the chain must be exactly 3. So it has to be independent from all vectors in $\ker(B^2)$. It is sufficient that it is linearly independent from a basis of $\ker(B^2)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 3. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^3 together with the vectors in $\ker(B^2)$ and also the vectors, if any, of exactly height 3 chosen in previous Jordan chains must be a linearly independent set of vectors.

We have then the chain

$$\{\mathbf{w_4} = B^2\,\mathbf{w_6}, \mathbf{w_5} = B\,\mathbf{w_6}, \mathbf{w_6}.\}$$

We look for a starting $\mathbf{w_6}$. We see from the information table that there is a generating vector $\mathbf{w_6}$ for a chain of length 3.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^3)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^3)$.

2. The generating vector may not be in the $\ker(B^2)$ because the length of the chain must be exactly 3. So it has to be independent from all vectors in $\ker(B^2)$. It is sufficient that it is linearly independent from a basis of $\ker(B^2)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 3. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^3 together with the vectors in $\ker(B^2)$ and also the vectors, if any, of exactly height 3 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that the kernel $\ker(B^3)$ is generated by the following set of vectors.

$$\ker(B^3) = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}.$$

So the generating vector has the following generic form

$$(a, b, c, d, e, f, g).$$

The kernel of B^2.
We remember that the kernel $\ker(B^2)$ is

$$\ker(B^2) = \text{span}\{(1,0,0,0,0,0,0), (0,1,1,0,0,-1,0), (0,0,0,1,0,0,0), (0,0,0,0,1,-1,0), (0,0,0,0,0,0,1)\}.$$

Vectors chosen in previous Jordan chains.
We have previously chosen the vector $\mathbf{w}_3 = (0,1,0,0,0,0,0)$ of exactly height 3 in the $\ker(B^3)$ in a previous Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ a & b & c & d & e & f & g \end{pmatrix}.$$

If we impose the condition $c + e + f \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $c + e + f \neq 0$. We can choose $a = 0$, $b = 0$, $c = 1$, $d = 0$, $e = 0$, $f = 0$, $g = 0$.

We have the generating vector

$$\mathbf{w_6} = (0, 0, 1, 0, 0, 0, 0).$$

We calculate $\mathbf{w_5}$.

$$\mathbf{w_5} = B\,\mathbf{w_6} = \begin{pmatrix} -2 & 1 & 0 & -2 & 1 & 3 & 2 \\ 1 & -1 & 0 & 1 & -1 & -2 & -1 \\ 1 & -1 & 0 & 1 & -1 & -2 & -1 \\ 0 & -1 & 0 & 2 & 2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 2 & 2 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 1 & 5 & 5 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

We have $\mathbf{w_5} = (0, 0, 0, 0, 1, -1, 1)$.

We calculate $\mathbf{w_4}$.

$$\mathbf{w_4} = B^2\,\mathbf{w_6} = \begin{pmatrix} 0 & 2 & 0 & 0 & 2 & 2 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

We have $\mathbf{w_4} = (0, 0, 0, -1, 0, 0, -1)$.

These vectors $\mathbf{w_4}$ and $\mathbf{w_5}$ and $\mathbf{w_6}$ can be checked to be linearly independent, though this is not necessary. They are automatically linearly independent, see the appendix.

We have now found our second Jordan chain. It has length 3.

$$\{\mathbf{w_4} = (0, 0, 0, -1, 0, 0, -1), \mathbf{w_5} = (0, 0, 0, 0, 1, -1, 1), \mathbf{w_6} = (0, 0, 1, 0, 0, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	rem dim
$\ker(B)$	3	$\mathbf{w}_1$	$\mathbf{w}_4$	1
$\ker(B^2)$	5	$\mathbf{w}_2$	$\mathbf{w}_5$	0
$\ker(B^3)$	7	$\mathbf{w}_3$	$\mathbf{w}_6$	0

with

$$\mathbf{w}_1 = (2, -1, -1, -1, 0, 1, 0)$$
$$\mathbf{w}_2 = (1, -1, -1, -1, 1, 0, 1)$$
$$\mathbf{w}_3 = (0, 1, 0, 0, 0, 0, 0)$$
$$\mathbf{w}_4 = (0, 0, 0, -1, 0, 0, -1)$$
$$\mathbf{w}_5 = (0, 0, 0, 0, 1, -1, 1)$$
$$\mathbf{w}_6 = (0, 0, 1, 0, 0, 0, 0)$$

The third chain.
The last column in the information table tells us that there is still a chain of length 1 left to be found. This is an eigenvector $\mathbf{w}_7$. We know that we have in our two previous Jordan chains found two eigenvectors which have automatically height 1. So we have to be careful when choosing an eigenvector.

The vector must be an eigenvector. It must be in the space

$$\ker(B) = \text{span}\{(1, 0, 0, 0, -2, 2, -1), (0, 1, 1, 0, -4, 3, -3), (0, 0, 0, 1, 0, 0, 1)\}.$$

The vector must be of the generic form

$$a\,(1,0,0,0,-2,2,-1) + b\,(0,1,1,0,-4,3,-3) + c\,(0,0,0,1,0,0,1)$$
$$= (a, b, b, c, -2\,a - 4\,b, 2\,a + 3\,b, -a - 3\,b + c).$$

We have at this point chosen in ker(B) already the vectors

$$\begin{cases} \mathbf{w_1} = (2,-1,-1,-1,0,1,0), \\ \mathbf{w_4} = (0,0,0,-1,0,0,-1) \end{cases}$$

of height exactly 1. They must be linearly independent from the vector $\mathbf{w_7}$ we have to choose.

We have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 2 & -1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ a & b & b & c & -2\,a-4\,b & 2\,a+3\,b & -a-3\,b+c \end{pmatrix}.$$

We row reduce this matrix H and find then if we impose the condition that $a + 2\,b \neq 0$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & 1 & 0 & -4 & 3 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + 2\,b \neq 0$.

We can choose $a = 1$, $b = 0$, $c = 0$.

We have the valid generating vector

$$\mathbf{w_7} = (1,0,0,0,-2,2,-1).$$

We have found our third Jordan chain. It has length 1.

$$\{\mathbf{w_7} = (1,0,0,0,-2,2,-1)\}.$$

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	chain 3	rem dim
$\ker(B)$	3	$\mathbf{w_1}$	$\mathbf{w_4}$	$\mathbf{w_7}$	0
$\ker(B^2)$	5	$\mathbf{w_2}$	$\mathbf{w_5}$		0
$\ker(B^3)$	7	$\mathbf{w_3}$	$\mathbf{w_6}$		0

with

$\mathbf{w_1} = (2, -1, -1, -1, 0, 1, 0)$
$\mathbf{w_2} = (1, -1, -1, -1, 1, 0, 1)$
$\mathbf{w_3} = (0, 1, 0, 0, 0, 0, 0)$
$\mathbf{w_4} = (0, 0, 0, -1, 0, 0, -1)$
$\mathbf{w_5} = (0, 0, 0, 0, 1, -1, 1)$
$\mathbf{w_6} = (0, 0, 1, 0, 0, 0, 0)$
$\mathbf{w_7} = (1, 0, 0, 0, -2, 2, -1)$

The last line consists only 0's and we are done. We have now a basis of vectors consisting of independent Jordan chains.

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 \end{pmatrix}.$$

We check now.

$$A = P^{-1} B P$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 & -3 & -3 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & -1 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} -2 & 1 & 0 & -2 & 1 & 3 & 2 \\ 1 & -1 & 0 & 1 & -1 & -2 & -1 \\ 1 & -1 & 0 & 1 & -1 & -2 & -1 \\ 0 & -1 & 0 & 2 & 2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 2 & 2 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 1 & 5 & 5 & -1 \end{pmatrix}$$

$$\times \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 \end{pmatrix}.$$

$$
\begin{aligned}
A &= P^{-1} B P \\
&= \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
&= \begin{pmatrix}
\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\
\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & (\mathbf{0}) \end{matrix}
\end{pmatrix}.
\end{aligned}
$$

5 exercise. (7×7); $(J_3(0), J_2(0), J_2(0))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is a matrix in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 3 & 3 & -4 & 2 & 1 & 1 & -1 \\ -3 & -3 & 3 & -2 & 0 & 0 & 1 \\ -1 & -1 & 1 & -1 & 0 & 0 & 1 \\ -1 & -1 & 3 & -1 & -2 & -2 & 1 \\ -5 & -5 & 6 & -4 & 0 & 0 & 3 \\ 4 & 4 & -5 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_7| = -\lambda^7.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 7. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field **K**. It is thus possible to put the matrix B in Jordan normal form over the field **K**.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 2. The solution will be completely independent from this section.

We want to investigate the endomorphism A associated with this Jordan block. We compute also the powers of A.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We show this matrix in a way that emphasises the position of the elementary Jordan blocks.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{matrix} \end{pmatrix},$$

$$A^2 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{matrix} \end{pmatrix},$$

$$A^3 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{matrix} \end{pmatrix}.$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the third power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 3, is called the height of nilpotency of the matrix A.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_6\}; \\ \ker(A^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}; \\ \ker(A^3) = \mathbf{K}^7. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	3	$3 = \dim(\ker(A))$
$\ker(A^2)$	6	$3 = \dim(\ker(A^2)) - \dim(\ker(A))$
$\ker(A^3)$	7	$1 = \dim(\ker(A^3)) - \dim(\ker(A^2))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the third power onwards

$$\ker(A) \subsetneq \ker(A^2) \subsetneq \ker(A^3) = \ker(A^4) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 + n_3 = 3 = \dim(\ker(A)), \\ \qquad n_2 + n_3 = 3 = \dim(\ker(A^2)) - \dim(\ker(A)), \\ \qquad\qquad n_3 = 1 = \dim(\ker(A^3)) - \dim(\ker(A^2)). \end{cases}$$

Solving this system, we have $n_1 = 0$, $n_2 = 2$, $n_3 = 1$.

A consequence from this fact is that the numbers in the last column are descending.

We remark by looking at the matrices A^i that the vector $\mathbf{e_3}$ satisfies

$$\begin{cases} A\,\mathbf{e_3} = \mathbf{e_2}, \\ A^2\,\mathbf{e_3} = \mathbf{e_1}, \\ A^3\,\mathbf{e_3} = \mathbf{0}. \end{cases}$$

We see that we have a first Jordan chain. It has length three. We write a Jordan chain in reverse order.

$$\{\mathbf{e_1} = A^2\,\mathbf{e_3}, \mathbf{e_2} = A\,\mathbf{e_3}, \mathbf{e_3}\}.$$

After we have found the first Jordan chain of length 2, we have the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A)$	3	$\mathbf{e_1}$	2
$\ker(A^2)$	6	$\mathbf{e_2}$	2
$\ker(A^3)$	7	$\mathbf{e_3}$	0

Because the last column consists now of at least one number that is not 0, we are **not** done with looking for Jordan chains. We see that we find another Jordan chain with length 2.

We remark by looking at the matrices A^i that the vector $\mathbf{e_5}$ satisfies

$$\begin{cases} A\,\mathbf{e}_5 = \mathbf{e}_4, \\ A^2\,\mathbf{e}_5 = \mathbf{0}. \end{cases}$$

We see that we have found a second Jordan chain. It has length two. We write a Jordan chain traditionally in reverse order.

$$\{\mathbf{e}_4 = A\,\mathbf{e}_5, \mathbf{e}_5\}.$$

Let us note this in the new information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	rem dim
$\ker(A)$	3	$\mathbf{e}_1$	$\mathbf{e}_4$	1
$\ker(A^2)$	6	$\mathbf{e}_2$	$\mathbf{e}_5$	1
$\ker(A^3)$	7	$\mathbf{e}_3$		0

We remark by looking at the matrices A^i that the vector $\mathbf{e}_7$ satisfies

$$\begin{cases} A\,\mathbf{e}_7 = \mathbf{e}_6, \\ A^2\,\mathbf{e}_7 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain with length two of linearly independent vectors. We write a Jordan chain traditionally in reverse order.

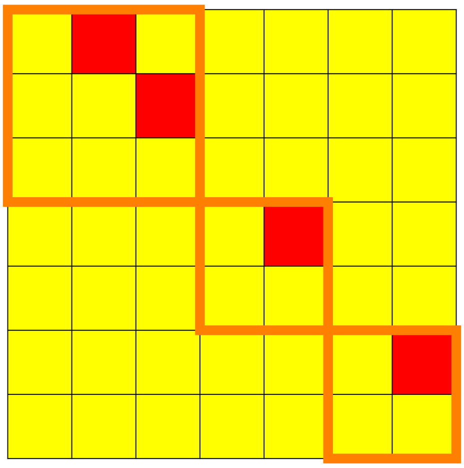
$$\{\mathbf{e}_6 = A\,\mathbf{e}_7, \mathbf{e}_7\}.$$

We have found a third Jordan chain. It has length 2. We have then the following table.

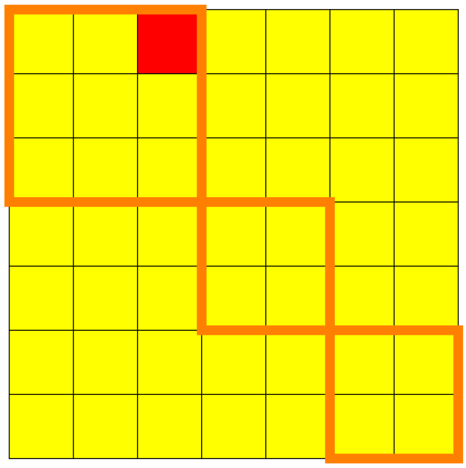
Keeping track of chains and dimensions.

	dim	chain 1	chain 2	chain 3	rem dim
$\ker(A)$	3	$\mathbf{e}_1$	$\mathbf{e}_4$	$\mathbf{e}_6$	0
$\ker(A^2)$	6	$\mathbf{e}_2$	$\mathbf{e}_5$	$\mathbf{e}_7$	0
$\ker(A^3)$	7	$\mathbf{e}_3$			0

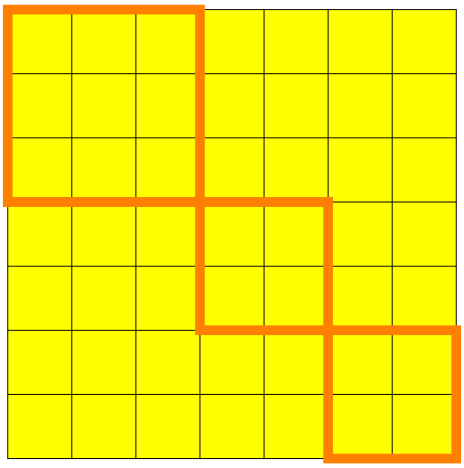
Let us express our findings in a way that emphasises the position of the elementary Jordan blocks and their relation with the powers of the matrix A^i.



(a) Visualisation of A.



(b) Visualisation of A^2.



(c) Visualisation of A^3.

Figure 6

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

Let us calculate the powers of the matrix B.

$$B = \begin{pmatrix} 3 & 3 & -4 & 2 & 1 & 1 & -1 \\ -3 & -3 & 3 & -2 & 0 & 0 & 1 \\ -1 & -1 & 1 & -1 & 0 & 0 & 1 \\ -1 & -1 & 3 & -1 & -2 & -2 & 1 \\ -5 & -5 & 6 & -4 & 0 & 0 & 3 \\ 4 & 4 & -5 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{pmatrix};$$

$$B^2 = \begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 & -1 \\ -1 & -1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & 2 & -2 & 0 & 0 & 2 \\ 2 & 2 & -2 & 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$B^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} 3 & 3 & -4 & 2 & 1 & 1 & -1 \\ -3 & -3 & 3 & -2 & 0 & 0 & 1 \\ -1 & -1 & 1 & -1 & 0 & 0 & 1 \\ -1 & -1 & 3 & -1 & -2 & -2 & 1 \\ -5 & -5 & 6 & -4 & 0 & 0 & 3 \\ 4 & 4 & -5 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} 3z_1 + 3z_2 - 4z_3 + 2z_4 + z_5 + z_6 - z_7 = 0, \\ -3z_1 - 3z_2 + 3z_3 - 2z_4 + z_7 = 0, \\ -z_1 - z_2 + z_3 - z_4 + z_7 = 0, \\ -z_1 - z_2 + 3z_3 - z_4 - 2z_5 - 2z_6 + z_7 = 0, \\ -5z_1 - 5z_2 + 6z_3 - 4z_4 + 3z_7 = 0, \\ 4z_1 + 4z_2 - 5z_3 + 3z_4 - 2z_7 = 0, \\ z_3 - z_5 - z_6 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} &\ker(B) \\ &= \{(r_1, r_2, 0, -2r_1 - 2r_2, r_5, -r_5, -r_1 - r_2) \mid r_1, r_2, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,-2,0,0,-1), (0,1,0,-2,0,0,-1), (0,0,0,0,1,-1,0)\}. \end{aligned}$$

Kernel of B^2.

We calculate the kernel of B^2 and we have to solve the matrix equation

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 & -1 \\ -1 & -1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & 2 & -2 & 0 & 0 & 2 \\ 2 & 2 & -2 & 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} z_1 + z_2 - z_3 + z_4 - z_7 = 0, \\ -z_1 - z_2 + z_3 - z_4 + z_7 = 0, \\ -2z_1 - 2z_2 + 2z_3 - 2z_4 + 2z_7 = 0, \\ 2z_1 + 2z_2 - 2z_3 + 2z_4 - 2z_7 = 0. \end{cases}$$

This system can be solved and this gives us the solutions set

$$\begin{aligned} &\ker(B^2) \\ &= \{(r_1, r_2, r_3, r_4, r_5, r_6, r_1 + r_2 - r_3 + r_4) \mid r_1, r_2, r_3, r_4, r_5, r_6 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,0,0,1), (0,1,0,0,0,0,1), (0,0,1,0,0,0,-1), \\ &\qquad (0,0,0,1,0,0,1), (0,0,0,0,1,0,0), (0,0,0,0,0,1,0)\}. \end{aligned}$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	3	$3 = \dim(\ker(B))$
$\ker(B^2)$	6	$3 = \dim(\ker(B^2)) - \dim(\ker(B))$
$\ker(B^3)$	7	$1 = \dim(\ker(B^3)) - \dim(\ker(B^2))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(B))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column is 1 and this gives the information that there will be a Jordan chain of length 3. The first number 3 in the last column is the dimension of the kernel of B. This number says that there are three elementary Jordan chains to be found. After we have calculated the chain with length 3, the last column will be from top to bottom $\{2, 2, 0\}$. There are still linearly independent vectors left to be found. We see that we can calculate again a Jordan chain with length

2. We have at this moment 5 linearly independent vectors. The last column will then be from top to bottom $\{1, 1, 0\}$. There are still linearly independent vectors left to be found. We see that we can calculate again a Jordan chain with length 2.

The last column will then be from top to bottom $\{0, 0, 0\}$. We have at this moment 7 linearly independent vectors which form a base for this vector space.

Then the procedure will stop because all entries in the last column are 0's.

The first Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ satisfying

$$\begin{cases} B^3\,\mathbf{w}_3 = \mathbf{0}, \\ B^2\,\mathbf{w}_3 = \mathbf{w}_1, \\ B\,\mathbf{w}_3 = \mathbf{w}_2, \end{cases}$$

where $\mathbf{w}_3$ is in the vector space $\ker(B^3)$ but not in $\ker(B^2)$.

We look for a generating vector $\mathbf{w}_3$. We see from the information table that there is a generating vector $\mathbf{w}_3$ for a chain of length 3.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^3)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^3)$.

2. The generating vector may not be in the $\ker(B^2)$ because the length of the chain must be exactly 3. So it has to be independent from all vectors in $\ker(B^2)$. It is sufficient that it is linearly independent from a basis of $\ker(B^2)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 3. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^3 together with the vectors in $\ker(B^2)$ and also the vectors, if any, of exactly height 3 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that the kernel $\ker(B^3)$ is generated by the following set of vectors.

$$\ker(B^3) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}.$$

So the generating vector has the following generic form

$$(a, b, c, d, e, f, g).$$

The kernel of B^2.
We remember that the kernel $\ker(B^2)$ is

$$\begin{aligned}&\ker(B^2)\\ &\quad = \operatorname{span}\{(1,0,0,0,0,0,1),(0,1,0,0,0,0,1),(0,0,1,0,0,0,-1),\\ &\qquad (0,0,0,1,0,0,1),(0,0,0,0,1,0,0),(0,0,0,0,0,1,0)\}.\end{aligned}$$

Vectors chosen in previous Jordan chains.
We have previously not chosen a vector in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ a & b & c & d & e & f & g \end{pmatrix}.$$

If we impose the condition $a + b - c + d - g \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + b - c + d - g \neq 0$. We can choose $a = 1$, $b = 0$, $c = 0$, $d = 0$, $e = 0$, $f = 0$, $g = 0$.

We have the generating vector

$$\mathbf{w_3} = (1, 0, 0, 0, 0, 0, 0).$$

We calculate then $B\,\mathbf{w_3}$, $B^2\,\mathbf{w_3}$ and we know that the vectors

$$\{\mathbf{w_1} = B^2\,\mathbf{w_3}, \mathbf{w_2} = B\,\mathbf{w_3}, \mathbf{w_3}\}.$$

are a Jordan chain of 3 linearly independent vectors. Because $B^3\,\mathbf{e_3} = \mathbf{0}$ we know that the length of the chain is exactly 3.

We calculate $\mathbf{w_2}$.

$$\mathbf{w_2} = B\,\mathbf{w_3} = \begin{pmatrix} 3 & 3 & -4 & 2 & 1 & 1 & -1 \\ -3 & -3 & 3 & -2 & 0 & 0 & 1 \\ -1 & -1 & 1 & -1 & 0 & 0 & 1 \\ -1 & -1 & 3 & -1 & -2 & -2 & 1 \\ -5 & -5 & 6 & -4 & 0 & 0 & 3 \\ 4 & 4 & -5 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -1 \\ -1 \\ -5 \\ 4 \\ 0 \end{pmatrix}.$$

We calculate now $\mathbf{w_1}$.

$$\mathbf{w_1} = B^2\,\mathbf{w_3} = \begin{pmatrix} 3 & 3 & -4 & 2 & 1 & 1 & -1 \\ -3 & -3 & 3 & -2 & 0 & 0 & 1 \\ -1 & -1 & 1 & -1 & 0 & 0 & 1 \\ -1 & -1 & 3 & -1 & -2 & -2 & 1 \\ -5 & -5 & 6 & -4 & 0 & 0 & 3 \\ 4 & 4 & -5 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \\ -1 \\ -1 \\ -5 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ -2 \\ 2 \\ 0 \end{pmatrix}.$$

We have now the first chain.

$$\{\mathbf{w_1} = B^2\,\mathbf{w_3} = (1,-1,0,0,-2,2,0), \mathbf{w_2} = B\,\mathbf{w_3} = (3,-3,-1,-1,-5,4,0), \\ \mathbf{w_3} = (1,0,0,0,0,0,0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	3	$\mathbf{w_1}$	2
$\ker(B^2)$	6	$\mathbf{w_2}$	2
$\ker(B^3)$	7	$\mathbf{w_3}$	0

with

$$\mathbf{w_1} = (1,-1,0,0,-2,2,0)$$
$$\mathbf{w_2} = (3,-3,-1,-1,-5,4,0)$$
$$\mathbf{w_3} = (1,0,0,0,0,0,0)$$

We see that we have two Jordan chains with both length 2 left.

The second Jordan chain.
So we have to find a generating vector $\mathbf{w_5} \in \ker(B^2)$ satisfying

1. It has to be in the kernel $\ker(B^2)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^2)$.

2. The generating vector may not be in the $\ker(B)$ because the length of the chain must be exactly 2. So it has to be independent from all vectors in $\ker(B)$. It is sufficient that it is linearly independent from a basis of $\ker(B)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 2. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^2 together with the vectors in $\ker(B)$ and also the vectors, if any, of exactly height 2 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that the kernel $\ker(B^2)$ is generated by the following set of vectors.

$$\begin{aligned}&\ker(B^2)\\ &\quad = \operatorname{span}\{(1,0,0,0,0,0,1),(0,1,0,0,0,0,1),(0,0,1,0,0,0,-1),\\ &\qquad (0,0,0,1,0,0,1),(0,0,0,0,1,0,0),(0,0,0,0,0,1,0)\}.\end{aligned}$$

So the generating vector has the following generic form

$$\begin{aligned}&a\,(1,0,0,0,0,0,1)+b\,(0,1,0,0,0,0,1)+c\,(0,0,1,0,0,0,-1)\\ &\qquad +d\,(0,0,0,1,0,0,1)+e\,(0,0,0,0,1,0,0)+f\,(0,0,0,0,0,1,0)\\ &\quad =(a,b,c,d,e,f,a+b-c+d).\end{aligned}$$

The kernel of B.
We remember that the kernel $\ker(B)$ is

$$\begin{aligned}&\ker(B)\\ &\quad = \operatorname{span}\{(1,0,0,-2,0,0,-1),(0,1,0,-2,0,0,-1),(0,0,0,0,1,-1,0)\}.\end{aligned}$$

Vectors chosen in previous Jordan chains.
We have previously chosen a vector of exactly height 2 in a previous Jordan chain. This is

$$\mathbf{w}_2 = (3, -3, -1, -1, -5, 4, 0).$$

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 3 & -3 & -1 & -1 & -5 & 4 & 0 \\ a & b & c & d & e & f & a+b-c+d \end{pmatrix}.$$

If we impose the condition $2\,a + 2\,b - c + d \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{2(c-e-f)}{2a+2b-c+d} & 1 \\ 0 & 1 & 0 & 0 & 0 & -\frac{2(c-e-f)}{2a+2b-c+d} & 1 \\ 0 & 0 & 1 & 0 & 0 & \frac{2a+2b+d-e-f}{2a+2b-c+d} & -1 \\ 0 & 0 & 0 & 1 & 0 & \frac{-c+e+f}{2a+2b-c+d} & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $2\,a + 2\,b - c + d \neq 0$. We can choose $a = 1$, $b = 0$, $c = 0$, $d = 0$, $e = 0$, $f = 0$.

We have the valid generating vector

$$\mathbf{w}_5 = (1, 0, 0, 0, 0, 0, 1).$$

We calculate $\mathbf{w}_4$.

$$\mathbf{w}_4 = B\,\mathbf{w}_5 = \begin{pmatrix} 3 & 3 & -4 & 2 & 1 & 1 & -1 \\ -3 & -3 & 3 & -2 & 0 & 0 & 1 \\ -1 & -1 & 1 & -1 & 0 & 0 & 1 \\ -1 & -1 & 3 & -1 & -2 & -2 & 1 \\ -5 & -5 & 6 & -4 & 0 & 0 & 3 \\ 4 & 4 & -5 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \\ -2 \\ 2 \\ 0 \end{pmatrix}.$$

We have now $\mathbf{w_4} = (2, -2, 0, 0, -2, 2, 0)$.

We have now the second chain. It has length 2.

$$\{\mathbf{w_4} = B\,\mathbf{w_5} = (2, -2, 0, 0, -2, 2, 0), \mathbf{w_5} = (1, 0, 0, 0, 0, 0, 1)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	rem dim
$\ker(B)$	3	$\mathbf{w_1}$	$\mathbf{w_4}$	1
$\ker(B^2)$	6	$\mathbf{w_2}$	$\mathbf{w_5}$	1
$\ker(B^3)$	7	$\mathbf{w_3}$		0

with

$\mathbf{w_1} = (1, -1, 0, 0, -2, 2, 0)$
$\mathbf{w_2} = (3, -3, -1, -1, -5, 4, 0)$
$\mathbf{w_3} = (1, 0, 0, 0, 0, 0, 0)$
$\mathbf{w_4} = (2, -2, 0, 0, -2, 2, 0)$
$\mathbf{w_5} = (1, 0, 0, 0, 0, 0, 1)$

The last column tells us that there is still a chain of length 2 left to be found.

The third Jordan chain.
We have to find a generating vector $\mathbf{w_7} \in \ker(B^2)$ satisfying

1. It has to be in the kernel $\ker(B^2)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^2)$.

2. The generating vector may not be in the $\ker(B)$ because the length of the chain must be exactly 2. So it has to be independent from all vectors in $\ker(B)$. It is sufficient that it is linearly independent from a basis of $\ker(B)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 2. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^2 together with the vectors in $\ker(B)$ and also the vectors, if any, of exactly height 2 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that the kernel $\ker(B^2)$ is generated by the following set of vectors.

$$\begin{aligned}&\ker(B^2)\\&\quad = \operatorname{span}\{(1,0,0,0,0,0,1),(0,1,0,0,0,0,1),(0,0,1,0,0,0,-1),\\&\qquad\quad (0,0,0,1,0,0,1),(0,0,0,0,1,0,0),(0,0,0,0,0,1,0)\}.\end{aligned}$$

So the generating vector has the following generic form

$$\begin{aligned}&a\,(1,0,0,0,0,0,1)+b\,(0,1,0,0,0,0,1)+c\,(0,0,1,0,0,0,-1)\\&\qquad\quad +d\,(0,0,0,1,0,0,1)+e\,(0,0,0,0,1,0,0)+f\,(0,0,0,0,0,1,0)\\&= (a,b,c,d,e,f,a+b-c+d).\end{aligned}$$

The kernel of B.
We remember that the kernel $\ker(B)$ is

$$\begin{aligned}&\ker(B)\\&\quad = \operatorname{span}\{(1,0,0,-2,0,0,-1),(0,1,0,-2,0,0,-1),(0,0,0,0,1,-1,0)\}.\end{aligned}$$

Vectors chosen in previous Jordan chains.
We have previously chosen two vectors of exactly height 2 in previous Jordan chains. These are

$$\begin{cases} \mathbf{w_2} = (3, -3, -1, -1, -5, 4, 0), \\ \mathbf{w_5} = (1, 0, 0, 0, 0, 0, 1). \end{cases}$$

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 3 & -3 & -1 & -1 & -5 & 4 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ a & b & c & d & e & f & a+b-c+d \end{pmatrix}.$$

If we impose the condition $c - e - f \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $c - e - f \neq 0$. So we can choose $a = 0$, $b = 0$, $c = 0$, $d = 0$, $e = 0$, $f = 1$.

We can choose the generating vector

$$\mathbf{w_7} = (0, 0, 0, 0, 0, 1, 0).$$

We make the choice $\mathbf{w_7} = (0, 0, 0, 0, 0, 1, 0)$.

We calculate $\mathbf{w_6}$.

$$\mathbf{w_6} = B\,\mathbf{w_7} = \begin{pmatrix} 3 & 3 & -4 & 2 & 1 & 1 & -1 \\ -3 & -3 & 3 & -2 & 0 & 0 & 1 \\ -1 & -1 & 1 & -1 & 0 & 0 & 1 \\ -1 & -1 & 3 & -1 & -2 & -2 & 1 \\ -5 & -5 & 6 & -4 & 0 & 0 & 3 \\ 4 & 4 & -5 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

We have now $\mathbf{w_6} = (1, 0, 0, -2, 0, 0, -1)$.

We have now found the third Jordan chain. It has length 2.

$$\{\mathbf{w_6} = (1, 0, 0, -2, 0, 0, -1), \mathbf{w_7} = (0, 0, 0, 0, 0, 1, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	chain 3	rem dim
$\ker(B)$	3	$\mathbf{w_1}$	$\mathbf{w_4}$	$\mathbf{w_6}$	0
$\ker(B^2)$	6	$\mathbf{w_2}$	$\mathbf{w_5}$	$\mathbf{w_7}$	0
$\ker(B^3)$	7	$\mathbf{w_3}$			0

with

$\mathbf{w_1} = (1, -1, 0, 0, -2, 2, 0)$

$\mathbf{w_2} = (3, -3, -1, -1, -5, 4, 0)$

$\mathbf{w_3} = (1, 0, 0, 0, 0, 0, 0)$

$\mathbf{w_4} = (2, -2, 0, 0, -2, 2, 0)$

$\mathbf{w_5} = (1, 0, 0, 0, 0, 0, 1)$

$\mathbf{w_6} = (1, 0, 0, -2, 0, 0, -1)$

$\mathbf{w_7} = (0, 0, 0, 0, 0, 1, 0)$

The last column consists only 0's and we are done. We have now a basis of vectors consisting of independent Jordan chains.

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} 1 & 3 & 1 & 2 & 1 & 1 & 0 \\ -1 & -3 & 0 & -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -2 & 0 \\ -2 & -5 & 0 & -2 & 0 & 0 & 0 \\ 2 & 4 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

$$A = P^{-1} B P$$

$$= \begin{pmatrix} 0 & 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

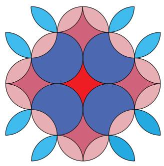
$$\times \begin{pmatrix} 3 & 3 & -4 & 2 & 1 & 1 & -1 \\ -3 & -3 & 3 & -2 & 0 & 0 & 1 \\ -1 & -1 & 1 & -1 & 0 & 0 & 1 \\ -1 & -1 & 3 & -1 & -2 & -2 & 1 \\ -5 & -5 & 6 & -4 & 0 & 0 & 3 \\ 4 & 4 & -5 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 3 & 1 & 2 & 1 & 1 & 0 \\ -1 & -3 & 0 & -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -2 & 0 \\ -2 & -5 & 0 & -2 & 0 & 0 & 0 \\ 2 & 4 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

$$A = P^{-1} B P$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{matrix} \end{pmatrix}.$$



6 exercise. (7×7); $(J_2(0), J_2(0), J_2(0), J_1(0))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is a matrix in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -4 & 3 & -1 & -1 \\ 0 & -2 & -1 & -5 & 4 & -1 & -1 \\ 1 & -1 & -2 & -5 & 4 & -1 & -1 \\ 1 & -2 & -3 & -9 & 7 & -2 & -2 \\ -1 & -1 & 2 & 3 & -2 & 1 & 1 \\ 0 & 2 & -1 & -1 & 0 & -1 & -1 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_7| = -\lambda^7.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 7. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field **K**. It is thus possible to put the matrix B in Jordan normal form over the field **K**.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to

this subsection 2. The solution will be completely independent from this section.

We want to investigate the endomorphism A associated with this Jordan block. We compute also the powers of A.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We show this matrix in a way that emphasises the position of the elementary Jordan blocks.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \; \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \; \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} & (\mathbf{0}) \end{pmatrix}.$$

We compute also the powers of A.

$$A = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (0) \end{pmatrix},$$

$$A^2 = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (0) \end{pmatrix}.$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the second power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 2, is called the height of nilpotency of the matrix A.

It is interesting to observe how the kernels change. We can see almost without calculation that $\ker(A) = \text{span}\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5, \mathbf{e}_7\}$. We have $\ker(A^2) = \mathbf{K}^7$.

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	4	$4 = \dim(\ker(A))$
$\ker(A^2)$	7	$3 = \dim(\ker(A^2)) - \dim(\ker(A))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.
2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.
3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.
4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.
5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A) \subsetneq \ker(A^2) = \ker(A^3) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 = 4 = \dim(\ker(A)), \\ \qquad\; n_2 = 3 = \dim(\ker(A^2)) - \dim(\ker(A)). \end{cases}$$

Solving this system, we have $n_1 = 1$, $n_2 = 3$.

A consequence from this fact is that the numbers in the last column are descending.

We remark by looking at the matrices A^i that the vector $\mathbf{e}_2$ satisfies

$$\begin{cases} A\,\mathbf{e}_2 = \mathbf{e}_1, \\ A^2\,\mathbf{e}_2 = \mathbf{0}. \end{cases}$$

We see that we have a first Jordan chain. It has length two. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = A\,\mathbf{e}_2, \mathbf{e}_2\}.$$

After we have found the first Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A)$	4	$\mathbf{e}_1$	3
$\ker(A^2)$	7	$\mathbf{e}_2$	2

Because the last column consists now of at least one number that is not 0, we are **not** done with looking for Jordan chains. We see that we find another Jordan chain with length 2.

We remark by looking at the matrices A^i that we have

$$\begin{cases} A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A^2\,\mathbf{e}_4 = \mathbf{0}. \end{cases}$$

We have a second Jordan chain. It has length two. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_3 = A\,\mathbf{e}_4, \mathbf{e}_4\}.$$

Let us note this in the new information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	rem dim
$\ker(A)$	3	$\mathbf{e}_1$	$\mathbf{e}_3$	2
$\ker(A^2)$	6	$\mathbf{e}_2$	$\mathbf{e}_4$	1

We remark by looking at the matrices A^i that the vector $\mathbf{e}_6$ satisfies

$$\begin{cases} A\,\mathbf{e}_6 = \mathbf{e}_5, \\ A^2\,\mathbf{e}_6 = \mathbf{0}. \end{cases}$$

We see that we have a third Jordan chain. It has length two. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_5 = A\,\mathbf{e}_6, \mathbf{e}_6\}.$$

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	chain 3	rem dim
$\ker(A)$	4	$\mathbf{e}_1$	$\mathbf{e}_3$	$\mathbf{e}_5$	1
$\ker(A^2)$	7	$\mathbf{e}_2$	$\mathbf{e}_4$	$\mathbf{e}_6$	0

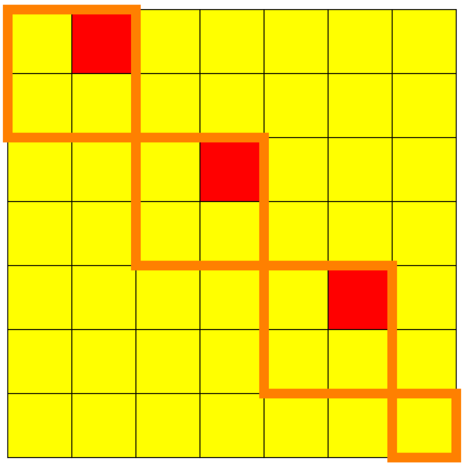
Now we have one vector $\mathbf{e}_7$ left that is an eigenvector. We have now our fourth Jordan chain. It has length 1.

$$\{\mathbf{e}_7\}.$$

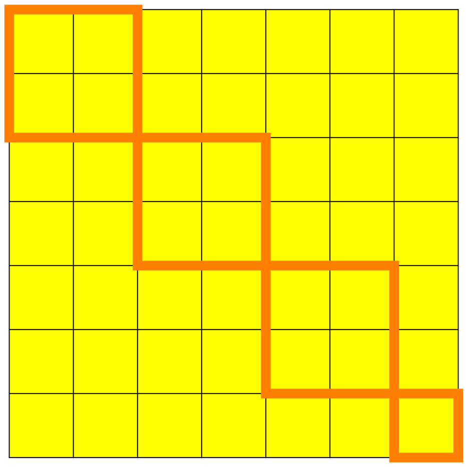
Keeping track of chains and dimensions.

	dim	chain 1	chain 2	chain 3	chain 4	rem dim
$\ker(A)$	4	$\mathbf{e}_1$	$\mathbf{e}_3$	$\mathbf{e}_5$	$\mathbf{e}_7$	0
$\ker(A^2)$	7	$\mathbf{e}_2$	$\mathbf{e}_4$	$\mathbf{e}_6$		0

Let us express our findings in a way that emphasises the position of the elementary Jordan blocks and their relation with the powers of the matrix A.



(a) Visualisation of A.



(b) Visualisation of A^2.

Figure 7

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

Let us calculate the powers of the matrix B.

$$B = \begin{pmatrix} 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -4 & 3 & -1 & -1 \\ 0 & -2 & -1 & -5 & 4 & -1 & -1 \\ 1 & -1 & -2 & -5 & 4 & -1 & -1 \\ 1 & -2 & -3 & -9 & 7 & -2 & -2 \\ -1 & -1 & 2 & 3 & -2 & 1 & 1 \\ 0 & 2 & -1 & -1 & 0 & -1 & -1 \end{pmatrix};$$

$$B^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -4 & 3 & -1 & -1 \\ 0 & -2 & -1 & -5 & 4 & -1 & -1 \\ 1 & -1 & -2 & -5 & 4 & -1 & -1 \\ 1 & -2 & -3 & -9 & 7 & -2 & -2 \\ -1 & -1 & 2 & 3 & -2 & 1 & 1 \\ 0 & 2 & -1 & -1 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} -z_2 - z_4 + z_5 = 0, \\ -z_2 - z_3 - 4z_4 + 3z_5 - z_6 - z_7 = 0, \\ -2z_2 - z_3 - 5z_4 + 4z_5 - z_6 - z_7 = 0, \\ z_1 - z_2 - 2z_3 - 5z_4 + 4z_5 - z_6 - z_7 = 0, \\ z_1 - 2z_2 - 3z_3 - 9z_4 + 7z_5 - 2z_6 - 2z_7 = 0, \\ -z_1 - z_2 + 2z_3 + 3z_4 - 2z_5 + z_6 + z_7 = 0, \\ +2z_2 - z_3 - z_4 - z_6 - z_7 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned}
&\ker(B)\\
&\quad = \{(r_1, r_2, r_1 + r_2, r_4, r_2 + r_4, r_6, -r_1 + r_2 - r_4 - r_6)\\
&\qquad\qquad \mid r_1, r_2, r_4, r_6 \in \mathbf{K}\}\\
&\quad = \operatorname{span}\{(1,0,1,0,0,0,-1),(0,1,1,0,1,0,1),\\
&\qquad\qquad (0,0,0,1,1,0,-1),(0,0,0,0,0,1,-1)\}.
\end{aligned}$$

Kernel of $\mathbf{B^2}$.
The kernel of B^2 is $\mathbf{K}^7$.

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	4	$4 = \dim(\ker(B))$
$\ker(B^2)$	7	$3 = \dim(\ker(B^2)) - \dim(\ker(B))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(B))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column is 3 and this gives the information that there will be three Jordan chains of length 2. The first number 4 in the last column is the dimension of the kernel of B. This number says that there will be 4 Jordan elementary block matrices. After we have calculated the chain with length 2, the last column will be from top to bottom $\{3, 2\}$. There will still be a chain of length 2. After we have found that chain the last column will be from top to bottom $\{2, 1\}$ We see that we can calculate again a Jordan chain with length 2. After we have found that chain the last column will be from top to bottom $\{1, 0\}$. We see that we can calculate again a Jordan chain with length 1. After this calculation, the last column will be from top to bottom $\{0, 0\}$. We have at that moment 7 linearly independent vectors which form a basis for this vector space. There are no chains left to look for.

4. Calculation of the Jordan chains.

The first Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w_1}, \mathbf{w_2}\}$ satisfying

$$B\,\mathbf{w_2} = \mathbf{w_1},$$

where $\mathbf{w_2}$ is in the vector space $\ker(B^2)$ but not in $\ker(B)$.

We choose a vector $\mathbf{w_2}$ that is in the kernel $\ker(B^2)$ but not in the kernel of $\ker(B)$.

We look for a starting $\mathbf{w_2}$. We see from the information table that there is a generating vector $\mathbf{w_2}$ for a chain of length 2.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^2)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^2)$.

2. The generating vector may not be in the $\ker(B)$ because the length of the chain must be exactly 2. So it has to be independent from all vectors in $\ker(B)$. It is sufficient that it is linearly independent from a basis of $\ker(B)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 2. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^2 together with the vectors in $\ker(B)$ and also the vectors, if any, of exactly height 2 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that $\ker(B^2) = \mathbf{K}^7$. So the generating vector has the following generic form

$$(a, b, c, d, e, f, g).$$

The kernel of B.
We remember that the kernel $\ker(B)$ is

$$\ker(B) = \operatorname{span}\{(1,0,1,0,0,0,-1),(0,1,1,0,1,0,1),\\ (0,0,0,1,1,0,-1),(0,0,0,0,0,1,-1)\}.$$

Vectors chosen in previous Jordan chains.
We have previously not chosen a vector in a previous Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ a & b & c & d & e & f & g \end{pmatrix}.$$

If we impose the condition $a + b - c \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{-b-d+e}{a+b-c} & 0 & \frac{-2b+c+d+f+g}{a+b-c} \\ 0 & 1 & 0 & 0 & \frac{a-c-d+e}{a+b-c} & 0 & \frac{2a-c+d+f+g}{a+b-c} \\ 0 & 0 & 1 & 0 & \frac{b+d-e}{a+b-c} & 0 & \frac{-a+b-d-f-g}{a+b-c} \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + b - c \neq 0$.

So we can choose $a = 1$, $b = 0$, $c = 0$, $d = 0$, $e = 0$, $f = 0$, $g = 0$.

We have the valid generating vector

$$\mathbf{w_2} = (1, 0, 0, 0, 0, 0, 0).$$

We calculate then $B(1, 0, 0, 0, 0, 0, 0)$, and we know that the vectors

$$\{\mathbf{w_1} = B\,\mathbf{w_2}, \mathbf{w_2}\}$$

are a Jordan chain of 2 linearly independent vectors and because $B^2\,\mathbf{w_2} = \mathbf{0}$, we know that the length of the chain is exactly 2.

We calculate $\mathbf{w_1}$.

$$\mathbf{w_1} = B\,\mathbf{w_2} = \begin{pmatrix} 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -4 & 3 & -1 & -1 \\ 0 & -2 & -1 & -5 & 4 & -1 & -1 \\ 1 & -1 & -2 & -5 & 4 & -1 & -1 \\ 1 & -2 & -3 & -9 & 7 & -2 & -2 \\ -1 & -1 & 2 & 3 & -2 & 1 & 1 \\ 0 & 2 & -1 & -1 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

We have now found the first Jordan chain. It has length 2.

$$\{\mathbf{w_1} = (0, 0, 0, 1, 1, -1, 0), \mathbf{w_2} = (1, 0, 0, 0, 0, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
ker(B)	4	$\mathbf{w}_1$	3
ker(B^2)	7	$\mathbf{w}_2$	2

with

$$\mathbf{w}_1 = (0,0,0,1,1,-1,0)$$
$$\mathbf{w}_2 = (1,0,0,0,0,0,0)$$

We see that we have two Jordan chains with both length 2 left.

The second Jordan chain.
So we have to find a generating vector $\mathbf{w}_4 \in \ker(B^2)$ satisfying

1. It has to be in the kernel $\ker(B^2)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^2)$.

2. The generating vector may not be in the $\ker(B)$ because the length of the chain must be exactly 2. So it has to be independent from all vectors in $\ker(B)$. It is sufficient that it is linearly independent from a basis of $\ker(B)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 2. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^2 together with the vectors in $\ker(B)$ and also the vectors, if any, of exactly height 2 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that $\ker(B^2) = \mathbf{K}^7$.

So the generating vector has the following generic form

$$(a, b, c, d, e, f, g).$$

The kernel of B.
We remember that the kernel $\ker(B)$ is

$$\ker(B) = \operatorname{span}\{(1,0,1,0,0,0,-1),(0,1,1,0,1,0,1),\\ (0,0,0,1,1,0,-1),(0,0,0,0,0,1,-1)\}.$$

Vectors chosen in previous Jordan chains.
We have previously chosen a vector $\mathbf{w_2} = (1,0,0,0,0,0,0)$ of exactly height 2 in a previous Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & b & c & d & e & f & g \end{pmatrix}.$$

If we impose the condition $b + d - e \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{c+3d-2e+f+g}{b+d-e} \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{-3b+c+e+f+g}{b+d-e} \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{2b-c-d-f-g}{b+d-e} \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $b + d - e \neq 0$. We can choose $a = 0$, $b = 1$, $c = 0$, $d = 0$, $e = 0$, $f = 0$, $g = 0$.

We have now the generating vector

$$\mathbf{w}_4 = (0, 1, 0, 0, 0, 0, 0).$$

We calculate $\mathbf{w}_3$.

$$\mathbf{w}_3 = B\,\mathbf{w}_4 = \begin{pmatrix} 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -4 & 3 & -1 & -1 \\ 0 & -2 & -1 & -5 & 4 & -1 & -1 \\ 1 & -1 & -2 & -5 & 4 & -1 & -1 \\ 1 & -2 & -3 & -9 & 7 & -2 & -2 \\ -1 & -1 & 2 & 3 & -2 & 1 & 1 \\ 0 & 2 & -1 & -1 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -2 \\ -1 \\ -2 \\ -1 \\ 2 \end{pmatrix}.$$

We have now $\mathbf{w}_3 = (-1, -1, -2, -1, -2, -1, 2)$.

We have now found a second Jordan chain. It has length 2.

$$\{\mathbf{w}_3 = (-1, -1, -2, -1, -2, -1, 2), \mathbf{w}_4 = (0, 1, 0, 0, 0, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	rem dim
$\ker(B)$	4	$\mathbf{w}_1$	$\mathbf{w}_3$	2
$\ker(B^2)$	7	$\mathbf{w}_2$	$\mathbf{w}_4$	1

$\mathbf{w}_1 = (0, 0, 0, 1, 1, -1, 0)$
$\mathbf{w}_2 = (1, 0, 0, 0, 0, 0, 0)$
$\mathbf{w}_3 = (-1, -1, -2, -1, -2, -1, 2)$
$\mathbf{w}_4 = (0, 1, 0, 0, 0, 0, 0)$

The last column tells us that there is still a chain of length 2 remaining left to be found.

The third Jordan chain.
We look for a starting $\mathbf{w_6}$. We see from the information table that there is a generating vector $\mathbf{w_6}$ for a chain of length 2.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^2)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^2)$.

2. The generating vector may not be in the $\ker(B)$ because the length of the chain must be exactly 2. So it has to be independent from all vectors in $\ker(B)$. It is sufficient that it is linearly independent from a basis of $\ker(B)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 2. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^2 together with the vectors in $\ker(B)$ and also the vectors, if any, of exactly height 2 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that $\ker(B^2) = \mathbf{K}^7$.

So the generating vector has the following generic form

$$(a, b, c, d, e, f, g).$$

The kernel of B.
We remember that the kernel $\ker(B)$ is

$$\begin{aligned}\ker(B) = \text{span}\{&(1,0,1,0,0,0,-1),(0,1,1,0,1,0,1),\\ &(0,0,0,1,1,0,-1),(0,0,0,0,0,1,-1)\}.\end{aligned}$$

Vectors chosen in previous Jordan chains.
We have previously chosen two vectors

$$\begin{cases} \mathbf{w}_2 = (1,0,0,0,0,0,0), \\ \mathbf{w}_4 = (0,1,0,0,0,0,0) \end{cases}$$

of exactly height 2 in a previous Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ a & b & c & d & e & f & g \end{pmatrix}.$$

If we impose the condition $c + 3d - 2e + f + g \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $c + 3d - 2e + f + g \neq 0$. We can choose $a = 0$, $b = 0$, $c = 0$, $d = 1$, $e = 0$, $f = 0$, $g = 0$.

So we can choose the generating vector

$$\mathbf{w}_6 = (0,0,0,1,0,0,0).$$

We calculate $\mathbf{w}_5$.

$$\mathbf{w_5} = B\,\mathbf{w_6} = \begin{pmatrix} 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -4 & 3 & -1 & -1 \\ 0 & -2 & -1 & -5 & 4 & -1 & -1 \\ 1 & -1 & -2 & -5 & 4 & -1 & -1 \\ 1 & -2 & -3 & -9 & 7 & -2 & -2 \\ -1 & -1 & 2 & 3 & -2 & 1 & 1 \\ 0 & 2 & -1 & -1 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ -5 \\ -5 \\ -9 \\ 3 \\ -1 \end{pmatrix}.$$

We have now $\mathbf{w_5} = (-1, -4, -5, -5, -9, 3, -1)$.

We have now found the third Jordan chain. It has length 2.

$$\{\mathbf{w_5} = (-1, -4, -5, -5, -9, 3, -1), \mathbf{w_6} = (0, 0, 0, 1, 0, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	chain 3	rem dim
$\ker(B)$	4	$\mathbf{w_1}$	$\mathbf{w_3}$	$\mathbf{w_5}$	1
$\ker(B^2)$	7	$\mathbf{w_2}$	$\mathbf{w_4}$	$\mathbf{w_6}$	0

with

$$\mathbf{w_1} = (0, 0, 0, 1, 1, -1, 0)$$
$$\mathbf{w_2} = (1, 0, 0, 0, 0, 0, 0)$$
$$\mathbf{w_3} = (-1, -1, -2, -1, -2, -1, 2)$$
$$\mathbf{w_4} = (0, 1, 0, 0, 0, 0, 0)$$
$$\mathbf{w_5} = (-1, -4, -5, -5, -9, 3, -1)$$
$$\mathbf{w_6} = (0, 0, 0, 1, 0, 0, 0)$$

The last column consists not entirely of 0's and we are **not** done.

The fourth Jordan chain.
The last column in the information table tells us that there is still a chain of length 1 left to be found. This is an eigenvector $\mathbf{w}_7$. We know that we have in our three previous Jordan chains found three eigenvectors which have height 1. So we have to be careful when choosing an eigenvector.

The vector must be an eigenvector. It must be in the space

$$\begin{aligned}\ker(B) = \operatorname{span}\{&(1,0,1,0,0,0,-1),(0,1,1,0,1,0,1),\\ &(0,0,0,1,1,0,-1),(0,0,0,0,0,1,-1)\}.\end{aligned}$$

The vector must be of the generic form

$$\begin{aligned}&a\,(1,0,1,0,0,0,-1) + b\,(0,1,1,0,1,0,1) + c\,(0,0,0,1,1,0,-1)\\ &\quad + d\,(0,0,0,0,0,1,-1) = (a,b,a+b,c,b+c,d,-a+b-c-d).\end{aligned}$$

We have at this point chosen in $\ker(B)$ already the vectors

$$\begin{cases}\mathbf{w}_1 = (0,0,0,1,1,-1,0),\\ \mathbf{w}_3 = (-1,-1,-2,-1,-2,-1,2),\\ \mathbf{w}_5 = (-1,-4,-5,-5,-9,3,-1)\end{cases}$$

of height exactly height 1. They must be linearly independent from the vector $\mathbf{w}_7$ we have to choose.

We have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & -1 & 0\\ -1 & -1 & -2 & -1 & -2 & -1 & 2\\ -1 & -4 & -5 & -5 & -9 & 3 & -1\\ a & b & a+b & c & b+c & d & -a+b-c-d \end{pmatrix}.$$

We row reduce this matrix H and find then if we impose the condition that $2\,a - c - d \neq 0$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & -1\\ 0 & 1 & 1 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 1 & 0 & -1\\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $2a - c - d \neq 0$.

We can choose $a = 0$, $b = 0$, $c = 0$, $d = 1$, $e = 0$, $f = 0$, $g = 0$.

With this choice we have now the generating vector

$$\mathbf{w}_7 = (0, 0, 0, 0, 0, 1, -1).$$

We have now the last Jordan chain. It has length 1.

$$\{\mathbf{w}_7 = (0, 0, 0, 0, 0, 1, -1)\}.$$

Our table is now

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	chain 3	chain 4	rem dim
$\ker(B)$	4	$\mathbf{w}_1$	$\mathbf{w}_3$	$\mathbf{w}_5$	$\mathbf{w}_7$	0
$\ker(B^2)$	7	$\mathbf{w}_2$	$\mathbf{w}_4$	$\mathbf{w}_6$		0

with

$\mathbf{w}_1 = (0, 0, 0, 1, 1, -1, 0)$
$\mathbf{w}_2 = (1, 0, 0, 0, 0, 0, 0)$
$\mathbf{w}_3 = (-1, -1, -2, -1, -2, -1, 2)$
$\mathbf{w}_4 = (0, 1, 0, 0, 0, 0, 0)$
$\mathbf{w}_5 = (-1, -4, -5, -5, -9, 3, -1)$
$\mathbf{w}_6 = (0, 0, 0, 1, 0, 0, 0)$
$\mathbf{w}_7 = (0, 0, 0, 0, 0, 1, -1)$

We have now a basis of vectors consisting of independent vectors in Jordan chains.

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -4 & 0 & 0 \\ 0 & 0 & -2 & 0 & -5 & 0 & 0 \\ 1 & 0 & -1 & 0 & -5 & 1 & 0 \\ 1 & 0 & -2 & 0 & -9 & 0 & 0 \\ -1 & 0 & -1 & 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 0 & -1 & 0 & -1 \end{pmatrix}.$$

$$A = P^{-1} B P$$

$$= \begin{pmatrix} 0 & 0 & -5/9 & 0 & 1/9 & -8/9 & -8/9 \\ 1 & 0 & -2/3 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & -7/9 & 0 & 5/9 & 5/9 & 5/9 \\ 0 & 1 & -1/3 & 0 & -1/3 & -1/3 & -1/3 \\ 0 & 0 & 1/9 & 0 & -2/9 & -2/9 & -2/9 \\ 0 & 0 & 1/3 & 1 & -2/3 & 1/3 & 1/3 \\ 0 & 0 & -5/3 & 0 & 4/3 & 4/3 & 1/3 \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -4 & 3 & -1 & -1 \\ 0 & -2 & -1 & -5 & 4 & -1 & -1 \\ 1 & -1 & -2 & -5 & 4 & -1 & -1 \\ 1 & -2 & -3 & -9 & 7 & -2 & -2 \\ -1 & -1 & 2 & 3 & -2 & 1 & 1 \\ 0 & 2 & -1 & -1 & 0 & -1 & -1 \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -4 & 0 & 0 \\ 0 & 0 & -2 & 0 & -5 & 0 & 0 \\ 1 & 0 & -1 & 0 & -5 & 1 & 0 \\ 1 & 0 & -2 & 0 & -9 & 0 & 0 \\ -1 & 0 & -1 & 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 0 & -1 & 0 & -1 \end{pmatrix}.$$

$$
\begin{aligned}
A &= P^{-1} B P \\
&= \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
&= \begin{pmatrix}
\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\
\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\
\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\
\begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} & (\mathbf{0})
\end{pmatrix}.
\end{aligned}
$$

7 exercise. (6×6); $(\mathbf{J}_5(0), \mathbf{J}_1(0))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is a matrix in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 1 & 2 & -1 & 0 & 2 & 1 \\ -1 & -1 & 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 2 & 1 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We compute the Cayley-Hamilton polynomial.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_6| = \lambda^6.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 6. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field $\mathbf{K}$. It is thus possible to put the matrix B in Jordan normal form over the field $\mathbf{K}$.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to

this subsection 2. The solution will be completely independent from this section.

We want to investigate the endomorphism A associated with this Jordan block. We compute also the powers of A.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We show this in a way that emphasises the position of the elementary Jordan blocks. We compute also the powers of A.

$$A = \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} & (0) \end{pmatrix};$$

$$A^2 = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} & (0) \end{pmatrix};$$

$$A^3 = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} & (0) \end{pmatrix};$$

$$A^4 = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} & (0) \end{pmatrix};$$

$$A^5 = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} & (0) \end{pmatrix}.$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix

A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the fifth power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 5, is called the height of nilpotency of the matrix A.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_6\}, \\ \ker(A^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_6\}, \\ \ker(A^3) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_6\}, \\ \ker(A^4) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_6\}, \\ \ker(A^5) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\}. \end{cases}$$

We have then $\ker(A^5) = \mathbf{K}^6$.

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	2	$2 = \dim(\ker(A))$
$\ker(A^2)$	3	$1 = \dim(\ker(A^2)) - \dim(\ker(A))$
$\ker(A^3)$	4	$1 = \dim(\ker(A^3)) - \dim(\ker(A^2))$
$\ker(A^4)$	5	$1 = \dim(\ker(A^4)) - \dim(\ker(A^3))$
$\ker(A^5)$	6	$1 = \dim(\ker(A^5)) - \dim(\ker(A^4))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have dim(ker(A)). This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the fifth power onwards.

$$\ker(A) \subsetneq \ker(A^2) \subsetneq \ker(A^3) \subsetneq \ker(A^4) \subsetneq \ker(A^5) = \ker(A^6) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 + n_3 + n_4 + n_5 = 2 = \dim(\ker(A)), \\ n_2 + n_3 + n_4 + n_5 = 1 = \dim(\ker(A^2)) - \dim(\ker(A)), \\ n_3 + n_4 + n_5 = 1 = \dim(\ker(A^3)) - \dim(\ker(A^2)), \\ n_4 + n_5 = 1 = \dim(\ker(A^4)) - \dim(\ker(A^3)), \\ n_5 = 1 = \dim(\ker(A^5)) - \dim(\ker(A^4)). \end{cases}$$

Solving this system, we have $n_1 = 1$, $n_2 = 0$, $n_3 = 0$, $n_4 = 0$, $n_5 = 1$.

A consequence from this fact is that the numbers in the last column are descending.

We remark by looking at the matrices A^i that the vector $\mathbf{e}_5$ satisfies

$$\begin{cases} A\,\mathbf{e}_5 = \mathbf{e}_4, \\ A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A\,\mathbf{e}_3 = \mathbf{e}_2, \\ A\,\mathbf{e}_2 = \mathbf{e}_1, \\ A\,\mathbf{e}_1 = \mathbf{0}. \end{cases}$$

or

$$\begin{cases} A\,\mathbf{e}_5 = \mathbf{e}_4, \\ A^2\,\mathbf{e}_5 = \mathbf{e}_3, \\ A^3\,\mathbf{e}_5 = \mathbf{e}_2, \\ A^4\,\mathbf{e}_5 = \mathbf{e}_1, \\ A^5\,\mathbf{e}_5 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain with length five of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{A^4\,\mathbf{e}_5 = \mathbf{e}_1, A^3\,\mathbf{e}_5 = \mathbf{e}_2, A^2\,\mathbf{e}_5 = \mathbf{e}_3, A\,\mathbf{e}_5 = \mathbf{e}_4, \mathbf{e}_5\}.$$

After we have found the first Jordan chain of length 5, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A)$	2	$\mathbf{e}_1$	1
$\ker(A^2)$	3	$\mathbf{e}_2$	0
$\ker(A^3)$	4	$\mathbf{e}_3$	0
$\ker(A^4)$	5	$\mathbf{e}_4$	0
$\ker(A^5)$	6	$\mathbf{e}_5$	0

Because the last column consists now of at least one number that is not 0, we are **not** done with looking for Jordan chains. We see that we find another Jordan chain with length 1.

We have the eigenvector $\mathbf{e}_6$. This vector gives us another elementary Jordan chain with length 1.

$$\{\mathbf{e}_6\}.$$

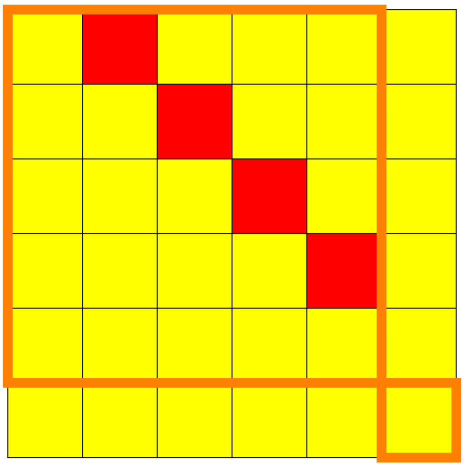
Let us note this in the new information table.

Keeping track of chains and dimensions.

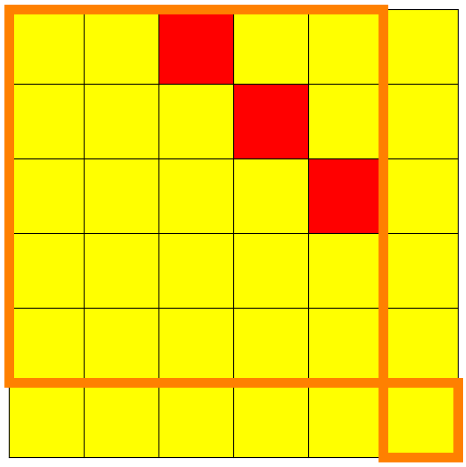
	dim	chain 1	chain 2	rem dim
$\ker(A)$	2	$\mathbf{e}_1$	$\mathbf{e}_6$	0
$\ker(A^2)$	3	$\mathbf{e}_2$		0
$\ker(A^3)$	4	$\mathbf{e}_3$		0
$\ker(A^4)$	5	$\mathbf{e}_4$		0
$\ker(A^5)$	6	$\mathbf{e}_5$		0

The last column consists of only 0's and this ends the search for Jordan chains. We have indeed found 6 linearly independent vectors and these form a basis for $\mathbf{K}^6$.

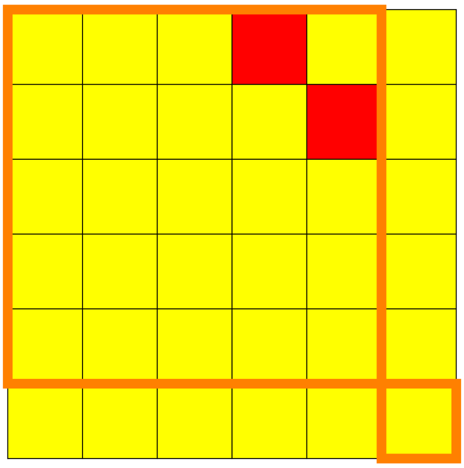
Let us express our findings in a way that emphasises the position of the elementary Jordan blocks and their relation with the powers of the matrix A.



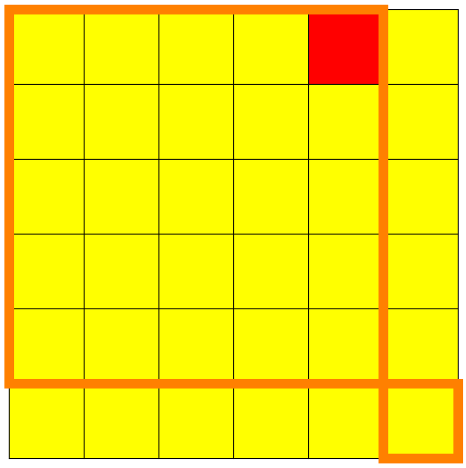
(a) Visualisation of A.



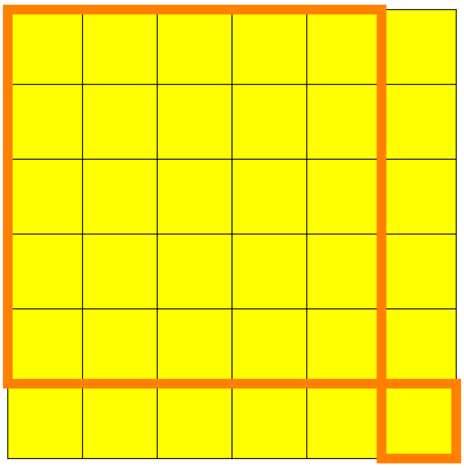
(b) Visualisation of A^2.



(c) Visualisation of A^3.



(d) Visualisation of A^4.



(e) Visualisation of A^5.

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

Let us calculate the powers of the matrix B.

$$B = \begin{pmatrix} 1 & 2 & -1 & 0 & 2 & 1 \\ -1 & -1 & 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 2 & 1 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \end{pmatrix};$$

$$B^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & -1 & 1 & 0 & -1 & 0 \\ -1 & -1 & 1 & 0 & -2 & -1 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix};$$

$$B^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$B^4 = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$B^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 2 & 1 \\ -1 & -1 & 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 2 & 1 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} z_1 + 2z_2 - z_3 + 2z_5 + z_6 = 0, \\ -z_1 - z_2 + z_3 - 2z_5 - z_6 = 0, \\ + z_2 = 0, \\ z_1 + z_2 - z_3 + 2z_5 + z_6 = 0, \\ z_1 + z_2 - z_3 = 0, \\ -z_1 - z_2 + z_3 + z_5 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B) &= \{(r_1, 0, r_1, r_4, 0, 0) \mid r_1, r_4 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,1,0,0,0), (0,0,0,1,0,0)\}. \end{aligned}$$

Kernel of B^2.
We calculate the kernel of B^2 and we have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & -1 & 1 & 0 & -1 & 0 \\ -1 & -1 & 1 & 0 & -2 & -1 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} \quad\quad\quad\quad\quad - z_5 - z_6 = 0, \\ -z_1 - z_2 + z_3 - z_5 = 0, \\ -z_1 - z_2 + z_3 - 2z_5 - z_6 = 0, \\ z_1 + z_2 - z_3 + z_5 = 0, \\ z_1 + z_2 - z_3 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B^2) &= \{(r_1, r_2, r_1 + r_2, r_4, 0, 0) \mid r_1, r_2, r_4 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,1,0,0,0), (0,1,1,0,0,0), (0,0,0,1,0,0)\}. \end{aligned}$$

Kernel of B^3.

We calculate the kernel of B^3 and we have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} \quad\quad\quad\quad\quad - z_5 = 0, \\ -z_1 - z_2 + z_3 = 0, \\ -z_1 - z_2 + z_3 - z_5 = 0, \\ z_1 + z_2 - z_3 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned}\ker(B^3) &= \{(r_1, r_2, r_1 + r_2, r_4, 0, r_6) \mid r_1, r_2, r_4, r_6 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,1,0,0,0), (0,1,1,0,0,0), (0,0,0,1,0,0), \\ &\qquad (0,0,0,0,0,1)\}.\end{aligned}$$

Kernel of B⁴.

We calculate the kernel of B^4 and we have to solve the matrix equation

$$\begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} -z_1 - z_2 + z_3 = 0, \\ -z_1 - z_2 + z_3 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned}\ker(B^4) &= \{(r_1, r_2, r_1 + r_2, r_4, r_5, r_6) \mid r_1, r_2, r_4, r_5, r_6 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,1,0,0,0), (0,1,1,0,0,0), (0,0,0,1,0,0), \\ &\qquad (0,0,0,0,1,0), (0,0,0,0,0,1)\}.\end{aligned}$$

Kernel of B⁵.

This is the vector space $\mathbf{K}^6$.

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	2	$2 = \dim(\ker(B))$
$\ker(B^2)$	3	$1 = \dim(\ker(B^2)) - \dim(\ker(B))$
$\ker(B^3)$	4	$1 = \dim(\ker(B^3)) - \dim(\ker(B^2))$
$\ker(B^4)$	5	$1 = \dim(\ker(B^4)) - \dim(\ker(B^3))$
$\ker(B^5)$	6	$1 = \dim(\ker(B^5)) - \dim(\ker(B^4))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.
2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.
3. In the first row, we have $\dim(\ker(B))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.
4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.
5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column and fifth line is 1 and this gives the information that there will be one Jordan chain of length 5. The first number 2 in the last column the dimension of the kernel of B. After we have calculated the chain with length 5, the last column will be from top to bottom $\{1, 0, 0, 0, 0\}$. There is still one linearly independent vector left to be found. After this calculation, the last column will be from top to bottom $\{0,0,0,0,0,0\}$. We have at that moment 6 linearly independent vectors which form a basis for this vector space.

4. Calculation of the Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$ satisfying

$$\begin{cases} B\,\mathbf{w}_5 = \mathbf{w}_4, \\ B\,\mathbf{w}_4 = \mathbf{w}_3, \\ B\,\mathbf{w}_3 = \mathbf{w}_2, \\ B\,\mathbf{w}_2 = \mathbf{w}_1, \\ B\,\mathbf{w}_1 = \mathbf{0} \end{cases}$$

or

$$\begin{cases} B\,\mathbf{w}_5 = \mathbf{w}_4, \\ B^2\,\mathbf{w}_5 = \mathbf{w}_3, \\ B^3\,\mathbf{w}_5 = \mathbf{w}_2, \\ B^4\,\mathbf{w}_5 = \mathbf{w}_1, \\ B^5\,\mathbf{w}_5 = \mathbf{0}. \end{cases}$$

The first Jordan chain.
We look for a starting $\mathbf{w}_5$. We see from the information table that there is a generating vector $\mathbf{w}_5$ for a chain of length 5.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^5)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^5)$.

2. The generating vector may not be in the $\ker(B^4)$ because the length of the chain must be exactly 5. So it has to be independent from all vectors in $\ker(B^4)$. It is sufficient that it is linearly independent from a basis of $\ker(B^4)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 5. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^5 together with the vectors in $\ker(B^4)$ and also the vectors, if any, of exactly height 5 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that $\ker(B^5) = \mathbf{K}^6$.
So the generating vector has the following generic form

$$(a, b, c, d, e, f).$$

The kernel of B⁴.
We remember that the kernel $\ker(B^4)$ is

$$\ker(B^4) = \text{span}\{(1,0,1,0,0,0),(0,1,1,0,0,0),(0,0,0,1,0,0),\\ (0,0,0,0,1,0),(0,0,0,0,0,1)\}.$$

Vectors chosen in previous Jordan chains.
We have previously chosen no vector of exactly height 5 in a previous Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ a & b & c & d & e & f \end{pmatrix}$$

If we impose the condition $a + b - c \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + b - c \neq 0$.

So we can choose $a = 1$, $b = 0$, $c = 0$, $d = 0$, $e = 0$ and $f = 0$.

Then we have the valid generating vector

$$\mathbf{w}_5 = (1, 0, 0, 0, 0, 0).$$

We calculate then $B^i(1, 0, 0, 0, 0, 0)$ from $i = 1$ to $i = 4$, and we know that the vectors

$$\{\mathbf{w}_1 = B^4\mathbf{w}_5, \mathbf{w}_2 = B^3\mathbf{w}_5, \mathbf{w}_3 = B^2\mathbf{w}_5, \mathbf{w}_4 = B\mathbf{w}_5, \mathbf{w}_5\}$$

are a Jordan chain of 5 linearly independent vectors and because $B^5\mathbf{w}_5 = \mathbf{0}$, we know that the length of the chain is exactly 5.

We calculate $\mathbf{w}_4$.

$$\mathbf{w}_4 = B\mathbf{w}_5 = \begin{pmatrix} 1 & 2 & -1 & 0 & 2 & 1 \\ -1 & -1 & 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 2 & 1 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}.$$

We calculate now $\mathbf{w}_3$.

$$\mathbf{w}_3 = B^2\mathbf{w}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & -1 & 1 & 0 & -1 & 0 \\ -1 & -1 & 1 & 0 & -2 & -1 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

We calculate now $\mathbf{w_2}$.

$$\mathbf{w_2} = B^3\,\mathbf{w_5} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

We calculate now $\mathbf{w_1}$.

$$\mathbf{w_1} = B^4\,\mathbf{w_5} = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We have now found the Jordan chain

$$\{\mathbf{w_1} = (-1,0,-1,0,0,0), \mathbf{w_2} = (0,-1,-1,1,0,0), \mathbf{w_3} = (0,-1,-1,1,0,1), \\ \mathbf{w_4} = (1,-1,0,1,1,-1), \mathbf{w_5} = (1,0,0,0,0,0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	2	$\mathbf{w_1}$	1
$\ker(B^2)$	3	$\mathbf{w_2}$	0
$\ker(B^3)$	4	$\mathbf{w_3}$	0
$\ker(B^4)$	5	$\mathbf{w_4}$	0
$\ker(B^5)$	6	$\mathbf{w_5}$	0

with

$$\mathbf{w_1} = (-1, 0, -1, 0, 0, 0)$$
$$\mathbf{w_2} = (0, -1, -1, 1, 0, 0)$$
$$\mathbf{w_3} = (0, -1, -1, 1, 0, 1)$$
$$\mathbf{w_4} = (1, -1, 0, 1, 1, -1)$$
$$\mathbf{w_5} = (1, 0, 0, 0, 0, 0)$$

We see at the number 1 at the top of the last column that we have still to find a chain of length 1.

The second Jordan chain.
The last column in the information table tells us that there is still a chain of length 1 left to be found. This is an eigenvector $\mathbf{w_6}$. We know that we have in our previous Jordan chain found an eigenvector which have height 1. So we have to be careful when choosing an eigenvector.

The vector must be an eigenvector. It must be in the space

$$\ker(B) = \text{span}\{(1, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0)\}.$$

The vector must be of the generic form

$$a\,(1, 0, 1, 0, 0, 0) + b\,(0, 0, 0, 1, 0, 0) = (a, 0, a, b, 0, 0).$$

We have at this point chosen in $\ker(B)$ already the vector

$$\mathbf{w_1} = (-1, 0, -1, 0, 0, 0)$$

of height exactly 1. This vector must be linearly independent from the vector $\mathbf{w_6}$ we have to choose.

We have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} -1 & 0 & -1 & 0 & 0 & 0 \\ a & 0 & a & b & 0 & 0 \end{pmatrix}.$$

We row reduce this matrix H and find then if we impose the condition that $b \neq 0$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $b \neq 0$.

So we can choose $a = 0$ and $b = 1$.

We have then generating vector

$$\mathbf{w_6} = (0, 0, 0, 1, 0, 0).$$

This vector forms a second Jordan chain on its own.

$$\{(0, 0, 0, 1, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	rem dim
$\ker(B)$	2	$\mathbf{w_1}$	$\mathbf{w_6}$	0
$\ker(B^2)$	3	$\mathbf{w_2}$		0
$\ker(B^3)$	4	$\mathbf{w_3}$		0
$\ker(B^4)$	5	$\mathbf{w_4}$		0
$\ker(B^5)$	6	$\mathbf{w_5}$		0

with

$$\mathbf{w_1} = (-1, 0, -1, 0, 0, 0)$$
$$\mathbf{w_2} = (0, -1, -1, 1, 0, 0)$$
$$\mathbf{w_3} = (0, -1, -1, 1, 0, 1)$$
$$\mathbf{w_4} = (1, -1, 0, 1, 1, -1)$$
$$\mathbf{w_5} = (1, 0, 0, 0, 0, 0)$$
$$\mathbf{w_6} = (0, 0, 0, 1, 0, 0)$$

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}.$$

We check this solution.

$$A = P^{-1} B P$$

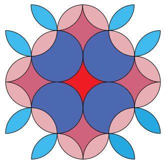
$$= \begin{pmatrix} 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 2 & -1 & 0 & 2 & 1 \\ -1 & -1 & 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 2 & 1 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \left(\begin{array}{ccccc|c} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & (\mathbf{0}) \end{array}\right).$$



8 exercise. (6×6); $(J_4(0), J_2(0))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is a matrix in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 2 & -2 & -1 & 2 & 1 & 1 \\ 4 & -5 & -3 & 5 & 1 & 2 \\ -1 & 2 & 1 & -3 & 1 & 0 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 3 & -4 & -2 & 2 & 0 & 1 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial $p_{\text{C-H}}(\lambda)$.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_6| = \lambda^6.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 6. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field **K**. It is thus possible to put the matrix B in Jordan normal form over the field **K**.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to

this subsection 2. The solution will be completely independent from this section.

We want to investigate the endomorphism A. We compute also the powers of A.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We show this matrix in a way that emphasises the position of the elementary Jordan blocks.

$$\begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}.$$

We compute also the powers of A.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix};$$

$$A^2 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix};$$

$$A^3 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix};$$

$$A^4 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}.$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the fourth power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 4, is called the height of nilpotency of the matrix A.

$$\begin{cases} \ker(A) = \text{span}\{\mathbf{e}_1, \mathbf{e}_5\}, \\ \ker(A^2) = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5, \mathbf{e}_6\}, \\ \ker(A^3) = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_5, \mathbf{e}_6\}, \\ \ker(A^4) = \mathbf{K}^6. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	2	$2 = \dim(\ker(A))$
$\ker(A^2)$	4	$2 = \dim(\ker(A^2)) - \dim(\ker(A))$
$\ker(A^3)$	5	$1 = \dim(\ker(A^3)) - \dim(\ker(A^2))$
$\ker(A^4)$	6	$1 = \dim(\ker(A^4)) - \dim(\ker(A^3))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the fourth power onwards.

$$\ker(A) \subsetneq \ker(A^2) \subsetneq \ker(A^3) \subsetneq \ker(A^4) = \ker(A^5) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 + n_3 + n_4 = 2 = \dim(\ker(A)), \\ n_2 + n_3 + n_4 = 2 = \dim(\ker(A^2)) - \dim(\ker(A)), \\ n_3 + n_4 = 1 = \dim(\ker(A^3)) - \dim(\ker(A^2)), \\ n_4 = 1 = \dim(\ker(A^4)) - \dim(\ker(A^3)). \end{cases}$$

Solving this system, we have $n_1 = 0$, $n_2 = 1$, $n_3 = 0$, $n_4 = 1$.

A consequence from this fact is that the numbers in the last column are descending.

We observe from the figures that

$$\begin{cases} A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A\,\mathbf{e}_3 = \mathbf{e}_2, \\ A\,\mathbf{e}_2 = \mathbf{e}_1, \\ A\,\mathbf{e}_1 = \mathbf{0}. \end{cases}$$

or

$$\begin{cases} A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A^2\,\mathbf{e}_4 = \mathbf{e}_2, \\ A^3\,\mathbf{e}_4 = \mathbf{e}_1, \\ A^4\,\mathbf{e}_4 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain with length four of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{A^3\,\mathbf{e}_4 = \mathbf{e}_1, A^2\,\mathbf{e}_4 = \mathbf{e}_2, A\,\mathbf{e}_4 = \mathbf{e}_3, \mathbf{e}_4\}.$$

After we have found the first Jordan chain of length 4, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A)$	2	$\mathbf{e}_1$	1
$\ker(A^2)$	4	$\mathbf{e}_2$	1
$\ker(A^3)$	5	$\mathbf{e}_3$	0
$\ker(A^4)$	6	$\mathbf{e}_4$	0

One sees that we have still left a Jordan chain with length two of 2 linearly independent vectors. We write a Jordan chain in reverse order. We take the vector $\mathbf{e}_6$ linearly independent from $\mathbf{e}_2$ in the the kernel of A^2 and not in the kernel of A.

$$\{A\,\mathbf{e}_6 = \mathbf{e}_5, \mathbf{e}_6\}.$$

We have found a second Jordan chain. It has length 2.

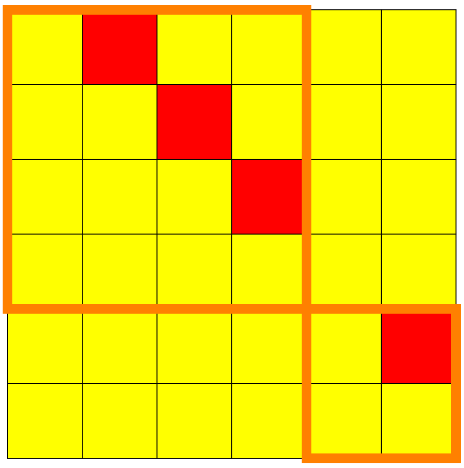
Let us note this in the new information table.

Keeping track of chains and dimensions.

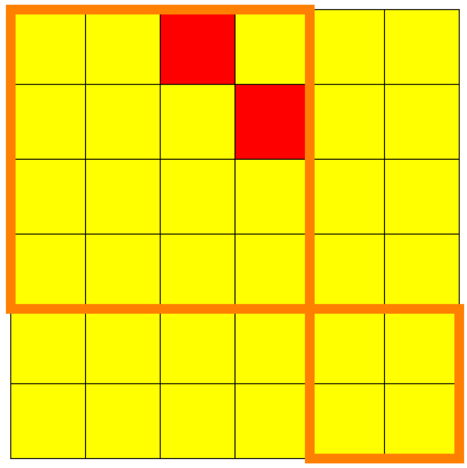
	dim	chain 1	chain 2	rem dim
$\ker(A)$	2	$\mathbf{e}_1$	$\mathbf{e}_5$	0
$\ker(A^2)$	4	$\mathbf{e}_2$	$\mathbf{e}_6$	0
$\ker(A^3)$	5	$\mathbf{e}_3$		0
$\ker(A^4)$	6	$\mathbf{e}_4$		0

The last column consists of only 0's and this ends the search for Jordan chains. We have indeed found 6 linearly independent vectors and these form a basis for $\mathbf{K}^6$.

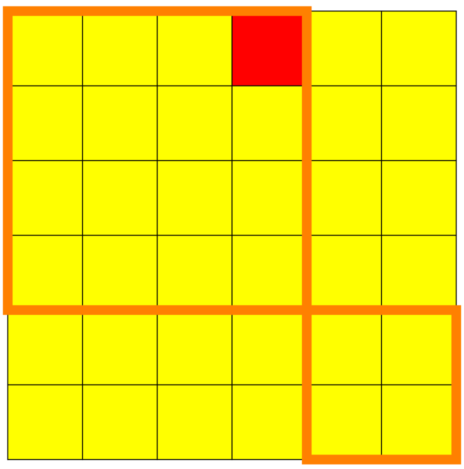
Let us express our findings in a way that emphasises the position of the elementary Jordan blocks and their relation with the powers of the matrix A.



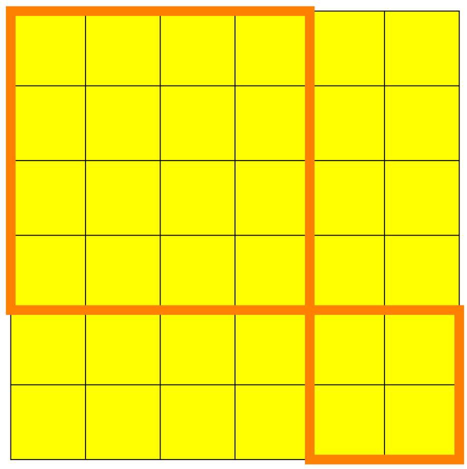
(a) Visualisation of A.



(b) Visualisation of A^2.



(c) Visualisation of A^3.



(d) Visualisation of A^4.

Figure 9

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

Let us calculate the powers of the matrix B.

$$B = \begin{pmatrix} 2 & -2 & -1 & 2 & 1 & 1 \\ 4 & -5 & -3 & 5 & 1 & 2 \\ -1 & 2 & 1 & -3 & 1 & 0 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 3 & -4 & -2 & 2 & 0 & 1 \end{pmatrix};$$

$$B^2 = \begin{pmatrix} 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -2 & -1 & 2 & -1 & 0 \\ 2 & -3 & -2 & 3 & 0 & 1 \\ 2 & -3 & -2 & 3 & 0 & 1 \\ -3 & 4 & 3 & -4 & -1 & -2 \end{pmatrix};$$

$$B^3 = \begin{pmatrix} 2 & -3 & -2 & 3 & 0 & 1 \\ 2 & -3 & -2 & 3 & 0 & 1 \\ -2 & 3 & 2 & -3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 3 & 2 & -3 & 0 & -1 \end{pmatrix};$$

$$B^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} 2 & -2 & -1 & 2 & 1 & 1 \\ 4 & -5 & -3 & 5 & 1 & 2 \\ -1 & 2 & 1 & -3 & 1 & 0 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 3 & -4 & -2 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} 2z_1 - 2z_2 - z_3 + 2z_4 + z_5 + z_6 = 0, \\ 4z_1 - 5z_2 - 3z_3 + 5z_4 + z_5 + 2z_6 = 0, \\ -z_1 + 2z_2 + z_3 - 3z_4 + z_5 = 0, \\ z_1 - z_2 - z_3 + z_4 + z_5 + z_6 = 0, \\ -z_1 + z_2 = 0, \\ 3z_1 - 4z_2 - 2z_3 + 2z_4 + z_6 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B) &= \{(r_1, r_1, r_3, 0, -r_1 - r_3, r_1 + 2\,r_3) \mid r_1, r_3 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,1,0,0,-1,1),(0,0,1,0,-1,2)\}. \end{aligned}$$

Kernel of B^2.
We calculate the kernel of B^2 and we have to solve the matrix equation

$$\begin{pmatrix} 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -2 & -1 & 2 & -1 & 0 \\ 2 & -3 & -2 & 3 & 0 & 1 \\ 2 & -3 & -2 & 3 & 0 & 1 \\ -3 & 4 & 3 & -4 & -1 & -2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} z_1 - z_2 - z_3 + z_4 + z_5 + z_6 = 0, \\ z_1 - z_2 - z_3 + z_4 + z_5 + z_6 = 0, \\ z_1 - 2z_2 - z_3 + 2z_4 - z_5 = 0, \\ 2z_1 - 3z_2 - 2z_3 + 3z_4 + z_6 = 0, \\ 2z_1 - 3z_2 - 2z_3 + 3z_4 + z_6 = 0, \\ -3z_1 + 4z_2 + 3z_3 - 4z_4 - z_5 - 2z_6 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B^2) &= \{(r_1, r_2, r_3, r_4, r_1 - 2\,r_2 - r_3 + 2\,r_4, -2\,r_1 + 3\,r_2 + 2\,r_3 - 3\,r_4) \\ &\qquad \mid r_1, r_2, r_3, r_4 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,1,-2),(0,1,0,0,-2,3),(0,0,1,0,-1,2), \\ &\qquad (0,0,0,1,2,-3)\}. \end{aligned}$$

Kernel of B^3.
We calculate the kernel of B^3 and we have to solve the matrix equation

$$\begin{pmatrix} 2 & -3 & -2 & 3 & 0 & 1 \\ 2 & -3 & -2 & 3 & 0 & 1 \\ -2 & 3 & 2 & -3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 3 & 2 & -3 & 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} 2z_1 - 3z_2 - 2z_3 + 3z_4 + z_6 = 0, \\ 2z_1 - 3z_2 - 2z_3 + 3z_4 + z_6 = 0, \\ -2z_1 + 3z_2 + 2z_3 - 3z_4 - z_6 = 0, \\ -2z_1 + 3z_2 + 2z_3 - 3z_4 - z_6 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B^3) &= \{(r_1, r_2, r_3, r_4, r_5, -2r_1 + 3r_2 + 2r_3 - 3r_4) \\ &\qquad \mid r_1, r_2, r_3, r_4, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,0,-2), (0,1,0,0,0,3), (0,0,1,0,0,2), \\ &\qquad (0,0,0,1,0,-3), (0,0,0,0,1,0)\}. \end{aligned}$$

Kernel of B^4.
We calculate the kernel of B^4 and we have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is the vector space $\mathbf{K}^6$.

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	2	$2 = \dim(\ker(B))$
$\ker(B^2)$	4	$2 = \dim(\ker(B^2)) - \dim(\ker(B))$
$\ker(B^3)$	5	$1 = \dim(\ker(B^3)) - \dim(\ker(B^2))$
$\ker(B^4)$	6	$1 = \dim(\ker(B^4)) - \dim(\ker(B^3))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(B))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column and fourth line is 1 and this gives the information that there will be a Jordan chain of length 4. The first number 2 in the last column is the dimension of the kernel of B. After we have calculated the chain with length 4, the last column will be from top to bottom $\{1, 1, 0, 0\}$. There is still one chain of length 2 with two linearly independent vectors left to be found. After this calculation, the last column will be from top to bottom $\{0, 0, 0, 0, 0, 0\}$. We have at this moment 6 linearly independent vectors which form a basis for this vector space.

4. Calculation of the Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ satisfying

$$\begin{cases} B\,\mathbf{w}_4 = \mathbf{w}_3, \\ B\,\mathbf{w}_3 = \mathbf{w}_2, \\ B\,\mathbf{w}_3 = \mathbf{w}_1, \\ B\,\mathbf{w}_1 = \mathbf{0} \end{cases}$$

or

$$\begin{cases} B\,\mathbf{w}_4 = \mathbf{w}_3, \\ B^2\,\mathbf{w}_4 = \mathbf{w}_2, \\ B^3\,\mathbf{w}_4 = \mathbf{w}_1, \\ B^4\,\mathbf{w}_4 = \mathbf{0}. \end{cases}$$

The first Jordan chain.
We look for a generating vector $\mathbf{w}_4$. We see from the information table that there is a generating vector $\mathbf{w}_4$ for a chain of length 4.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^4)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^4)$.

2. The generating vector may not be in the $\ker(B^3)$ because the length of the chain must be exactly 4. So it has to be independent from all vectors in $\ker(B^3)$. It is sufficient that it is linearly independent from a basis of $\ker(B^3)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 4. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^4 together with the vectors in $\ker(B^3)$ and also the vectors, if any, of exactly height 4 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that $\ker(B^4) = \mathbf{K}^6$.

The generating vector has the following generic form

$$(a, b, c, d, e, f).$$

The kernel of $\mathbf{B^3}$.
We remember that the kernel $\ker(B^3)$ is

$$\ker(B^3) = \operatorname{span}\{(1,0,0,0,0,-2),(0,1,0,0,0,3),(0,0,1,0,0,2),\\ (0,0,0,1,0,-3),(0,0,0,0,1,0)\}.$$

Vectors chosen in previous Jordan chains.
We have previously chosen no vector of exactly height 4 in a previous Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ a & b & c & d & e & f \end{pmatrix}.$$

If we impose the condition $2\,a - 3\,b - 2\,c + 3\,d + f \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $2a - 3b - 2c + 3d + f \neq 0$.

So we can choose $a = 1$, $b = 0$, $c = 0$, $d = 0$, $e = 0$, $f = 0$.

We can choose the generating vector of the chain as follows

$$\mathbf{w_4} = (1, 0, 0, 0, 0, 0).$$

We calculate then $B^i\,(1, 0, 0, 0, 0, 0)$ from $i = 1$ to $i = 4$, and we know that the vectors

$$\{\mathbf{w_1} = B^3\,\mathbf{w_4}, \mathbf{w_2} = B^2\,\mathbf{w_4}, \mathbf{w_3} = B\,\mathbf{w_4}, \mathbf{w_4}\}$$

are a Jordan chain of 4 linearly independent vectors and because $B^4\,\mathbf{w_4} = \mathbf{0}$ we know that the length of the chain is exactly 4.

We calculate $\mathbf{w_3}$.

$$\mathbf{w_3} = B\,\mathbf{w_4} = \begin{pmatrix} 2 & -2 & -1 & 2 & 1 & 1 \\ 4 & -5 & -3 & 5 & 1 & 2 \\ -1 & 2 & 1 & -3 & 1 & 0 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 3 & -4 & -2 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -1 \\ 1 \\ -1 \\ 3 \end{pmatrix}.$$

We calculate $\mathbf{w_2}$.

$$\mathbf{w_2} = B^2\,\mathbf{w_4} = \begin{pmatrix} 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -2 & -1 & 2 & -1 & 0 \\ 2 & -3 & -2 & 3 & 0 & 1 \\ 2 & -3 & -2 & 3 & 0 & 1 \\ -3 & 4 & 3 & -4 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ -3 \end{pmatrix}.$$

We calculate $\mathbf{w_1}$.

$$\mathbf{w_1} = B^3\mathbf{w_4} = \begin{pmatrix} 2 & -3 & -2 & 3 & 0 & 1 \\ 2 & -3 & -2 & 3 & 0 & 1 \\ -2 & 3 & 2 & -3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 3 & 2 & -3 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ 0 \\ 0 \\ -2 \end{pmatrix}.$$

We have now found the first Jordan chain. It has length 4.

$$\{\mathbf{w_1} = (2,2,-2,0,0,-2), \mathbf{w_2} = (1,1,1,2,2,-3),$$
$$\mathbf{w_3} = (2,4,-1,1,-1,3), \mathbf{w_4} = (1,0,0,0,0,0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	2	$\mathbf{w_1}$	1
$\ker(B^2)$	4	$\mathbf{w_2}$	1
$\ker(B^3)$	5	$\mathbf{w_3}$	0
$\ker(B^4)$	6	$\mathbf{w_4}$	0

with

$$\mathbf{w_1} = (2,2,-2,0,0,-2)$$
$$\mathbf{w_2} = (1,1,1,2,2,-3)$$
$$\mathbf{w_3} = (2,4,-1,1,-1,3)$$
$$\mathbf{w_4} = (1,0,0,0,0,0)$$

We see at the number 1 at the second line of the last column that we have still a chain left of length 2.

We look for a generating vector $\mathbf{w_6}$. We see from the information table that there is a generating vector $\mathbf{w_6}$ for a chain of length 2.

The second Jordan chain.
The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^2)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^2)$.
2. The generating vector may not be in the $\ker(B)$ because the length of the chain must be exactly 2. So it has to be independent from all vectors in $\ker(B)$. It is sufficient that it is linearly independent from a basis of $\ker(B)$.
3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 2. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.
4. We summarise: the generating vector in B^2 together with the vectors in $\ker(B)$ and also the vectors, if any, of exactly height 2 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^2) = \operatorname{span}\{(1,0,0,0,1,-2),(0,1,0,0,-2,3),(0,0,1,0,-1,2),\\ (0,0,0,1,2,-3)\}.$$

So the generating vector has the following generic form

$$\begin{aligned} a\,(1,0,0,0,1,-2) + b\,(0,1,0,0,-2,3) + c\,(0,0,1,0,-1,2)\\ + d\,(0,0,0,1,2,-3)\\ = (a,b,c,d,a-2\,b-c+2\,d,-2\,a+3\,b+2\,c-3\,d). \end{aligned}$$

The kernel of B.
We remember that the kernel $\ker(B)$ is

$$\ker(B) = \operatorname{span}\{(1,1,0,0,-1,1),(0,0,1,0,-1,2)\}.$$

Vectors chosen in previous Jordan chains.
We have previously chosen the vector $\mathbf{w_2} = (1,1,1,2,2,-3)$ of exactly height 2 in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 1 & 1 & 1 & 2 & 2 & -3 \\ a & b & c & d & a-2b-c+2d & -2a+3b+2c-3d \end{pmatrix}.$$

If we impose the condition $a - b \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 & -2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 & -3 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a - b \neq 0$.

So we can choose $a = 0$, $b = 1$, $c = 0$, $d = 0$. We have now the valid generating vector

$$\mathbf{w_6} = (0,1,0,0,-2,3).$$

We calculate now our chain with length 2.

$$\{B\,\mathbf{w_6}, \mathbf{w_6}\}.$$

We calculate $\mathbf{w_5}$.

$$\mathbf{w}_5 = B\,\mathbf{w}_6 = \begin{pmatrix} 2 & -2 & -1 & 2 & 1 & 1 \\ 4 & -5 & -3 & 5 & 1 & 2 \\ -1 & 2 & 1 & -3 & 1 & 0 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 3 & -4 & -2 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

We have now found the second Jordan chain. It has length 2.

$$\{\mathbf{w}_5 = (-1,-1,0,0,1,-1), \mathbf{w}_6 = (0,1,0,0,-2,3)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	rem dim
$\ker(B)$	2	$\mathbf{w}_1$	$\mathbf{w}_5$	0
$\ker(B^2)$	4	$\mathbf{w}_2$	$\mathbf{w}_6$	0
$\ker(B^3)$	5	$\mathbf{w}_3$		0
$\ker(B^4)$	6	$\mathbf{w}_4$		0

with

$$\mathbf{w}_1 = (2,2,-2,0,0,-2)$$
$$\mathbf{w}_2 = (1,1,1,2,2,-3)$$
$$\mathbf{w}_3 = (2,4,-1,1,-1,3)$$
$$\mathbf{w}_4 = (1,0,0,0,0,0)$$
$$\mathbf{w}_5 = (-1,-1,0,0,1,-1)$$
$$\mathbf{w}_6 = (0,1,0,0,-2,3)$$

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} 2 & 1 & 2 & 1 & -1 & 0 \\ 2 & 1 & 4 & 0 & -1 & 1 \\ -2 & 1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & -2 \\ -2 & -3 & 3 & 0 & -1 & 3 \end{pmatrix}.$$

We check this solution.

$$
\begin{aligned}
A &= P^{-1} B P \\
&= \begin{pmatrix} 0 & -3/16 & -1/2 & 7/16 & -3/8 & -3/16 \\ 0 & -1/8 & 0 & 5/8 & -1/4 & -1/8 \\ 0 & 1/4 & 0 & -1/4 & 1/2 & 1/4 \\ 1 & -3/2 & -1 & 3/2 & 0 & 1/2 \\ 0 & -3/2 & -2 & 5/2 & 0 & 1/2 \\ 0 & -1 & -1 & 2 & -1 & 0 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 2 & -2 & -1 & 2 & 1 & 1 \\ 4 & -5 & -3 & 5 & 1 & 2 \\ -1 & 2 & 1 & -3 & 1 & 0 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 3 & -4 & -2 & 2 & 0 & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 2 & 1 & 2 & 1 & -1 & 0 \\ 2 & 1 & 4 & 0 & -1 & 1 \\ -2 & 1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & -2 \\ -2 & -3 & 3 & 0 & -1 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}.
\end{aligned}
$$

9 exercise. (6×6); $(\mathbf{J_3(0)}, \mathbf{J_3(0)})$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B so that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 3 & 3 & -1 & 2 & 2 & 1 \\ -5 & -5 & 1 & -3 & -3 & -1 \\ -12 & -10 & 3 & -8 & -5 & -3 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -4 & -4 & 1 & -3 & -2 & -1 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial $p_{\text{C-H}}(\lambda)$.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_6| = \lambda^6.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 6. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field $\mathbf{K}$. It is thus possible to put the matrix B in Jordan normal form over the field $\mathbf{K}$.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to

this subsection 2. The solution will be completely independent from this section.

We want to investigate the endomorphism A associated with this Jordan block.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We show this in a way that emphasises the position of the elementary Jordan blocks.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}.$$

We compute also the powers of A.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix};$$

$$A^2 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix};$$

$$A^3 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}.$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the third power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 3, is called the height of nilpotency of the matrix A.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_4\}, \\ \ker(A^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5\}, \\ \ker(A^3) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_5, \mathbf{e}_6\} = \mathbf{K}^6. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	2	$2 = \dim(\ker(A))$
$\ker(A^2)$	4	$2 = \dim(\ker(A^2)) - \dim(\ker(A))$
$\ker(A^3)$	6	$2 = \dim(\ker(A^3)) - \dim(\ker(A^2))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.
2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.
3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.
4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.
5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the third power onwards.

$$\ker(A) \subsetneq \ker(A^2) \subsetneq \ker(A^3) = \ker(A^4) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 + n_3 = 2 = \dim(\ker(A)), \\ \qquad n_2 + n_3 = 2 = \dim(\ker(A^2)) - \dim(\ker(A)), \\ \qquad\qquad n_3 = 2 = \dim(\ker(A^3)) - \dim(\ker(A^2)). \end{cases}$$

Solving this system, we have $n_1 = 0$, $n_2 = 0$, $n_3 = 2$.

A consequence from this fact is that the numbers in the last column are descending.

One sees that we have two Jordan chains with length three of linearly independent vectors.

Let us start with the first chain.

We observe from the matrices that

$$\begin{cases} A\,\mathbf{e}_3 = \mathbf{e}_2, \\ A\,\mathbf{e}_2 = \mathbf{e}_1, \\ A\,\mathbf{e}_1 = \mathbf{0}. \end{cases}$$

or

$$\begin{cases} A\,\mathbf{e}_3 = \mathbf{e}_2, \\ A^2\,\mathbf{e}_3 = \mathbf{e}_1, \\ A^3\,\mathbf{e}_3 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain with length three of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{A^2\,\mathbf{e}_3 = \mathbf{e}_1, A\,\mathbf{e}_3 = \mathbf{e}_2, \mathbf{e}_3\}.$$

After we have found the first Jordan chain of length 3, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A)$	2	$\mathbf{e}_1$	1
$\ker(A^2)$	4	$\mathbf{e}_2$	1
$\ker(A^3)$	6	$\mathbf{e}_3$	1

One sees that we have now a Jordan chain left with length three of 3 linearly independent vectors.

We observe from the matrices that

$$\begin{cases} A\,\mathbf{e}_6 = \mathbf{e}_5, \\ A\,\mathbf{e}_5 = \mathbf{e}_4, \\ A\,\mathbf{e}_4 = \mathbf{0}. \end{cases}$$

or

$$\begin{cases} A\,\mathbf{e}_6 = \mathbf{e}_5, \\ A^2\,\mathbf{e}_6 = \mathbf{e}_4, \\ A^3\,\mathbf{e}_6 = \mathbf{0}. \end{cases}$$

We write a Jordan chain in reverse order.

$$\{A^2\,\mathbf{e}_6 = \mathbf{e}_4, A\,\mathbf{e}_6 = \mathbf{e}_5, \mathbf{e}_6\}.$$

We have found the second Jordan chain. It has length 3. We have now the following table.

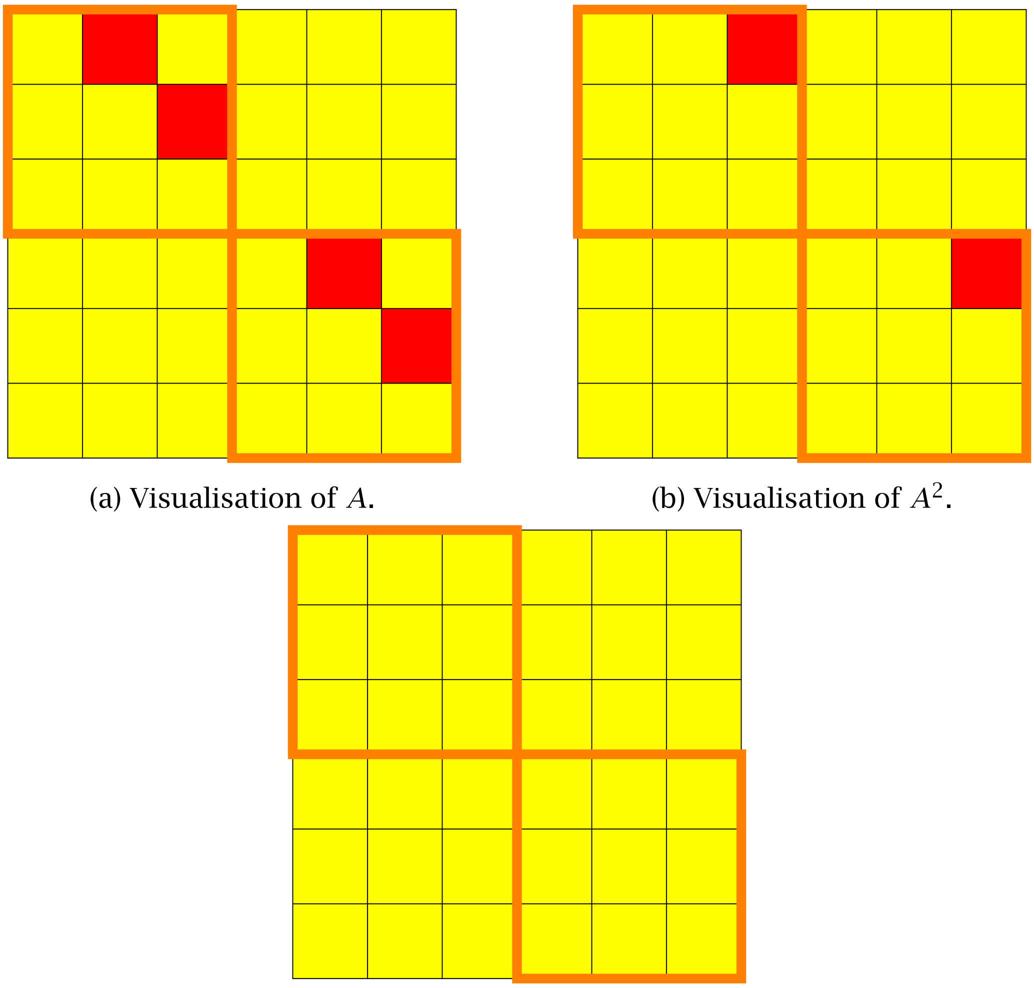
Let us note this in the new information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	rem dim
$\ker(A)$	2	$\mathbf{e}_1$	$\mathbf{e}_4$	0
$\ker(A^2)$	4	$\mathbf{e}_2$	$\mathbf{e}_5$	0
$\ker(A^3)$	5	$\mathbf{e}_3$	$\mathbf{e}_6$	0

Because the last column consists now of 0's, we are done with looking for Jordan chains. We have found at this point a basis of V consisting entirely of vectors in Jordan chains.

Let us express our findings in a way that emphasises the position of the elementary Jordan blocks and their relation with the powers of the matrix A.



(a) Visualisation of A.

(b) Visualisation of A^2.

(c) Visualisation of A^3.

Figure 10

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red,

and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

Let us calculate the powers of the matrix B.

$$B = \begin{pmatrix} 3 & 3 & -1 & 2 & 2 & 1 \\ -5 & -5 & 1 & -3 & -3 & -1 \\ -12 & -10 & 3 & -8 & -5 & -3 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -4 & -4 & 1 & -3 & -2 & -1 \end{pmatrix};$$

$$B^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 4 & -1 & 3 & 2 & 1 \\ -2 & -1 & 0 & -1 & 0 & 0 \\ -5 & -4 & 1 & -3 & -2 & -1 \\ -5 & -4 & 1 & -3 & -2 & -1 \\ 3 & 3 & -1 & 2 & 2 & 1 \end{pmatrix};$$

$$B^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} 3 & 3 & -1 & 2 & 2 & 1 \\ -5 & -5 & 1 & -3 & -3 & -1 \\ -12 & -10 & 3 & -8 & -5 & -3 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -4 & -4 & 1 & -3 & -2 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} 3z_1 + 3z_2 - z_3 + 2z_4 + 2z_5 + z_6 = 0, \\ -5z_1 - 5z_2 + z_3 - 3z_4 - 3z_5 - z_6 = 0, \\ -12z_1 - 10z_2 + 3z_3 - 8z_4 - 5z_5 - 3z_6 = 0, \\ -z_1 - z_2 + z_3 - z_4 - z_5 - z_6 = 0, \\ z_2 + z_5 = 0, \\ -4z_1 - 4z_2 + z_3 - 3z_4 - 2z_5 - z_6 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned}\ker(B) &= \{(0, r_2, r_3, -r_2, -r_2, r_2 + r_3) \mid r_2, r_3 \in \mathbf{K}\}\\ &= \operatorname{span}\{(0,1,0,-1,-1,1),(0,0,1,0,0,1)\}.\end{aligned}$$

Kernel of B^2.
We calculate the kernel of B^2 and we have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 5 & 4 & -1 & 3 & 2 & 1\\ -2 & -1 & 0 & -1 & 0 & 0\\ -5 & -4 & 1 & -3 & -2 & -1\\ -5 & -4 & 1 & -3 & -2 & -1\\ 3 & 3 & -1 & 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\\ z_5\\ z_6\end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} 5z_1 + 4z_2 - z_3 + 3z_4 + 2z_5 + z_6 = 0,\\ -2z_1 - z_2 - z_4 = 0,\\ -5z_1 - 4z_2 + z_3 - 3z_4 - 2z_5 - z_6 = 0,\\ -5z_1 - 4z_2 + z_3 - 3z_4 - 2z_5 - z_6 = 0,\\ 3z_1 + 3z_2 - z_3 + 2z_4 + 2z_5 + z_6 = 0.\end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned}\ker(B^2) &= \{(r_1, r_2, r_3, -2r_1 - r_2, r_5, r_1 - r_2 + r_3 - 2r_5) \mid r_1, r_2, r_3, r_5 \in \mathbf{K}\}\\ &= \operatorname{span}\{(1,0,0,-2,0,1),(0,1,0,-1,0,-1),\\ &\qquad (0,0,1,0,0,1),(0,0,0,0,1,-2)\}.\end{aligned}$$

Kernel of B^3.
We calculate the kernel of B^3 and we have to solve the matrix equation

$$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\end{pmatrix}\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\\ z_5\\ z_6\end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\end{pmatrix}.$$

This is the vector space $\mathbf{K}^6$.

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	2	$2 = \dim(\ker(B))$
$\ker(B^2)$	4	$2 = \dim(\ker(B^2)) - \dim(\ker(B))$
$\ker(B^3)$	6	$2 = \dim(\ker(B^3)) - \dim(\ker(B^2))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(B))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column is 2 and this gives the information that there will be two Jordan chains of length 3. The first number 2 in the last column is the dimension of the kernel of B. After we have calculated the chain with length 3, the last column will be from top to bottom $\{1, 1, 1\}$. There is still one chain with three linearly independent vectors left to be found. After we have found these three vectors, the last column will be from top to bottom $\{0, 0, 0\}$. We have at that moment 6 linearly independent vectors which form a basis for this vector space.

4. Calculation of the Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ satisfying

$$\begin{cases} B\,\mathbf{w}_3 = \mathbf{w}_2, \\ B^2\,\mathbf{w}_3 = \mathbf{w}_1, \\ B^3\,\mathbf{w}_3 = \mathbf{0}. \end{cases}$$

The first Jordan chain.
We look for a starting $\mathbf{w}_3$. We see from the information table that there is a generating vector $\mathbf{w}_3$ for a chain of length 3.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^3)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^3)$.

2. The generating vector may not be in the $\ker(B^2)$ because the length of the chain must be exactly 3. So it has to be independent from all vectors in $\ker(B^2)$. It is sufficient that it is linearly independent from a basis of $\ker(B^2)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 3. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^3 together with the vectors in $\ker(B^2)$ and also the vectors, if any, of exactly height 3 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^3) = \mathbf{K}^6.$$

So the generating vector has the following generic form

$$(a, b, c, d, e, f).$$

The kernel of B^2.
We remember that the kernel $\ker(B^2)$ is

$$\ker(B^2) = \text{span}\{(1,0,0,-2,0,1),(0,1,0,-1,0,-1),\\ (0,0,1,0,0,1),(0,0,0,0,1,-2)\}.$$

Vectors chosen in previous Jordan chains.
We have previously not chosen a vector of exactly height 3 in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ a & b & c & d & e & f \end{pmatrix}.$$

If we impose the condition $2\,a + b + d \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \frac{3\,b-2c+d+4e+2\,f}{2\,a+b+d} \\ 0 & 1 & 0 & 0 & 0 & \frac{-3\,a-c-d+2\,e+f}{2\,a+b+d} \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & \frac{-a+b-c+2\,e+f}{2\,a+b+d} \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $2\,a + b + d \neq 0$.

So we can choose $a = 1$, $b = 0$, $c = 0$, $d = 0$, $e = 0$, $f = 0$. We can choose a valid generating vector of the chain as follows

$$\mathbf{w_3} = (1,0,0,0,0,0).$$

We calculate then $B^i\,(1,0,0,0,0,0)$ from $i = 1$ to $i = 2$, and we know that the vectors

$$\{\mathbf{w_1} = B^2\,\mathbf{w_3}, \mathbf{w_2} = B\,\mathbf{w_3}, \mathbf{w_3}\}$$

are a Jordan chain of 3 linearly independent vectors and because $B^3\,\mathbf{w_4} = \mathbf{0}$, we know that the length of the chain is exactly 3.

We calculate $\mathbf{w_2}$

$$\mathbf{w_2} = B\,\mathbf{w_3} = \begin{pmatrix} 3 & 3 & -1 & 2 & 2 & 1 \\ -5 & -5 & 1 & -3 & -3 & -1 \\ -12 & -10 & 3 & -8 & -5 & -3 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -4 & -4 & 1 & -3 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ -12 \\ -1 \\ 0 \\ -4 \end{pmatrix}.$$

We calculate $\mathbf{w_1}$

$$\mathbf{w_1} = B^2\,\mathbf{w_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 4 & -1 & 3 & 2 & 1 \\ -2 & -1 & 0 & -1 & 0 & 0 \\ -5 & -4 & 1 & -3 & -2 & -1 \\ -5 & -4 & 1 & -3 & -2 & -1 \\ 3 & 3 & -1 & 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -2 \\ -5 \\ -5 \\ 3 \end{pmatrix}.$$

We have now found the Jordan chain

$$\{\mathbf{w_1} = (0,5,-2,-5,-5,3), \mathbf{w_2} = (3,-5,-12,-1,0,-4), \mathbf{w_3} = (1,0,0,0,0,0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	2	$\mathbf{w_1}$	1
$\ker(B^2)$	4	$\mathbf{w_2}$	1
$\ker(B^3)$	6	$\mathbf{w_3}$	1

with

$$\mathbf{w_1} = (0, 5, -2, -5, -5, 3)$$

$$\mathbf{w_2} = (3, -5, -12, -1, 0, -4)$$

$$\mathbf{w_3} = (1, 0, 0, 0, 0, 0)$$

We see at the number 1 at the third row of the last column that there is still a Jordan chain of length 3 left to be found.

The second Jordan chain.
We look for a starting $\mathbf{w_6}$. We see from the information table that there is a generating vector $\mathbf{w_6}$ for a chain of length 3.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^3)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^3)$.

2. The generating vector may not be in the $\ker(B^2)$ because the length of the chain must be exactly 3. So it has to be independent from all vectors in $\ker(B^2)$. It is sufficient that it is linearly independent from a basis of $\ker(B^2)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 3. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^3 together with the vectors in $\ker(B^2)$ and also the vectors, if any, of exactly height 3 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^3) = \mathbf{K}^6.$$

So the generating vector has the following generic form

$$(a, b, c, d, e, f).$$

The kernel of B^2.
We remember that the kernel $\ker(B^2)$ is

$$\ker(B^2) = \text{span}\{(1,0,0,-2,0,1),(0,1,0,-1,0,-1),\\ (0,0,1,0,0,1),(0,0,0,0,1,-2)\}.$$

Vectors chosen in previous Jordan chains.
We have previously chosen the vector $\mathbf{w_3} = (1,0,0,0,0,0)$ of exactly height 3 in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ a & b & c & d & e & f \end{pmatrix}.$$

If we impose the condition $3\,b - 2\,c + d + 4\,e + 2\,f \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $3\,b - 2\,c + d + 4\,e + 2\,f \neq 0$.

So we can choose $a = 0$, $b = 0$, $c = 0$, $d = 0$. $e = 0$ and $f = 1$.

We can choose a valid generating vector of the chain as follows

$$\mathbf{w_6} = (0,0,0,0,0,1).$$

We calculate now our chain with length 3.

$$\{B^2\,\mathbf{w_6}, B\,\mathbf{w_6}, \mathbf{w_6}\}.$$

We calculate $\mathbf{w_5}$.

$$\mathbf{w_5} = B\,\mathbf{w_6} = \begin{pmatrix} 3 & 3 & -1 & 2 & 2 & 1 \\ -5 & -5 & 1 & -3 & -3 & -1 \\ -12 & -10 & 3 & -8 & -5 & -3 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -4 & -4 & 1 & -3 & -2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -3 \\ -1 \\ 0 \\ -1 \end{pmatrix}.$$

We calculate $\mathbf{w_4}$

$$\mathbf{w_4} = B\,\mathbf{w_5} = \begin{pmatrix} 3 & 3 & -1 & 2 & 2 & 1 \\ -5 & -5 & 1 & -3 & -3 & -1 \\ -12 & -10 & 3 & -8 & -5 & -3 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -4 & -4 & 1 & -3 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -3 \\ -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

We have now found the second Jordan chain. It has length 3.

$$\{\mathbf{w_4} = (0,1,0,-1,-1,1), \mathbf{w_5} = (1,-1,-3,-1,0,-1), \mathbf{w_6} = (0,0,0,0,0,1)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	rem dim
$\ker(B)$	2	$\mathbf{w_1}$	$\mathbf{w_4}$	0
$\ker(B^2)$	4	$\mathbf{w_2}$	$\mathbf{w_5}$	0
$\ker(B^3)$	6	$\mathbf{w_3}$	$\mathbf{w_6}$	0

with

$$\mathbf{w_1} = (0, 5, -2, -5, -5, 3)$$

$$\mathbf{w_2} = (3, -5, -12, -1, 0, -4)$$

$$\mathbf{w_3} = (1, 0, 0, 0, 0, 0)$$

$$\mathbf{w_4} = (0, 1, 0, -1, -1, 1)$$

$$\mathbf{w_5} = (1, -1, -3, -1, 0, -1)$$

$$\mathbf{w_6} = (0, 0, 0, 0, 0, 1)$$

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} 0 & 3 & 1 & 0 & 1 & 0 \\ 5 & -5 & 0 & 1 & -1 & 0 \\ -2 & -12 & 0 & 0 & -3 & 0 \\ -5 & -1 & 0 & -1 & -1 & 0 \\ -5 & 0 & 0 & -1 & 0 & 0 \\ 3 & -4 & 0 & 1 & -1 & 1 \end{pmatrix}.$$

We check this solution.

$$A = P^{-1} B P$$

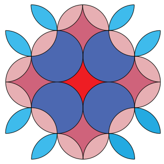
$$= \begin{pmatrix} 0 & 9/8 & -1/2 & 3/8 & 3/4 & 0 \\ 0 & -1/4 & 0 & 1/4 & -1/2 & 0 \\ 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & -45/8 & 5/2 & -15/8 & -19/4 & 0 \\ 0 & 1/4 & 0 & -5/4 & 3/2 & 0 \\ 0 & 3/2 & -1 & 1/2 & 2 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 3 & 3 & -1 & 2 & 2 & 1 \\ -5 & -5 & 1 & -3 & -3 & -1 \\ -12 & -10 & 3 & -8 & -5 & -3 \\ -1 & -1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -4 & -4 & 1 & -3 & -2 & -1 \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & 3 & 1 & 0 & 1 & 0 \\ 5 & -5 & 0 & 1 & -1 & 0 \\ -2 & -12 & 0 & 0 & -3 & 0 \\ -5 & -1 & 0 & -1 & -1 & 0 \\ -5 & 0 & 0 & -1 & 0 & 0 \\ 3 & -4 & 0 & 1 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}.$$



10 exercise. (5×5); $(\mathbf{J_4(0)}, \mathbf{J_1(0)})$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B so that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 0 & 2 & -1 & -4 & -2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & -2 & 0 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial $p_{\text{C-H}}(\lambda)$.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_5| = -\lambda^5.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 5. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field $\mathbf{K}$. It is thus possible to put the matrix B in Jordan normal form over the field $\mathbf{K}$.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to

this subsection 2. The solution will be completely independent from this section.

Before trying to find a solution of this exercise, let us try to see what is going on with a particular related type of a Jordan matrix. We want to investigate the endomorphism A associated with this Jordan block.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We show this in a way that emphasises the position of the elementary Jordan blocks.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} & (\mathbf{0}) \end{pmatrix}.$$

We compute also the powers of A.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} & (\mathbf{0}) \end{pmatrix};$$

$$A^2 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} & (\mathbf{0}) \end{pmatrix};$$

$$A^3 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} & (\mathbf{0}) \end{pmatrix};$$

$$A^4 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} & (\mathbf{0}) \end{pmatrix}.$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the fourth power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 4, is called the height of nilpotency of the matrix A.

It is interesting to observe how the kernels change. We have $\ker(A^4) = \mathbf{K}^5$.

$$\begin{cases} \ker(A) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_5\}, \\ \ker(A^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5\}, \\ \ker(A^3) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_5\}, \\ \ker(A^4) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	2	$2 = \dim(\ker(A))$
$\ker(A^2)$	3	$1 = \dim(\ker(A^2)) - \dim(\ker(A))$
$\ker(A^3)$	4	$1 = \dim(\ker(A^3)) - \dim(\ker(A^2))$
$\ker(A^4)$	5	$1 = \dim(\ker(A^4)) - \dim(\ker(A^3))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see that there is equality in the inclusion of sets from the fourth power onwards.

$$\ker(A) \subsetneq \ker(A^2) \subsetneq \ker(A^3) \subsetneq \ker(A^4) = \ker(A^5) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 + n_3 + n_4 = 2 = \dim(\ker(A)), \\ \qquad n_2 + n_3 + n_4 = 1 = \dim(\ker(A^2)) - \dim(\ker(A)), \\ \qquad\qquad n_3 + n_4 = 1 = \dim(\ker(A^3)) - \dim(\ker(A^2)), \\ \qquad\qquad\qquad n_4 = 1 = \dim(\ker(A^4)) - \dim(\ker(A^3)). \end{cases}$$

Solving this system, we have $n_1 = 1$, $n_2 = 0$, $n_3 = 0$, $n_4 = 1$.

A consequence from this fact is that the numbers in the last column are descending.

One sees by looking at the matrices A^i that we have two Jordan chains. Let us start with the first chain.

We observe from the matrices that

$$\begin{cases} A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A\,\mathbf{e}_3 = \mathbf{e}_2, \\ A\,\mathbf{e}_2 = \mathbf{e}_1, \\ A\,\mathbf{e}_1 = \mathbf{0}, \end{cases}$$

or

$$\begin{cases} A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A^2\,\mathbf{e}_4 = \mathbf{e}_2, \\ A^3\,\mathbf{e}_4 = \mathbf{e}_1, \\ A^4\,\mathbf{e}_4 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain with length four of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{A^3\,\mathbf{e}_4 = \mathbf{e}_1, A^2\,\mathbf{e}_4 = \mathbf{e}_2, A\,\mathbf{e}_4 = \mathbf{e}_3, \mathbf{e}_4\}.$$

After we have found the first Jordan chain of length 4, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
ker(A)	2	$\mathbf{e}_1$	1
ker(A^2)	3	$\mathbf{e}_2$	0
ker(A^3)	4	$\mathbf{e}_3$	0
ker(A^4)	5	$\mathbf{e}_4$	0

Because the last column consists now of at least one number that is not 0, we are **not** done with looking for Jordan chains. We see that we can find another Jordan chain with length 1.

We have found the eigenvector $\mathbf{e}_5$. This vector is an elementary Jordan chain on its own. It has length 1.

$$\{\mathbf{e}_5\}.$$

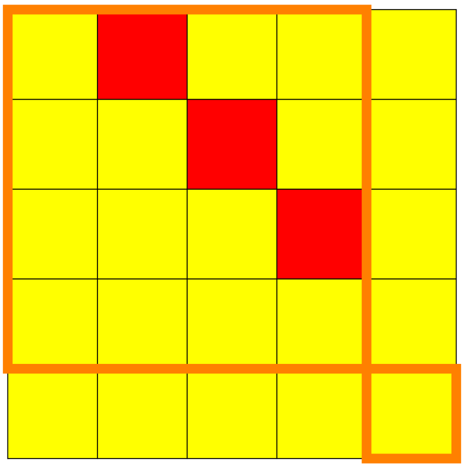
Let us note this in the new information table.

Keeping track of chains and dimensions.

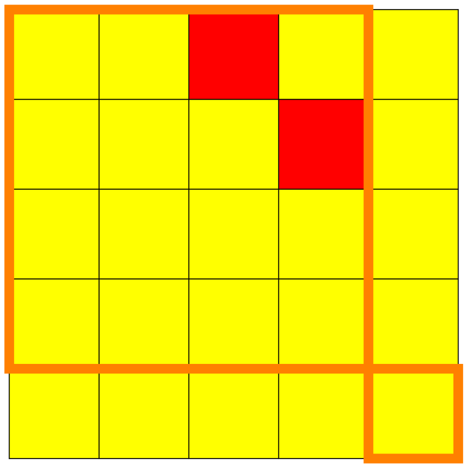
	dim	chain 1	chain 2	rem dim
ker(A)	2	$\mathbf{e}_1$	$\mathbf{e}_5$	0
ker(A^2)	3	$\mathbf{e}_2$		0
ker(A^3)	4	$\mathbf{e}_3$		0
ker(A^4)	5	$\mathbf{e}_4$		0

The last column consists of only 0's and this ends the search for Jordan chains. We have indeed found 5 linearly independent vectors and these form a basis for $\mathbf{K}^5$.

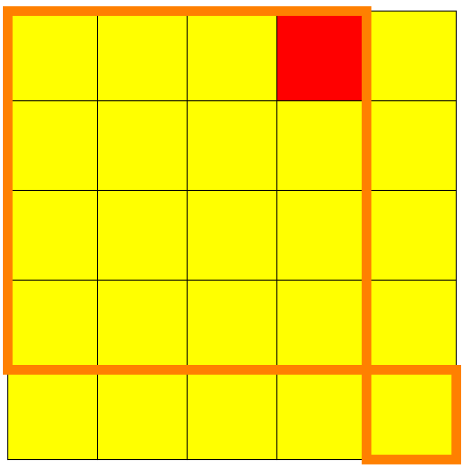
Let us express our findings in a way that emphasises the position of the elementary Jordan blocks and their relation with the powers of the matrix A.



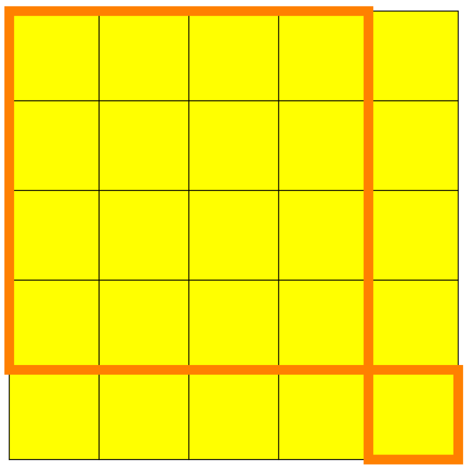
(a) Visualisation of A.



(b) Visualisation of A^2.



(c) Visualisation of A^3.



(d) Visualisation of A^4.

Figure 11

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

Let us calculate the powers of the matrix B.

$$B = \begin{pmatrix} 0 & 2 & -1 & -4 & -2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & -2 & 0 \end{pmatrix};$$

$$B^2 = \begin{pmatrix} 0 & -1 & 2 & 3 & 1 \\ 0 & -1 & 1 & 2 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \end{pmatrix};$$

$$B^3 = \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{pmatrix};$$

$$B^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} 0 & 2 & -1 & -4 & -2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & -2 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} 2z_2 - z_3 - 4z_4 - 2z_5 = 0, \\ z_2 - 2z_4 - z_5 = 0, \\ -z_2 + z_4 + z_5 = 0, \\ z_2 - z_4 - z_5 = 0, \\ -z_3 - 2z_4 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\ker(B) = \{(r_1, r_2, 0, 0, r_2) \mid r_1, r_2 \in \mathbf{K}\} = \operatorname{span}\{(1,0,0,0,0), (0,1,0,0,1)\}.$$

Kernel of B^2.

We calculate the kernel of B^2 and we have to solve the matrix equation

$$\begin{pmatrix} 0 & -1 & 2 & 3 & 1 \\ 0 & -1 & 1 & 2 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} -z_2 + 2z_3 + 3z_4 + z_5 = 0, \\ -z_2 + z_3 + 2z_4 + z_5 = 0, \\ -z_3 - z_4 = 0, \\ +z_3 + z_4 = 0, \\ -z_2 + z_4 + z_5 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B^2) &= \{(r_1, r_2, r_3, -r_3, r_2 + r_3) \mid r_1, r_2, r_3 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,0), (0,1,0,0,1), (0,0,1,-1,1)\}. \end{aligned}$$

Kernel of B^3.

We calculate the kernel of B^3 and we have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} -z_3 - z_4 = 0, \\ -z_3 - z_4 = 0, \\ -z_3 - z_4 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B^3) &= \{(r_1, r_2, r_3, -r_3, r_5) \mid r_1, r_2, r_3, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,0), (0,1,0,0,0), \\ &\qquad\qquad (0,0,1,-1,0), (0,0,0,0,1)\}. \end{aligned}$$

Kernel of B^4.
We calculate the kernel of B^4 and we have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The solution set is the vector space $\mathbf{K}^5$.

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	2	$2 = \dim(\ker(B))$
$\ker(B^2)$	3	$1 = \dim(\ker(B^2)) - \dim(\ker(B))$
$\ker(B^3)$	4	$1 = \dim(\ker(B^3)) - \dim(\ker(B^2))$
$\ker(B^4)$	5	$1 = \dim(\ker(B^4)) - \dim(\ker(B^3))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have dim(ker(B)). This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column and fourth line is 1 and this gives the information that there will be a Jordan chain of length 4. The first number 2 in the last column is the dimension of the kernel of B. After we have calculated the chain with length 4, the last column will be from top to bottom $\{1, 0, 0, 0, 0\}$. There is still one linearly independent vector left to be found. After this calculation, the last column will be from top to bottom $\{0, 0, 0, 0, 0\}$. We have at that moment 5 linearly independent vectors which form a basis for this vector space.

4. Calculation of the Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ satisfying

$$\begin{cases} B\,\mathbf{w}_4 = \mathbf{w}_3, \\ B^2\,\mathbf{w}_4 = \mathbf{w}_2, \\ B^3\,\mathbf{w}_4 = \mathbf{w}_1, \\ B^4\,\mathbf{w}_4 = \mathbf{0}. \end{cases}$$

where $\mathbf{w}_4$ is in the vector space ker(B^4) but not in ker(B^3).

We look for a starting $\mathbf{w}_4$. We see from the information table that there is a generating vector $\mathbf{w}_4$ for a chain of length 4.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^4)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^4)$.

2. The generating vector may not be in the $\ker(B^3)$ because the length of the chain must be exactly 4. So it has to be independent from all vectors in $\ker(B^3)$. It is sufficient that it is linearly independent from a basis of $\ker(B^3)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 4. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^4 together with the vectors in $\ker(B^3)$ and also the vectors, if any, of exactly height 4 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^4) = \mathbf{K}^5.$$

So the generating vector has the following generic form

$$(a, b, c, d, e).$$

The kernel of B^3.
We remember that the kernel $\ker(B^3)$ is

$$\begin{aligned}\ker(B^3) = \operatorname{span}\{&(1,0,0,0,0),(0,1,0,0,0),\\ &(0,0,1,-1,0),(0,0,0,0,1)\}.\end{aligned}$$

Vectors chosen in previous Jordan chains.
We have previously chosen no vector of exactly height 4 in a Jordan chain.

Condition of linear independency of the vectors.

We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ a & b & c & d & e \end{pmatrix}.$$

If we impose the condition $c + d \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $c + d \neq 0$.

So we can choose $a = 0$, $b = 0$, $c = 1$, $d = 0$, $e = 0$.

We can choose a valid generating vector of the chain as follows

$$\mathbf{w_4} = (0, 0, 1, 0, 0).$$

We calculate then $B^i(0, 0, 1, 0, 0)$ from $i = 1$ to $i = 3$, and we know that the vectors

$$\{\mathbf{w_1} = B^3\,\mathbf{w_4}, \mathbf{w_2} = B^2\,\mathbf{w_4}, \mathbf{w_3} = B\,\mathbf{w_4}, \mathbf{w_4}\}$$

are a Jordan chain of 4 linearly independent vectors and because $B^4\,\mathbf{w_4} = \mathbf{0}$, we know that the length of the chain is exactly 4.

We calculate $\mathbf{w_3}$.

$$\mathbf{w_3} = B\,\mathbf{w_4} = \begin{pmatrix} 0 & 2 & -1 & -4 & -2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

We calculate $\mathbf{w_2}$.

$$\mathbf{w_2} = B^2\mathbf{w_4} = \begin{pmatrix} 0 & -1 & 2 & 3 & 1 \\ 0 & -1 & 1 & 2 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

We calculate $\mathbf{w_1}$.

$$\mathbf{w_1} = B^3\mathbf{w_4} = \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

We have now found the Jordan chain

$$\{\mathbf{w_1} = (-1,-1,0,0,-1), \mathbf{w_2} = (2,1,-1,1,0),\\ \mathbf{w_3} = (-1,0,0,0,-1), \mathbf{w_4} = (0,0,1,0,0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	2	$\mathbf{w_1}$	1
$\ker(B^2)$	3	$\mathbf{w_2}$	0
$\ker(B^3)$	4	$\mathbf{w_3}$	0
$\ker(B^4)$	5	$\mathbf{w_4}$	0

with

$$\mathbf{w_1} = (-1,-1,0,0,-1)$$

$$\mathbf{w_2} = (2,1,-1,1,0)$$

$$\mathbf{w_3} = (-1,0,0,0,-1)$$

$$\mathbf{w_4} = (0,0,1,0,0)$$

The last column in the information table tells us that there is still a chain of length 1 left to be found. This is an eigenvector $\mathbf{w_5}$. We know that we have in our previous Jordan chain found an eigenvector $\mathbf{w_1} = (-1,-1,0,0,-1)$ which has height 1. So we have to be careful when choosing an eigenvector.

The vector must be an eigenvector. It must be in the space

$$\ker(B) = \operatorname{span}\{(1,0,0,0,0),(0,1,0,0,1)\}.$$

The vector must be of the generic form

$$a\,(1,0,0,0,0) + b\,(0,1,0,0,1) = (a,b,0,0,b).$$

We have at this point chosen in $\ker(B)$ already the vector

$$\mathbf{w_1} = (-1,-1,0,0,-1)$$

of height exactly height 1.

We have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} -1 & -1 & 0 & 0 & -1 \\ a & b & 0 & 0 & b \end{pmatrix}.$$

We row reduce this matrix H and find then if we impose the condition that $a - b \neq 0$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a - b \neq 0$.

So we can choose $a = 0$, $b = 1$.

We have then the generating vector for the chain

$$\mathbf{w_5} = (0, 1, 0, 0, 1).$$

This vector is an eigenvector. It forms a Jordan chain on its own. It has length 1.

$$\{\mathbf{w_5} = (0, 1, 0, 0, 1)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(B)$	2	$\mathbf{w_1}$	$\mathbf{w_5}$	0
$\ker(B^2)$	3	$\mathbf{w_2}$		0
$\ker(B^3)$	4	$\mathbf{w_3}$		0
$\ker(B^4)$	5	$\mathbf{w_4}$		0

with

$$\mathbf{w_1} = (-1, -1, 0, 0, -1)$$
$$\mathbf{w_2} = (2, 1, -1, 1, 0)$$
$$\mathbf{w_3} = (-1, 0, 0, 0, -1)$$
$$\mathbf{w_4} = (0, 0, 1, 0, 0)$$
$$\mathbf{w_5} = (0, 1, 0, 0, 1)$$

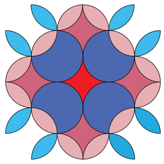
5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} -1 & 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

We check our calculations.

$$\begin{aligned} A &= P^{-1} B P \\ &= \begin{pmatrix} -1 & -1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 2 & 1 \end{pmatrix} \\ &\times \begin{pmatrix} 0 & 2 & -1 & -4 & -2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & -2 & 0 \end{pmatrix} \\ &\times \begin{pmatrix} -1 & 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} & (\mathbf{0}) \end{pmatrix}. \end{aligned}$$



11 exercise. (5×5); $(J_3(0), J_2(0))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B so that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 & 1 \\ 3 & -1 & 1 & 3 & 2 \\ -2 & 1 & -1 & -2 & -1 \\ 2 & -1 & 0 & 1 & 1 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_5| = -\lambda^5.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 5. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field $\mathbf{K}$. It is thus possible to put the matrix B in Jordan normal form over the field $\mathbf{K}$.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 2. The solution will be completely independent from this section.

Before trying to find a solution of this exercise, let us try to see what is going on with a particular type of an elementary Jordan block. We want to investigate the endomorphism A associated with this Jordan block.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We show this in a way that emphasises the position of the elementary Jordan blocks.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}.$$

We compute also the powers of A.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix};$$

$$A^2 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix};$$

$$A^3 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}.$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the third power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 3, is called the height of nilpotency of the matrix A.

It is interesting to observe how the kernels change.

$$\begin{cases} \ker(A) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_4\}, \\ \ker(A^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5\}, \\ \ker(A^3) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\} = \mathbf{K}^5. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	2	$2 = \dim(\ker(A))$
$\ker(A^2)$	4	$2 = \dim(\ker(A^2)) - \dim(\ker(A))$
$\ker(A^3)$	5	$1 = \dim(\ker(A^3)) - \dim(\ker(A^2))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so

that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the third power onwards.

$$\ker(A) \subsetneq \ker(A^2) \subsetneq \ker(A^3) = \ker(A^4) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 + n_3 = 2 = \dim(\ker(A)), \\ \quad\quad n_2 + n_3 = 2 = \dim(\ker(A^2)) - \dim(\ker(A)), \\ \quad\quad\quad\quad n_3 = 1 = \dim(\ker(A^3)) - \dim(\ker(A^2)). \end{cases}$$

Solving this system, we have $n_1 = 0$, $n_2 = 1$, $n_3 = 1$.

A consequence from this fact is that the numbers in the last column are descending.

One sees by looking at the matrices A^i that we have two Jordan chains.

Let us start with the first chain.

We observe from the matrices that

$$\begin{cases} A\,\mathbf{e}_3 = \mathbf{e}_2, \\ A\,\mathbf{e}_2 = \mathbf{e}_1, \\ A\,\mathbf{e}_1 = \mathbf{0}, \end{cases}$$

or

$$\begin{cases} A\,\mathbf{e}_3 = \mathbf{e}_2, \\ A^2\,\mathbf{e}_3 = \mathbf{e}_1, \\ A^3\,\mathbf{e}_3 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain with length three of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{A^2\,\mathbf{e}_3 = \mathbf{e}_1, A\,\mathbf{e}_3 = \mathbf{e}_2, \mathbf{e}_3\}.$$

After we have found the first Jordan chain of length 4, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
ker(A)	2	$\mathbf{e}_1$	1
ker(A^2)	4	$\mathbf{e}_2$	1
ker(A^3)	5	$\mathbf{e}_3$	0

Because the last column consists now of at least one number that is not 0, we are **not** done with looking for Jordan chains.

Let us start with the second chain.

We observe from the matrices that

$$\begin{cases} A\,\mathbf{e}_5 = \mathbf{e}_4, \\ A\,\mathbf{e}_4 = \mathbf{0}, \end{cases}$$

or

$$\begin{cases} A\,\mathbf{e}_5 = \mathbf{e}_4, \\ A^2\,\mathbf{e}_5 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain with length two of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_4 = A\,\mathbf{e}_5, \mathbf{e}_5\}.$$

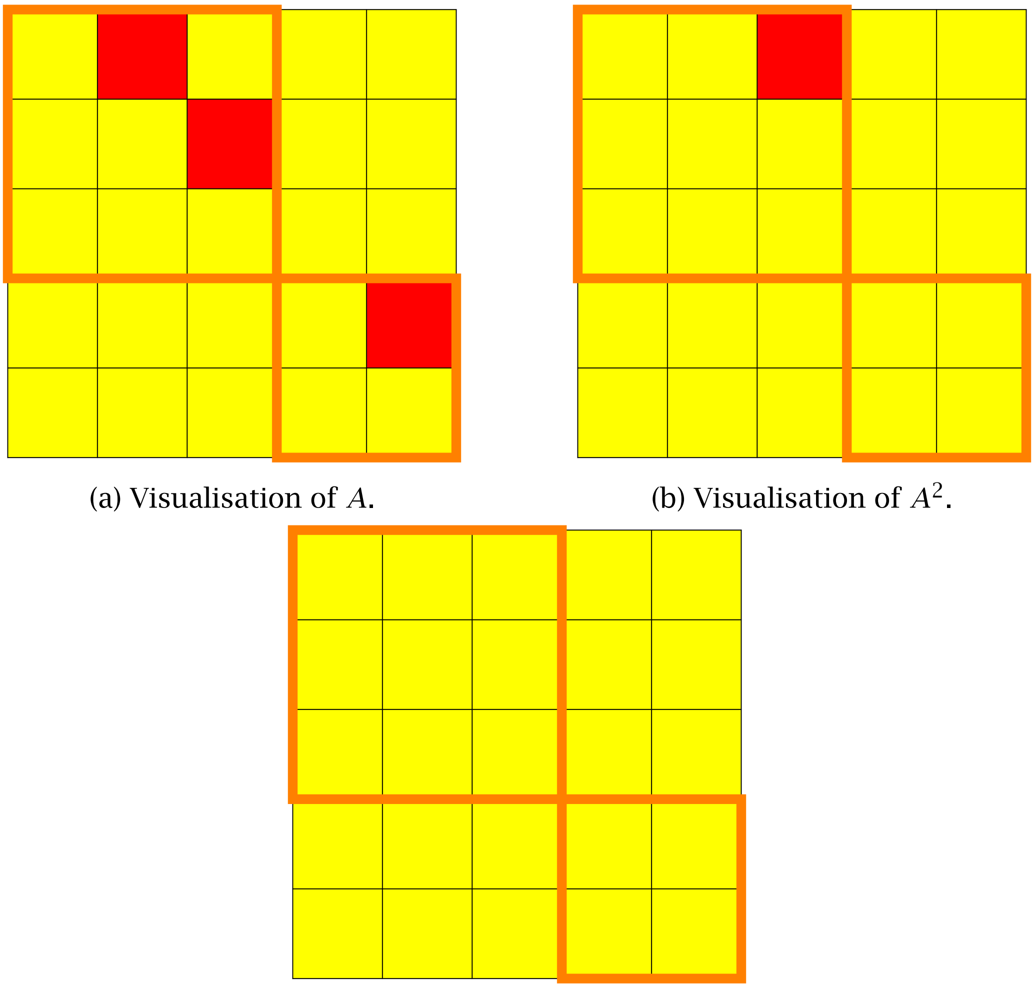
Let us note this in the new information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(A)$	2	$\mathbf{e}_1$	$\mathbf{e}_4$	0
$\ker(A^2)$	4	$\mathbf{e}_2$	$\mathbf{e}_5$	0
$\ker(A^3)$	5	$\mathbf{e}_3$		0

The last column consists of only 0's and this ends the search for Jordan chains. We have indeed found 5 linearly independent vectors and these form a basis for $\mathbf{K}^5$.

Let us express our findings in a way that emphasises the position of the elementary Jordan blocks and their relation with the powers of the matrix A.



(a) Visualisation of A.

(b) Visualisation of A^2.

(c) Visualisation of A^3.

Figure 12

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up

into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

Let us calculate the powers of the matrix B.

$$B = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 & 1 \\ 3 & -1 & 1 & 3 & 2 \\ -2 & 1 & -1 & -2 & -1 \\ 2 & -1 & 0 & 1 & 1 \end{pmatrix};$$

$$B^2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 \end{pmatrix};$$

$$B^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 & 1 \\ 3 & -1 & 1 & 3 & 2 \\ -2 & 1 & -1 & -2 & -1 \\ 2 & -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} z_3 + z_4 = 0, \\ z_1 + z_3 + 2z_4 + z_5 = 0, \\ 3z_1 - z_2 + z_3 + 3z_4 + 2z_5 = 0, \\ -2z_1 + z_2 - z_3 - 2z_4 - z_5 = 0, \\ 2z_1 - z_2 + z_4 + z_5 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned}\ker(B) &= \{(r_1, r_1, r_3, -r_3, -r_1 + r_3) \mid r_1, r_3 \in \mathbf{K}\}\\ &= \operatorname{span}\{(1,1,0,0,-1),(0,0,1,-1,1)\}.\end{aligned}$$

Kernel of B².
We calculate the kernel of B^2 and we have to solve the matrix equation

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1\\ 1 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\\ z_5\end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} z_1 + z_4 + z_5 = 0,\\ z_1 + z_4 + z_5 = 0,\\ -z_1 - z_4 - z_5 = 0.\end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned}\ker(B^2) &= \{(r_1, r_2, r_3, r_4, -r_1 - r_4) \mid r_1, r_2, r_3, r_4 \in \mathbf{K}\}\\ &= \operatorname{span}\{(1,0,0,0,-1),(0,1,0,0,0),\\ &\qquad (0,0,1,0,0),(0,0,0,1,-1)\}.\end{aligned}$$

Kernel of B³.
We calculate the kernel of B^3 and we have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\\ z_5\end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\end{pmatrix}.$$

The solution is the vector space $\mathbf{K}^5$.

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	2	$2 = \dim(\ker(B))$
$\ker(B^2)$	4	$2 = \dim(\ker(B^2)) - \dim(\ker(B))$
$\ker(B^3)$	5	$1 = \dim(\ker(B^3)) - \dim(\ker(B^2))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(B))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column on the third line is 1 and this gives the information that there will be one Jordan chain of length 3. The first number 2 in the last column is the dimension of the kernel of B. After we have calculated the chain with length 3, the last column will be from top to bottom $\{1, 1, 0\}$. There is still one chain of length two left to be found. After this calculation, the last column will be from top to bottom $\{0, 0, 0\}$. We have at that moment 5 linearly independent vectors which form a basis for this vector space.

4. Calculation of the Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ satisfying

$$\begin{cases} B\,\mathbf{w}_3 = \mathbf{w}_2, \\ B^2\,\mathbf{w}_3 = \mathbf{w}_1 \\ B^3\,\mathbf{w}_3 = \mathbf{0}. \end{cases}$$

where $\mathbf{w}_3$ is in the vector space $\ker(B^3)$ but not in $\ker(B^2)$.

The first Jordan chain.
We look for a starting $\mathbf{w}_3$. We see from the information table that there is a generating vector $\mathbf{w}_3$ for a chain of length 3.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^3)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^3)$.

2. The generating vector may not be in the $\ker(B^2)$ because the length of the chain must be exactly 3. So it has to be independent from all vectors in $\ker(B^2)$. It is sufficient that it is linearly independent from a basis of $\ker(B^2)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 3. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^3 together with the vectors in $\ker(B^2)$ and also the vectors, if any, of exactly height 3 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^3) = \mathbf{K}^5.$$

The generating vector has the following generic form

$$(a, b, c, d, e).$$

The kernel of B^2.
We remember that the kernel $\ker(B^2)$ is

$$\ker(B^2) = \operatorname{span}\{(1,0,0,0,-1),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,-1)\}.$$

Vectors chosen in previous Jordan chains.
We have previously chosen no vector of exactly height 3 in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ a & b & c & d & e \end{pmatrix}.$$

If we impose the condition $a + d + e \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + d + e \neq 0$. So we can choose $a = 1$, $b = 0$, $c = 0$, $d = 0$, $e = 0$.

We can choose a valid generating vector of the chain as follows

$$\mathbf{w_3} = (1,0,0,0,0).$$

We calculate then $B^i(1,0,0,0,0)$ from $i = 1$ to $i = 2$, and we know that the vectors

$$\{\mathbf{w_1} = B^2\mathbf{w_3}, \mathbf{w_2} = B\mathbf{w_3}, \mathbf{w_3}\}$$

are a Jordan chain of 3 linearly independent vectors and because $B^3\mathbf{w_3} = \mathbf{0}$, we know that the length of the chain is exactly 3.

We calculate $\mathbf{w}_2$.

$$\mathbf{w}_2 = B\,\mathbf{w}_3 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 & 1 \\ 3 & -1 & 1 & 3 & 2 \\ -2 & 1 & -1 & -2 & -1 \\ 2 & -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ -2 \\ 2 \end{pmatrix}.$$

We calculate $\mathbf{w}_1$.

$$\mathbf{w}_1 = B^2\,\mathbf{w}_3 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

We have now found the Jordan chain

$$\{\mathbf{w}_1 = (1,1,0,0,-1), \mathbf{w}_2 = (0,1,3,-2,2), \mathbf{w}_3 = (1,0,0,0,0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	2	$\mathbf{w}_1$	1
$\ker(B^2)$	4	$\mathbf{w}_2$	1
$\ker(B^3)$	5	$\mathbf{w}_3$	0

with

$$\mathbf{w}_1 = (1,1,0,0,-1)$$

$$\mathbf{w}_2 = (0,1,3,-2,2)$$

$$\mathbf{w}_3 = (1,0,0,0,0)$$

We see at the number 1 in the second row of the last column that we have still to find a vector $\mathbf{w}_5$.

The second Jordan chain.
We look for a starting $\mathbf{w}_5$. We see from the information table that there is a generating vector $\mathbf{w}_5$ for a chain of length 2.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^2)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^2)$.

2. The generating vector may not be in the $\ker(B)$ because the length of the chain must be exactly 2. So it has to be independent from all vectors in $\ker(B)$. It is sufficient that it is linearly independent from a basis of $\ker(B)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 2. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^2 together with the vectors in $\ker(B)$ and also the vectors, if any, of exactly height 2 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^2) = \operatorname{span}\{(1,0,0,0,-1),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,-1)\}.$$

So the generating vector has the following generic form

$$a\,(1,0,0,0,-1) + b\,(0,1,0,0,0) + c\,(0,0,1,0,0) + d\,(0,0,0,1,-1)$$
$$= (a,b,c,d,-a-d).$$

The kernel of B.
We remember that the kernel $\ker(B)$ is

$$\ker(B) = \text{span}\{(1,1,0,0,-1),(0,0,1,-1,1)\}.$$

Vectors chosen in previous Jordan chains.
We have previously chosen the vector $\mathbf{w}_2 = (0,1,3,-2,2)$ of exactly height 2 in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ a & b & c & d & -a-d \end{pmatrix}.$$

If we impose the condition $a - b + c + d \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a - b + c + d \neq 0$. So we can choose $a = 0$, $b = 1$, $c = 0$, $d = 0$.

We can choose a valid generating vector of the chain as follows

$$\mathbf{w}_5 = (0,1,0,0,0).$$

We calculate then $B\,(0,1,0,0,0)$, and we know that the vectors

$$\{B\,\mathbf{w}_5, \mathbf{w}_5\}$$

form a Jordan chain of two linearly independent vectors.

We calculate $\mathbf{w}_4$.

$$\mathbf{w}_4 = B\,\mathbf{w}_5 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 & 1 \\ 3 & -1 & 1 & 3 & 2 \\ -2 & 1 & -1 & -2 & -1 \\ 2 & -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

We have now found the second Jordan chain. It has length 2.

$$\{\mathbf{w_4} = (0, 0, -1, 1, -1), \mathbf{w_5} = (0, 1, 0, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(B)$	2	$\mathbf{w_1}$	$\mathbf{w_4}$	0
$\ker(B^2)$	4	$\mathbf{w_2}$	$\mathbf{w_5}$	0
$\ker(B^3)$	5	$\mathbf{w_3}$		0

with

$\mathbf{w_1} = (1, 1, 0, 0, -1)$
$\mathbf{w_2} = (0, 1, 3, -2, 2)$
$\mathbf{w_3} = (1, 0, 0, 0, 0)$
$\mathbf{w_4} = (0, 0, -1, 1, -1)$
$\mathbf{w_5} = (0, 1, 0, 0, 0)$

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ -1 & 2 & 0 & -1 & 0 \end{pmatrix}.$$

We check our calculations.

$$
\begin{aligned}
A &= P^{-1} B P \\
&= \begin{pmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 & 1 \\ 3 & -1 & 1 & 3 & 2 \\ -2 & 1 & -1 & -2 & -1 \\ 2 & -1 & 0 & 1 & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ -1 & 2 & 0 & -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.
\end{aligned}
$$

12 exercise. (5×5); $(\mathbf{J_2(0), J_2(0), J_1(0)})$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B so that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 1 & 0 & 1 & -1 & -2 \\ -1 & 0 & -1 & 1 & 2 \\ 1 & -1 & 0 & 0 & -2 \\ 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 & -2 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial $p_{\text{C-H}}(\lambda)$.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_5| = -\lambda^5.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 5. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field $\mathbf{K}$. It is thus possible to put the matrix B in Jordan normal form over the field $\mathbf{K}$.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to

this subsection 2. The solution will be completely independent from this section.

We want to investigate the endomorphism A associated with this Jordan block.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We show this matrix in a way that emphasises the position of the elementary Jordan blocks. We compute also the powers of A.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & (\mathbf{0}) \end{matrix} \end{pmatrix},$$

$$A^2 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & (\mathbf{0}) \end{matrix} \end{pmatrix}.$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the second power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 2, is called the height of nilpotency of the matrix A.

It is interesting to observe how the kernels change.

$$\begin{cases} \ker(A) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5\}, \\ \ker(A^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\} = \mathbf{K}^5. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	3	$3 = \dim(\ker(A))$
$\ker(A^2)$	5	$2 = \dim(\ker(A^2)) - \dim(\ker(A))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.
2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.
3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.
4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.
5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A) \subsetneq \ker(A^2) = \ker(A^3) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 = 3 = \dim(\ker(A)), \\ \qquad n_2 = 2 = \dim(\ker(A^2)) - \dim(\ker(A)). \end{cases}$$

Solving this system, we have $n_1 = 1$, $n_2 = 2$.

A consequence from this fact is that the numbers in the last column are descending.

Let us start with the first chain.

We observe from the matrices that

$$\begin{cases} A\,\mathbf{e}_2 = \mathbf{e}_1, \\ A\,\mathbf{e}_1 = \mathbf{0}, \end{cases}$$

or

$$\begin{cases} \;A\,\mathbf{e}_2 = \mathbf{e}_1, \\ A^2\,\mathbf{e}_2 = \mathbf{0}. \end{cases}$$

We have the first Jordan chain

$$\{A\,\mathbf{e}_2 = \mathbf{e}_1, \mathbf{e}_2\}.$$

After we have found the first Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A)$	3	$\mathbf{e}_1$	2
$\ker(A^2)$	5	$\mathbf{e}_2$	1

Because the last column consists now of at least one number that is not 0, we are **not** done with looking for Jordan chains. One sees that we have another Jordan chain with length two of linearly independent vectors. Let us start with the second chain.

We observe from the matrices that

$$\begin{cases} A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A\,\mathbf{e}_3 = \mathbf{0}, \end{cases}$$

or

$$\begin{cases} A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A^2\,\mathbf{e}_4 = \mathbf{0}. \end{cases}$$

We write a Jordan chain in reverse order.

$$\{\mathbf{e}_3 = A\,\mathbf{e}_4, \mathbf{e}_4\}.$$

Let us note this in the new information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(A)$	3	$\mathbf{e}_1$	$\mathbf{e}_3$	1
$\ker(A^2)$	5	$\mathbf{e}_2$	$\mathbf{e}_4$	0

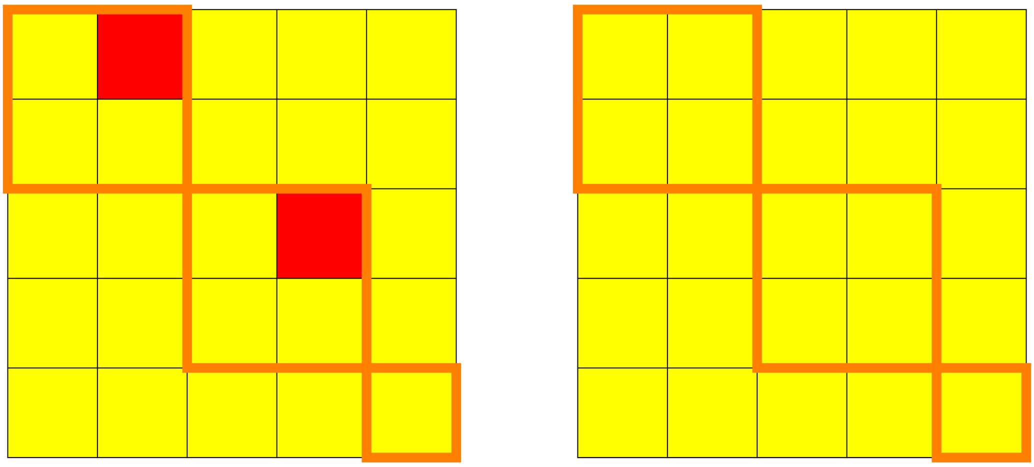
The last column does not consist of only 0's and we are not finished with the search for Jordan chains. We have indeed a third Jordan chain with length 1. It consists of a single eigenvector.

$$\{\mathbf{e}_5\}.$$

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	chain 3	remaining dim
$\ker(A)$	3	$\mathbf{e}_1$	$\mathbf{e}_3$	$\mathbf{e}_5$	0
$\ker(A^2)$	5	$\mathbf{e}_2$	$\mathbf{e}_4$		0

Let us express our findings in a way that emphasises the position of the elementary Jordan blocks and their relation with the powers of the matrix A.



(a) Visualisation of A. (b) Visualisation of A^2.

Figure 13

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

Let us calculate the powers of the matrix B.

$$B = \begin{pmatrix} 1 & 0 & 1 & -1 & -2 \\ -1 & 0 & -1 & 1 & 2 \\ 1 & -1 & 0 & 0 & -2 \\ 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 & -2 \end{pmatrix};$$

$$B^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} 1 & 0 & 1 & -1 & -2 \\ -1 & 0 & -1 & 1 & 2 \\ 1 & -1 & 0 & 0 & -2 \\ 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} z_1 + z_3 - z_4 - 2z_5 = 0, \\ -z_1 - z_3 + z_4 + 2z_5 = 0, \\ z_1 - z_2 - 2z_5 = 0, \\ -z_2 - z_3 + z_4 = 0, \\ z_1 + z_3 - z_4 - 2z_5 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B) &= \{(r_1, r_2, r_3, r_2 + r_3, r_1/2 - r_2/2) \mid r_1, r_2, r_3 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,1/2), (0,1,0,1,-1/2), (0,0,1,1,0)\}. \end{aligned}$$

Kernel of B^2.
We calculate the kernel of B^2 and we have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is the vector space $\mathbf{K}^5$.

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	3	$3 = \dim(\ker(B))$
$\ker(B^2)$	5	$2 = \dim(\ker(B^2)) - \dim(\ker(B))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(B))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The “remaining dim” give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column and on the second line is 2 and this gives the information that there will be two Jordan chains of length 2. The first number 3 in the last column is the dimension of the kernel of B. After we have calculated the chain with length 2, the last column will be from top to bottom $\{2,1\}$. There is still one two chain of independent vectors left to be found. After this calculation, the last column will be from top to bottom $\{1,0\}$. So there is one chain with length 1 left. We have at that moment 5 linearly independent vectors which form a basis for this vector space.

4. Calculation of the Jordan chains.

The first Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w_1}, \mathbf{w_2}\}$ satisfying

$$\begin{cases} B\,\mathbf{w_2} = \mathbf{w_1}, \\ B\,\mathbf{w_1} = \mathbf{0} \end{cases}$$

or

$$\begin{cases} B\,\mathbf{w_2} = \mathbf{w_1}, \\ B^2\,\mathbf{w_2} = \mathbf{0} \end{cases}$$

where $\mathbf{w_2}$ is in the vector space $\ker(B^2)$ but not in $\ker(B)$.

We look for a starting $\mathbf{w_2}$. We see from the information table that there is a generating vector $\mathbf{w_2}$ for a chain of length 2.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^2)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^2)$.

2. The generating vector may not be in the $\ker(B)$ because the length of the chain must be exactly 2. So it has to be independent from all vectors in $\ker(B)$. It is sufficient that it is linearly independent from a basis of $\ker(B)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 2. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^2 together with the vectors in $\ker(B)$ and also the vectors, if any, of exactly height 2 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^2) = \mathbf{K}^5.$$

So the generating vector has the following generic form

$$(a, b, c, d, e).$$

The kernel of B.
We remember that the kernel $\ker(B)$ is

$$\ker(B) = \operatorname{span}\{(2,0,0,0,1),(0,2,0,2,-1),(0,0,1,1,0)\}.$$

Vectors chosen in previous Jordan chains.
We have previously not chosen a vector of exactly height 2 in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 2 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ a & b & c & d & e \end{pmatrix}.$$

If we impose the condition $b + c - d \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{-a-c+d+2e}{2(b+c-d)} \\ 0 & 0 & 1 & 0 & \frac{-a+b+2e}{2(b+c-d)} \\ 0 & 0 & 0 & 1 & \frac{a-b-2e}{2(b+c-d)} \end{pmatrix}$$

We see that these vectors are independent if we impose the condition $b + c - d \neq 0$.

So we can choose $a = 0$, $b = 1$, $c = 0$, $d = 0$, $e = 0$.

We can choose a valid generating vector of the chain as follows

$$\mathbf{w_2} = (0, 1, 0, 0, 0).$$

We calculate then $B\,(0, 1, 0, 0, 0)$ and we know that the vectors

$$\{\mathbf{w_1} = B\,\mathbf{w_2}, \mathbf{w_2}\}$$

are a Jordan chain of 2 linearly independent vectors and because $B^2\,\mathbf{w_2} = \mathbf{0}$ we know that the length of the chain is exactly 2.

We calculate $\mathbf{w_1}$.

$$\mathbf{w_1} = B\,\mathbf{w_2} = \begin{pmatrix} 1 & 0 & 1 & -1 & -2 \\ -1 & 0 & -1 & 1 & 2 \\ 1 & -1 & 0 & 0 & -2 \\ 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}.$$

We have now found the first Jordan chain. It has length 2.

$$\{\mathbf{w_1} = (0, 0, -1, -1, 0), \mathbf{w_2} = (0, 1, 0, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	3	$\mathbf{w_1}$	2
$\ker(B^2)$	5	$\mathbf{w_2}$	1

with

$\mathbf{w_1} = (0, 0, -1, -1, 0)$
$\mathbf{w_2} = (0, 1, 0, 0, 0)$

We see at the number 1 at the second row of the last column that we have still to find another chain.

The second Jordan chain.
We look for a starting $\mathbf{w}_4$. We see from the information table that there is a generating vector $\mathbf{w}_4$ for a chain of length 2.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^2)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^2)$.

2. The generating vector may not be in the $\ker(B)$ because the length of the chain must be exactly 2. So it has to be independent from all vectors in $\ker(B)$. It is sufficient that it is linearly independent from a basis of $\ker(B)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 2. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^2 together with the vectors in $\ker(B)$ and also the vectors, if any, of exactly height 2 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^2) = \mathbf{K}^5.$$

So the generating vector has the following generic form

$$(a, b, c, d, e).$$

The kernel of B.
We remember that the kernel $\ker(B)$ is

$$\ker(B) = \operatorname{span}\{(2, 0, 0, 0, 1), (0, 2, 0, 2, -1), (0, 0, 1, 1, 0)\}.$$

Vectors chosen in previous Jordan chains.
We have previously chosen a vector $\mathbf{w}_2 = (0,1,0,0,0)$ of exactly height 2 in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 2 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ a & b & c & d & e \end{pmatrix}.$$

If we impose the condition $a + c - d - 2\,e \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + c - d - 2\,e \neq 0$.

So we can choose $a = 0$, $b = 0$, $c = 1$, $d = 0$, $e = 0$.

We can choose a valid generating vector of the chain as follows

$$\mathbf{w}_4 = (0,0,1,0,0).$$

We calculate then $B\,(0,0,1,0,0)$ and we know that the vectors

$$\{B\,\mathbf{w}_4, \mathbf{w}_4\}$$

form a Jordan chain of two linearly independent vectors.

We calculate $\mathbf{w}_3$.

$$\mathbf{w}_3 = B\,\mathbf{w}_4 = \begin{pmatrix} 1 & 0 & 1 & -1 & -2 \\ -1 & 0 & -1 & 1 & 2 \\ 1 & -1 & 0 & 0 & -2 \\ 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

We have now found the second Jordan chain. It has length 2.

$$\{\mathbf{w_3} = (1, -1, 0, -1, 1), \mathbf{w_4} = (0, 0, 1, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(B)$	3	$\mathbf{w_1}$	$\mathbf{w_3}$	1
$\ker(B^2)$	5	$\mathbf{w_2}$	$\mathbf{w_4}$	0

with

$\mathbf{w_1} = (0, 0, -1, -1, 0)$
$\mathbf{w_2} = (0, 1, 0, 0, 0)$
$\mathbf{w_3} = (1, -1, 0, -1, 1)$
$\mathbf{w_4} = (0, 0, 1, 0, 0)$

The third Jordan chain.
We see that we have still one Jordan chain with length 1 left. This is an eigenvector. We must choose this $\mathbf{w_5}$ in $\ker(B)$ but independent from the vectors already chosen in $\ker(B)$, these are $\mathbf{w_1}$ and $\mathbf{w_3}$.

The last column in the information table tells us that there is still a chain of length 1 left to be found. This is an eigenvector $\mathbf{w_5}$. We know that we have in our previous Jordan chain found eigenvectors $\mathbf{w_1} = (0, 0, -1, -1, 0)$ and $\mathbf{w_3} = (1, -1, 0, -1, 1)$ of height 1. So we have to be careful when choosing another eigenvector.

The vector must be an eigenvector. It must be in the space

$$\ker(B) = \text{span}\{(2, 0, 0, 0, 1), (0, 2, 0, 2, -1), (0, 0, 1, 1, 0)\}.$$

The vector must be of the generic form

$$a\,(2,0,0,0,1) + b\,(0,2,0,2,-1) + c\,(0,0,1,1,0) \\ = (2\,a, 2\,b, c, 2\,b + c, a - b).$$

We have at this point chosen in ker(B) already the vectors

$$\begin{cases} \mathbf{w_1} = (0,0,-1,-1,0), \\ \mathbf{w_3} = (1,-1,0,-1,1). \end{cases}$$

of height exactly height 1.

We have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 0 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 & 1 \\ 2\,a & 2\,b & c & 2\,b + c & a - b \end{pmatrix}.$$

We row reduce this matrix H and find then if we impose the condition that $a + b \neq 0$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1 & -1/2 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + b \neq 0$.

So we can choose $a = 0$, $b = 1$, $c = 0$.

We have the generating vector

$$\mathbf{w_5} = (0,2,0,2,-1).$$

This vector is an eigenvector. It forms a Jordan chain on its own. We have found our third Jordan chain.

$$\{\mathbf{w_5} = (0,2,0,2,-1)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	chain 3	remaining dim
$\ker(B)$	3	$\mathbf{w_1}$	$\mathbf{w_3}$	$\mathbf{w_5}$	0
$\ker(B^2)$	5	$\mathbf{w_2}$	$\mathbf{w_4}$		0

with

$$\mathbf{w_1} = (0, 0, -1, -1, 0)$$
$$\mathbf{w_2} = (0, 1, 0, 0, 0)$$
$$\mathbf{w_3} = (1, -1, 0, -1, 1)$$
$$\mathbf{w_4} = (0, 0, 1, 0, 0)$$
$$\mathbf{w_5} = (0, 2, 0, 2, -1)$$

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

We check our calculations.

$$
\begin{aligned}
A &= P^{-1} B P \\
&= \begin{pmatrix} 1 & 0 & 0 & -1 & -2 \\ -1 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & -2 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix} \\
&\times \begin{pmatrix} 1 & 0 & 1 & -1 & -2 \\ -1 & 0 & -1 & 1 & 2 \\ 1 & -1 & 0 & 0 & -2 \\ 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 & -2 \end{pmatrix} \\
&\times \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} & (\mathbf{0}) \end{pmatrix}.
\end{aligned}
$$

13 exercise. (3×3); $(\mathbf{J}_2(0), \mathbf{J}_1(0))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B so that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial $p_{\text{C-H}}(\lambda)$.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_3| = -\lambda^3.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 3. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field $\mathbf{K}$. It is thus possible to put the matrix B in Jordan normal form over the field $\mathbf{K}$.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 2. The solution will be completely independent from this section.

We want to investigate the endomorphism A. We look also at the powers A^i.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We emphasise the structure of this Jordan matrix. We compute also the powers of A.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & (\mathbf{0}) \end{pmatrix},$$

$$A^2 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & (\mathbf{0}) \end{pmatrix}.$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the second power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 2, is called the height of nilpotency of the matrix A.

We can see almost without calculation that

It is interesting to observe how the kernels change.

$$\begin{cases} \ker(A) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_3\}, \\ \ker(A^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \mathbf{K}^3. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	2	$2 = \dim(\ker(A))$
$\ker(A^2)$	3	$1 = \dim(\ker(A^2)) - \dim(\ker(A))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A) \subsetneq \ker(A^2) = \ker(A^3) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 = 2 = \dim(\ker(A)), \\ \qquad n_2 = 1 = \dim(\ker(A^2)) - \dim(\ker(A)). \end{cases}$$

Solving this system, we have $n_1 = 1$, $n_2 = 1$.

A consequence from this fact is that the numbers in the last column are descending.

We remark by looking at the matrices A^i that the vector $\mathbf{e}_2$ satisfies $A\,\mathbf{e}_2 = \mathbf{e}_1$. We have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = A\,\mathbf{e}_2, \mathbf{e}_2\}.$$

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A)$	2	$\mathbf{e}_1$	1
$\ker(A^2)$	3	$\mathbf{e}_2$	0

Because the last column consists now of at least one number that is not 0, we are **not** done with looking for Jordan chains. One sees that we have another Jordan chain with length one of linearly independent vectors. Let us start with the second chain.

We have immediately a Jordan chain of length 1 by looking at the eigenvectors. We have indeed

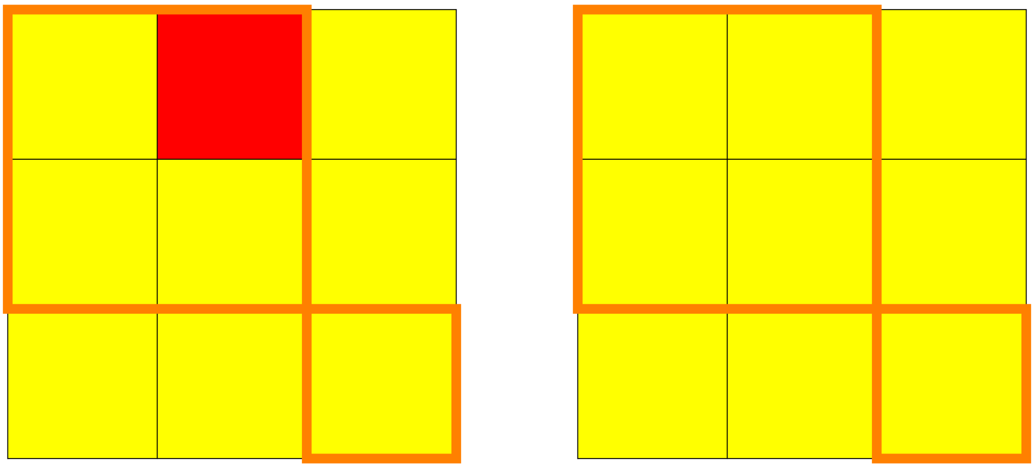
$$\{\mathbf{e}_3\}.$$

Let us note this in the new information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(A)$	2	$\mathbf{e}_1$	$\mathbf{e}_3$	0
$\ker(A^2)$	3	$\mathbf{e}_2$		0

Let us express our findings in a way that emphasises the position of the elementary Jordan blocks and their relation with the powers of the matrix A.



(a) Visualisation of A.

(b) Visualisation of A^2.

Figure 14

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

Let us calculate the powers of the matrix B.

$$B = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix};$$

$$B^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} -z_1 - z_2 + z_3 = 0, \\ z_1 + z_2 - z_3 = 0. \end{cases}$$

This system can be solved and this gives us the solutions set

$$\ker(B) = \{(r_1, r_2, r_1 + r_2) \mid r_1, r_2 \in \mathbf{K}\} = \operatorname{span}\{(1,0,1), (0,1,1)\}.$$

Kernel of B^2.
It is clear that $\ker(B^2) = \mathbf{K}^3$.

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	2	$2 = \dim(\ker(B))$
$\ker(B^2)$	3	$1 = \dim(\ker(B^2)) - \dim(\ker(B))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.
2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.
3. In the first row, we have dim(ker(B)). This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.
4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.
5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column is 1 and this gives the information that there will be a Jordan chain of length 2. The first number 2 in the last column is the dimension of the kernel of B. After we have calculated this chain, the last column will be from top to bottom $\{1, 0\}$. There is then one chain of one vector left to be found. After we have found this chain, we have at that moment already 3 linearly independent vectors which form a base for this vector space. The new last column will be from top to bottom $\{0, 0\}$.

4. Calculation of the Jordan chains.

The first Jordan chain.
We look for a starting $\mathbf{w}_2$. We see from the information table that there is a generating vector $\mathbf{w}_2$ for a chain of length 2.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel ker(B^2). So we can describe such a vector as a linear combination of vectors of a basis of the subspace ker(B^2).
2. The generating vector may not be in the ker(B) because the length of the chain must be exactly 2. So it has to be independent from all vectors in ker(B). It is sufficient that it is linearly independent from a basis of ker(B).

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 2. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^2 together with the vectors in $\ker(B)$ and also the vectors, if any, of exactly height 2 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^2) = \mathbf{K}^3.$$

So the generating vector has the following generic form

$$(a, b, c).$$

The kernel of B.
We remember that the kernel $\ker(B)$ is

$$\ker(B) = \operatorname{span}\{(1, 0, 1), (0, 1, 1)\}.$$

Vectors chosen in previous Jordan chains.
We have previously not chosen a vector of exactly height 2 in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ a & b & c \end{pmatrix}.$$

If we impose the condition $a + b - c \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + b - c \neq 0$.

We can choose $a = 0$, $b = 1$ and $c = 0$.

We can choose a valid generating vector of the chain as follows

$$\mathbf{w_2} = (0, 1, 0).$$

We calculate $\mathbf{w_1}$.

$$\mathbf{w_1} = B\,\mathbf{w_2} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

We know that the vectors

$$\{\mathbf{w_1} = B\,\mathbf{w_2}, \mathbf{w_2}\}$$

are a Jordan chain of 2 linearly independent vectors and because $B^2\,\mathbf{w_2} = \mathbf{0}$, we know that the length of the chain is exactly 2.

We have now found the first Jordan chain. It has length 2.

$$\{\mathbf{w_1} = (-1, 1, 0), \mathbf{w_2} = (0, 1, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	2	$\mathbf{w_1}$	1
$\ker(B^2)$	3	$\mathbf{w_2}$	0

with

$\mathbf{w_1} = (-1, 1, 0)$
$\mathbf{w_2} = (0, 1, 0)$

The second Jordan chain.
The last column in the information table tells us that there is still a chain of length 1 left to be found. This is an eigenvector $\mathbf{w_3}$. We know that we have in our previous Jordan chain found an eigenvector $\mathbf{w_1} = (-1, 1, 0)$. So we have to be careful when choosing another eigenvector.

The vector must be an eigenvector. It must be in the space

$$\ker(B) = \text{span}\{(1, 0, 1), (0, 1, 1)\}.$$

The vector must be of the generic form

$$a\,(1, 0, 1) + b\,(0, 1, 1) = (a, b, a + b).$$

We have at this point chosen in $\ker(B)$ already the vector

$$\mathbf{w_1} = (-1, 1, 0)$$

of height exactly 1.

We have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} -1 & 1 & 0 \\ a & b & a+b \end{pmatrix}.$$

We row reduce this matrix H and find then if we impose the condition that $a + b \neq 0$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + b \neq 0$.

So we can choose $a = 0$, $b = 1$.

We have then generating vector

$$\mathbf{w_3} = (0, 1, 1).$$

This vector is an eigenvector. This vector forms a Jordan chain on its own. We have the second Jordan chain

$$\{\mathbf{w_3} = (0, 1, 1)\}.$$

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(B)$	2	$\mathbf{w_1}$	$\mathbf{w_3}$	0
$\ker(B^2)$	3	$\mathbf{w_2}$		0

with

$\mathbf{w_1} = (-1, 1, 0)$
$\mathbf{w_2} = (0, 1, 0)$
$\mathbf{w_3} = (0, 1, 1)$

Because the last column consists now of 0's, we are done with looking for Jordan chains. We have found at this point a basis of V consisting entirely of vectors in Jordan chains.

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We check this solution.

$$
\begin{aligned}
A &= P^{-1}\,B\,P \\
&= \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & (\mathbf{0}) \end{pmatrix}.
\end{aligned}
$$

14 exercise. (3 × 3); (J₃(0)).

Use the relevant vector spaces $\ker(B-\lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B so that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1}BP$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} -2 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & -1 & 2 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial $p_{\text{C-H}}(\lambda)$.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_3| = -\lambda^3.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 3. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field **K**. It is thus possible to put the matrix B in Jordan normal form over the field **K**.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 2. The solution will be completely independent from this section.

We want to investigate the endomorphism A associated with this Jordan block.

Remark that A is itself already an elementary Jordan block. We look also at the powers A^i.

$$A = \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right);$$
$$A^2 = \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right);$$
$$A^3 = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right).$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the third power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 3, is called the height of nilpotency of the matrix A.

It is interesting to observe how the kernels change.

$$\begin{cases} \ker(A) = \operatorname{span}\{\mathbf{e}_1\}, \\ \ker(A^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}, \\ \ker(A^3) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \mathbf{K}^3. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	1	$1 = \dim(\ker(A))$
$\ker(A^2)$	2	$1 = \dim(\ker(A^2)) - \dim(\ker(A))$
$\ker(A^3)$	3	$1 = \dim(\ker(A^3)) - \dim(\ker(A^2))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.
2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.
3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.
4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.
5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the third power onwards.

$$\ker(A) \subsetneq \ker(A^2) \subsetneq \ker(A^3) = \ker(A^4) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 + n_3 = 1 = \dim(\ker(A)), \\ \qquad n_2 + n_3 = 1 = \dim(\ker(A^2)) - \dim(\ker(A)), \\ \qquad\qquad n_3 = 1 = \dim(\ker(A^3)) - \dim(\ker(A^2)). \end{cases}$$

Solving this system, we have $n_1 = 0$, $n_2 = 0$, $n_3 = 1$.

A consequence from this fact is that the numbers in the last column are descending.

We remark by looking at the matrices A^i

$$\begin{cases} A\,\mathbf{e}_3 = \mathbf{e}_2, \\ A\,\mathbf{e}_2 = \mathbf{e}_1, \\ A\,\mathbf{e}_1 = \mathbf{0}. \end{cases}$$

or

$$\begin{cases} A\,\mathbf{e}_3 = \mathbf{e}_2, \\ A^2\,\mathbf{e}_3 = \mathbf{e}_1, \\ A^3\,\mathbf{e}_3 = \mathbf{0}. \end{cases}$$

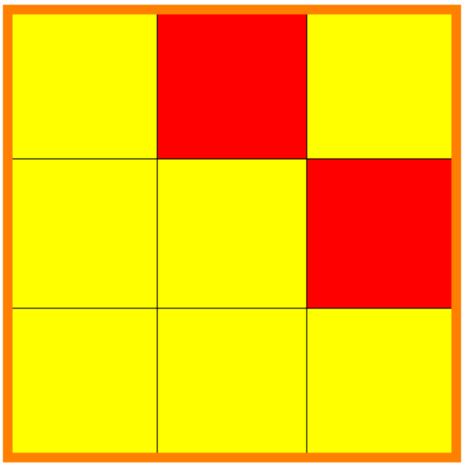
We have a Jordan chain of linearly independent vectors of length 3. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = A^2\,\mathbf{e}_3, \mathbf{e}_2 = A\,\mathbf{e}_3, \mathbf{e}_3\}.$$

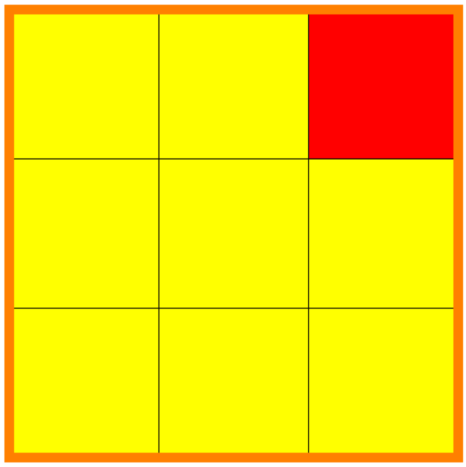
Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A)$	1	$\mathbf{e}_1$	0
$\ker(A^2)$	1	$\mathbf{e}_2$	0
$\ker(A^3)$	1	$\mathbf{e}_3$	0

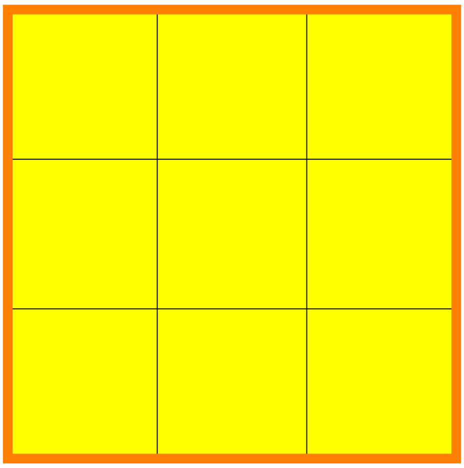
Because the last column consists now of 0's, we are done with looking for Jordan chains. We have found at this point a basis of V consisting entirely of vectors in Jordan chains.



(a) Visualisation of A.



(b) Visualisation of A^2.



(c) Visualisation of A^3.

Figure 15

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up

into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

Let us calculate the powers of the matrix B.

$$B = \begin{pmatrix} -2 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & -1 & 2 \end{pmatrix}; \qquad B^2 = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix};$$

$$B^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} -2 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} -2z_1 + z_2 - 2z_3 = 0, \\ -\ z_1 \qquad - \ z_3 = 0, \\ \ \ 2z_1 - z_2 + 2z_3 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\ker(B) = \{(r_1, 0, -r_1) \mid r_1 \in \mathbf{K}\} = \operatorname{span}\{(1, 0, -1)\}.$$

Kernel of B^2.
We calculate the kernel of B^2 and we have to solve the matrix equation

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} -z_1 - z_3 = 0, \\ \ \ z_1 + z_3 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\ker(B^2) = \{(r_1, r_2, -r_1) \mid r_1, r_2 \in \mathbf{K}\} = \text{span}\{(1,0,-1),(0,1,0)\}.$$

Kernel of B³.
This is

$$\ker(B^3) = \mathbf{K}^3.$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	1	$1 = \dim(\ker(B))$
$\ker(B^2)$	2	$1 = \dim(\ker(B^2)) - \dim(\ker(B))$
$\ker(B^3)$	3	$1 = \dim(\ker(B^3)) - \dim(\ker(B^2))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(B))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The “remaining dim” give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column and the third line is 1 and this gives the information that there will be a Jordan chain of length 3. The first number 1 in the last column is the dimension of the kernel of B. After we have calculated this chain, the last column will be from top to bottom $\{0, 0, 0\}$. There is no chain left to be found. We have indeed at that moment 3 linearly independent vectors which form a base for this vector space.

4. Calculation of the Jordan chains.

The first Jordan chain.
We look for a starting $\mathbf{w_3}$. We see from the information table that there is a generating vector $\mathbf{w_3}$ for a chain of length 3.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^3)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^3)$.

2. The generating vector may not be in the $\ker(B^2)$ because the length of the chain must be exactly 3. So it has to be independent from all vectors in $\ker(B^2)$. It is sufficient that it is linearly independent from a basis of $\ker(B^2)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 3. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^3 together with the vectors in $\ker(B^2)$ and also the vectors, if any, of exactly height 3 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^3) = \mathbf{K}^3.$$

So the generating vector has the following generic form

$$(a, b, c).$$

The kernel of B.
We remember that the kernel ker(B) is

$$\ker(B^2) = \operatorname{span}\{(1,0,-1),(0,1,0)\}$$

Vectors chosen in previous Jordan chains.
We have previously not chosen a vector of exactly height 3 in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}.$$

If we impose the condition $a+c \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + c \neq 0$. we can choose $a = 1$, $b = 1$, $c = 0$.

We can choose a valid generating vector of the chain as follows

$$\mathbf{w_3} = (1,1,0).$$

We calculate now $B\,(1,1,0)$, $B^2\,(1,1,0)$, and we know that the vectors

$$\{\mathbf{w_1} = B^2\,\mathbf{w_3}, \mathbf{w_2} = B\,\mathbf{w_3}, \mathbf{w_3}\}$$

are a Jordan chain of 3 linearly independent vectors and because $B^3\,\mathbf{w_3} = \mathbf{0}$, we know that the length of the chain is exactly 3.

We calculate $\mathbf{w_2}$.

$$\mathbf{w_2} = B\,\mathbf{w_3} = \begin{pmatrix} -2 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

We calculate $\mathbf{w_1}$.

$$\mathbf{w_1} = B^2\,\mathbf{w_3} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

We have now found the Jordan chain

$$\{\mathbf{w_1} = (-1, 0, 1), \mathbf{w_2} = (-1, -1, 1), \mathbf{w_3} = (1, 1, 0)\}.$$

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	1	$\mathbf{w_1}$	0
$\ker(B^2)$	2	$\mathbf{w_2}$	0
$\ker(B^2)$	3	$\mathbf{w_3}$	0

with

$\mathbf{w_1} = (-1, 0, 1)$
$\mathbf{w_2} = (-1, -1, 1)$
$\mathbf{w_3} = (1, 1, 0)$

Because the last column consists now of 0's, we are done with looking for Jordan chains. We have found at this point a basis of V consisting entirely of vectors in Jordan chains.

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

We check this solution.

$$
\begin{aligned}
A &= P^{-1}\,B\,P \\
&= \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \left(\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \right).
\end{aligned}
$$

15 exercise. (2×2); $(\mathbf{J_2(0)})$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B so that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial $p_{\text{C-H}}(\lambda)$.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_2| = \lambda^2.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 2. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field $\mathbf{K}$. It is thus possible to put the matrix B in Jordan normal form over the field $\mathbf{K}$.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 2. The solution will be completely independent from this section.

We want to investigate the endomorphism A associated with this Jordan block.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Remark that this is already an elementary Jordan block. We compute the powers of A.

$$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We emphasise the structure of this Jordan matrix and compute the powers of this matrix.

$$A = \left(\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right).$$

$$A^2 = \left(\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right),$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the second power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 2, is called the height of nilpotency of the matrix A.

We can see almost without calculation that

$$\begin{cases} \ker(A) = \operatorname{span}\{\mathbf{e}_1\}; \\ \ker(A^2) = \mathbf{K}^2. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	1	$1 = \dim(\ker(A))$
$\ker(A^2)$	2	$1 = \dim(\ker(A^2)) - \dim(\ker(A))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A) \subsetneq \ker(A^2) = \ker(A^3) \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 = 1 = \dim(\ker(A)), \\ \qquad\; n_2 = 1 = \dim(\ker(A^2)) - \dim(\ker(A)). \end{cases}$$

Solving this system, we have $n_1 = 0$, $n_2 = 1$.

A consequence from this fact is that the numbers in the last column are descending.

We remark by looking at the matrices A^i that the vector $\mathbf{e}_2$ satisfies $A\,\mathbf{e}_2 = \mathbf{e}_1$. We have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = A\,\mathbf{e}_2, \mathbf{e}_2\}.$$

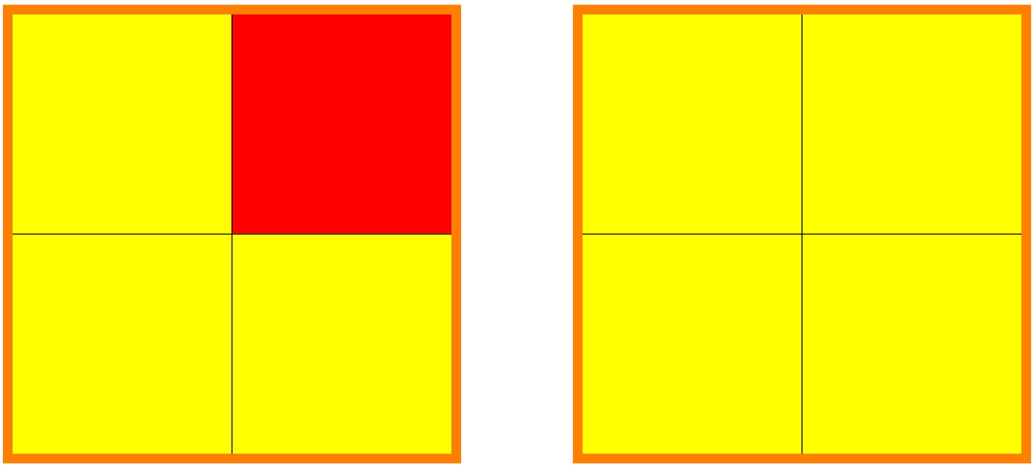
After we have found the first Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A)$	1	$\mathbf{e}_1$	0
$\ker(A^2)$	1	$\mathbf{e}_2$	0

Because the last column consists now of 0's, we are done with looking for Jordan chains. We have found at this point a basis of V consisting entirely of vectors in Jordan chains.

Let us try to visualise this.



(a) Visualisation of A. (b) Visualisation of A^2.

Figure 16

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

The powers of the matrix B.
Let us calculate the powers of the matrix B.

$$B = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}; \qquad B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} -z_1 + z_2 = 0, \\ -z_1 + z_2 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace $\ker(B) = \{(r_1, r_1) \mid r_1 \in \mathbf{K}\} = \text{span}\{(1,1)\}$.

Kernel of B^2.
We conclude immediately that $\ker(B^2) = \mathbf{K}^2$.

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	1	$1 = \dim(\ker(B))$
$\ker(B^2)$	2	$1 = \dim(\ker(B^2)) - \dim(\ker(B))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.
2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.
3. In the first row, we have $\dim(\ker(B))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.
4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.
5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column and second line is 1 and this gives the information that there will be a Jordan chain of length 2. The first number 1 in the last column is the dimension of the kernel of B. After we have calculated this chain, the last column will be from top to bottom $\{0, 0\}$. We have found a basis.

4. Calculation of the Jordan chains.

The first Jordan chain.
We look for a starting $\mathbf{w}_2$. We see from the information table that there is a generating vector $\mathbf{w}_2$ for a chain of length 2.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^2)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^2)$.

2. The generating vector may not be in the $\ker(B)$ because the length of the chain must be exactly 2. So it has to be independent from all vectors in $\ker(B)$. It is sufficient that it is linearly independent from a basis of $\ker(B)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 2. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^2 together with the vectors in $\ker(B)$ and also the vectors, if any, of exactly height 2 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^2) = \mathbf{K}^2.$$

So the generating vector has the following generic form

$$(a, b).$$

The kernel of B.
We remember that the kernel $\ker(B)$ is

$$\ker(B) = \text{span}\{(1, 1)\}.$$

Vectors chosen in previous Jordan chains.
We have previously not chosen a vector of exactly height 2 in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 1 \\ a & b \end{pmatrix}.$$

If we impose the condition $a - b \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a - b \neq 0$. We can choose $a = 1$, $b = 0$.

We can choose a valid generating vector of the chain as follows

$$\mathbf{w_2} = (1, 0).$$

We calculate $B\,(1, 0)$ and we know that

$$\{\mathbf{w_1} = B\,\mathbf{w_2}, \mathbf{w_2}\}$$

is a Jordan chain of 2 linearly independent vectors and because $B^2\,\mathbf{w_3} = \mathbf{0}$ we know that the length of the chain is exactly 2.

We calculate $\mathbf{w_1}$.

$$\mathbf{w_1} = B\,\mathbf{w_2} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

We have now found the Jordan chain

$$\{\mathbf{w_1} = (-1, -1), \mathbf{w_2} = (1, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	1	$\mathbf{w_1}$	0
$\ker(B^2)$	2	$\mathbf{w_2}$	0

with

$$\mathbf{w_1} = (-1, -1)$$

$$\mathbf{w_2} = (1, 0)$$

Because the last column consists now of 0's, we are done with looking for Jordan chains. We have found at this point a basis of V consisting entirely of vectors in Jordan chains.

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

We check this solution.

$$\begin{aligned} A &= P^{-1} B P \\ &= \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}. \end{aligned}$$

16 exercise. (8×8); $(\mathbf{J}_4(0), \mathbf{J}_2(0), \mathbf{J}_2(0))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B so that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 10 & -1 & 9 & -11 & 6 & 3 & -5 & -9 \\ 20 & 0 & 19 & -20 & 16 & 4 & -15 & -19 \\ -5 & 0 & -5 & 5 & -4 & -1 & 4 & 5 \\ 6 & -1 & 5 & -7 & 3 & 2 & -2 & -5 \\ -3 & 0 & -3 & 3 & -2 & -1 & 2 & 3 \\ -23 & 1 & -21 & 24 & -15 & -6 & 13 & 20 \\ 21 & -1 & 19 & -22 & 15 & 5 & -13 & -19 \\ -25 & 1 & -23 & 26 & -18 & -6 & 16 & 23 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial $p_{\text{C-H}}(\lambda)$.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_8| = \lambda^8.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 8. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field $\mathbf{K}$. It is thus possible to put the matrix B in Jordan normal form over the field $\mathbf{K}$.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely

this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 2. The solution will be completely independent from this section.

We want to investigate the endomorphism A associated with this Jordan block.

We compute also the powers of A.

$$A = \begin{pmatrix} 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} \end{pmatrix} & \begin{matrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{matrix} \\ \begin{matrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{matrix} & \begin{matrix} \begin{pmatrix} \mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0} \end{pmatrix} & \begin{matrix} 0&0\\ 0&0 \end{matrix} \\ \begin{matrix} 0&0\\ 0&0 \end{matrix} & \begin{pmatrix} \mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0} \end{pmatrix} \end{matrix} \end{pmatrix}.$$

$$A^2 = \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} \end{pmatrix} & \begin{matrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{matrix} \\ \begin{matrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{matrix} & \begin{matrix} \begin{pmatrix} \mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0} \end{pmatrix} & \begin{matrix} 0&0\\ 0&0 \end{matrix} \\ \begin{matrix} 0&0\\ 0&0 \end{matrix} & \begin{pmatrix} \mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0} \end{pmatrix} \end{matrix} \end{pmatrix}.$$

$$A^3 = \begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} \end{pmatrix} & \begin{matrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{matrix} \\ \begin{matrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{matrix} & \begin{matrix} \begin{pmatrix} \mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0} \end{pmatrix} & \begin{matrix} 0&0\\ 0&0 \end{matrix} \\ \begin{matrix} 0&0\\ 0&0 \end{matrix} & \begin{pmatrix} \mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0} \end{pmatrix} \end{matrix} \end{pmatrix}.$$

$$A^4 = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} \end{pmatrix} & \begin{matrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{matrix}\\ \begin{matrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{matrix} & \begin{matrix} \begin{pmatrix} \mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0} \end{pmatrix} & \begin{matrix} 0&0\\ 0&0 \end{matrix}\\ \begin{matrix} 0&0\\ 0&0 \end{matrix} & \begin{pmatrix} \mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0} \end{pmatrix} \end{matrix} \end{pmatrix}.$$

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_5, \mathbf{e}_7\};\\ \ker(A^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8\};\\ \ker(A^3) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8\};\\ \ker(A^4) = \mathbf{K}^8. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	3	$3 = \dim(\ker(A))$
$\ker(A^2)$	6	$3 = \dim(\ker(A^2)) - \dim(\ker(A))$
$\ker(A^3)$	7	$1 = \dim(\ker(A^3)) - \dim(\ker(A^2))$
$\ker(A^4)$	8	$1 = \dim(\ker(A^4)) - \dim(\ker(A^3))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The “remaining dim” give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the fourth power onwards.

$$\ker(A) \subsetneq \ker(A^2) \subsetneq \ker(A^3) \subsetneq \ker(A^4) = \ker(A^5) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 + n_3 + n_4 = 3 = \dim(\ker(A)), \\ n_2 + n_3 + n_4 = 3 = \dim(\ker(A^2)) - \dim(\ker(A)), \\ n_3 + n_4 = 1 = \dim(\ker(A^3)) - \dim(\ker(A^2)), \\ n_4 = 1 = \dim(\ker(A^4)) - \dim(\ker(A^3)). \end{cases}$$

Solving this system, we have $n_1 = 0$, $n_2 = 2$, $n_3 = 0$, $n_4 = 1$.

A consequence from this fact is that the numbers in the last column are descending.

We have

$$\begin{cases} A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A\,\mathbf{e}_3 = \mathbf{e}_2, \\ A\,\mathbf{e}_2 = \mathbf{e}_1, \\ A\,\mathbf{e}_1 = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices A^i that the vector $\mathbf{e}_4$ satisfies

$$\begin{cases} A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A^2\,\mathbf{e}_4 = \mathbf{e}_2, \\ A^3\,\mathbf{e}_4 = \mathbf{e}_1, \\ A^4\,\mathbf{e}_4 = \mathbf{0}. \end{cases}$$

We have a Jordan chain of 4 linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = A^3\,\mathbf{e}_4, \mathbf{e}_2 = A^2\,\mathbf{e}_4, \mathbf{e}_3 = A\,\mathbf{e}_4, \mathbf{e}_4\}.$$

We have found the first Jordan chain. It has length 4. We have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A)$	3	$\mathbf{e}_1$	2
$\ker(A^2)$	6	$\mathbf{e}_2$	2
$\ker(A^3)$	7	$\mathbf{e}_3$	0
$\ker(A^4)$	8	$\mathbf{e}_4$	0

We remark by looking at the matrices A^i that

$$\begin{cases} A\,\mathbf{e}_6 = \mathbf{e}_5, \\ A\,\mathbf{e}_5 = \mathbf{0}. \end{cases}$$

We see also that the vector $\mathbf{e}_6$ satisfies

$$\begin{cases} A\,\mathbf{e}_6 = \mathbf{e}_5, \\ A^2\,\mathbf{e}_6 = \mathbf{0}. \end{cases}$$

We have now a second Jordan chain. It has length 2. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_5 = A\,\mathbf{e}_6, \mathbf{e}_6\}.$$

We have now the following table

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(A)$	3	$\mathbf{e}_1$	$\mathbf{e}_5$	1
$\ker(A^2)$	6	$\mathbf{e}_2$	$\mathbf{e}_6$	1
$\ker(A^3)$	7	$\mathbf{e}_3$		0
$\ker(A^4)$	8	$\mathbf{e}_4$		0

We investigate now our third Jordan chain.

We remark by looking at the matrices A^i that

$$\begin{cases} A\,\mathbf{e}_8 = \mathbf{e}_7, \\ A\,\mathbf{e}_7 = \mathbf{0}. \end{cases}$$

We see also that the vector $\mathbf{e}_8$ satisfies

$$\begin{cases} A\,\mathbf{e}_8 = \mathbf{e}_7, \\ A^2\,\mathbf{e}_8 = \mathbf{0}. \end{cases}$$

We have now found a third Jordan chain. It has length 2. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_7 = A\,\mathbf{e}_8, \mathbf{e}_8\}.$$

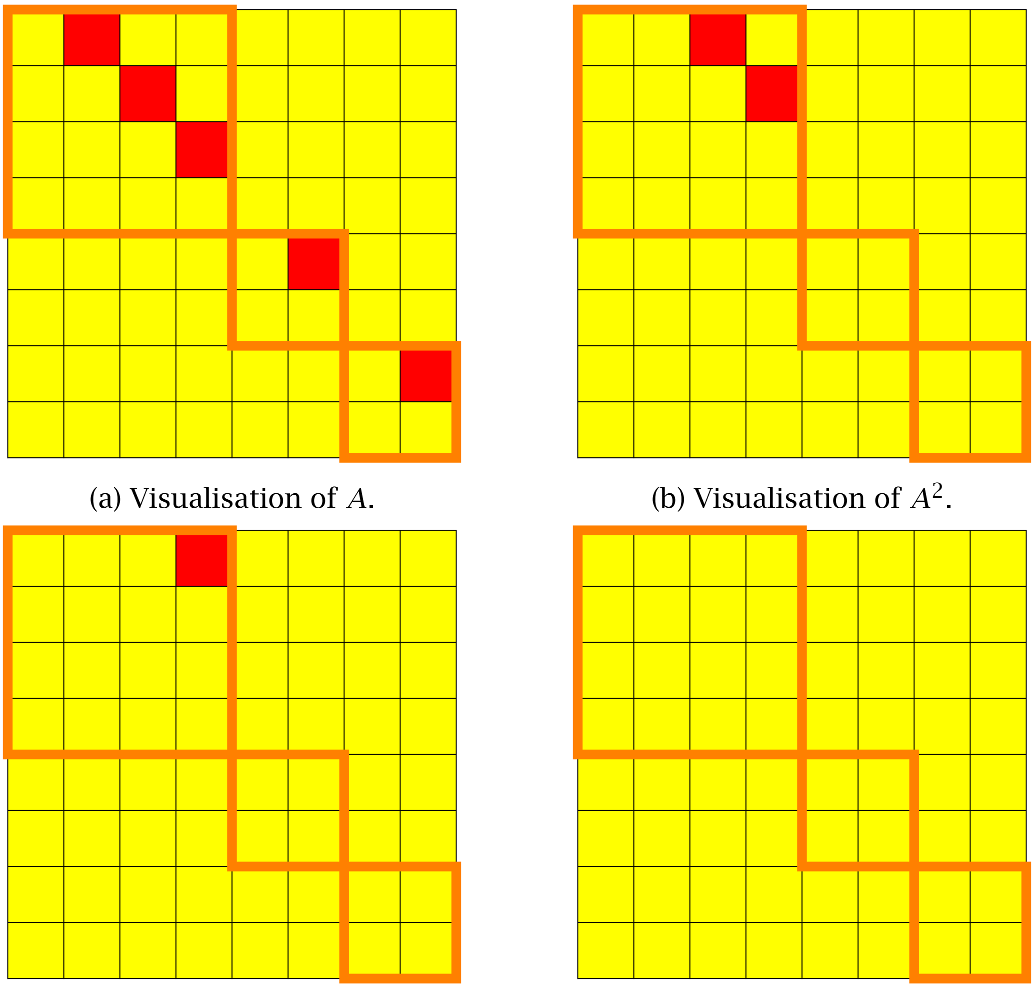
After we have found the third Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	chain 3	remaining dim
$\ker(A)$	3	$\mathbf{e}_1$	$\mathbf{e}_5$	$\mathbf{e}_7$	0
$\ker(A^2)$	6	$\mathbf{e}_2$	$\mathbf{e}_6$	$\mathbf{e}_8$	0
$\ker(A^3)$	7	$\mathbf{e}_3$			0
$\ker(A^4)$	8	$\mathbf{e}_4$			0

Because the last column consists now of 0's, we are done with looking for Jordan chains. We have found at this point a basis of V consisting entirely of vectors in Jordan chains.

Let us try to visualise our findings.



(a) Visualisation of A.

(b) Visualisation of A^2.

(c) Visualisation of A^3.

(d) Visualisation of A^4.

Figure 17

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

The powers of B.

$$B = \begin{pmatrix} 10 & -1 & 9 & -11 & 6 & 3 & -5 & -9 \\ 20 & 0 & 19 & -20 & 16 & 4 & -15 & -19 \\ -5 & 0 & -5 & 5 & -4 & -1 & 4 & 5 \\ 6 & -1 & 5 & -7 & 3 & 2 & -2 & -5 \\ -3 & 0 & -3 & 3 & -2 & -1 & 2 & 3 \\ -23 & 1 & -21 & 24 & -15 & -6 & 13 & 20 \\ 21 & -1 & 19 & -22 & 15 & 5 & -13 & -19 \\ -25 & 1 & -23 & 26 & -18 & -6 & 16 & 23 \end{pmatrix},$$

$$B^2 = \begin{pmatrix} 2 & 0 & 2 & -2 & 5 & 0 & -5 & -5 \\ 5 & 0 & 5 & -5 & 9 & 0 & -9 & -9 \\ -1 & 0 & -1 & 1 & -2 & 0 & 2 & 2 \\ 1 & 0 & 1 & -1 & 3 & 0 & -3 & -3 \\ -1 & 0 & -1 & 1 & -2 & 0 & 2 & 2 \\ -5 & 0 & -5 & 5 & -11 & 0 & 11 & 11 \\ 5 & 0 & 5 & -5 & 10 & 0 & -10 & -10 \\ -6 & 0 & -6 & 6 & -12 & 0 & 12 & 12 \end{pmatrix},$$

$$B^3 = \begin{pmatrix} 3 & 0 & 3 & -3 & 3 & 0 & -3 & -3 \\ 4 & 0 & 4 & -4 & 4 & 0 & -4 & -4 \\ -1 & 0 & -1 & 1 & -1 & 0 & 1 & 1 \\ 2 & 0 & 2 & -2 & 2 & 0 & -2 & -2 \\ -1 & 0 & -1 & 1 & -1 & 0 & 1 & 1 \\ -6 & 0 & -6 & 6 & -6 & 0 & 6 & 6 \\ 5 & 0 & 5 & -5 & 5 & 0 & -5 & -5 \\ -6 & 0 & -6 & 6 & -6 & 0 & 6 & 6 \end{pmatrix},$$

$$B^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

3. Calculation of the kernels of B^i.

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} 10 & -1 & 9 & -11 & 6 & 3 & -5 & -9 \\ 20 & 0 & 19 & -20 & 16 & 4 & -15 & -19 \\ -5 & 0 & -5 & 5 & -4 & -1 & 4 & 5 \\ 6 & -1 & 5 & -7 & 3 & 2 & -2 & -5 \\ -3 & 0 & -3 & 3 & -2 & -1 & 2 & 3 \\ -23 & 1 & -21 & 24 & -15 & -6 & 13 & 20 \\ 21 & -1 & 19 & -22 & 15 & 5 & -13 & -19 \\ -25 & 1 & -23 & 26 & -18 & -6 & 16 & 23 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} 10z_1 - z_2 + 9z_3 - 11z_4 + 6z_5 + 3z_6 - 5z_7 - 9z_8 = 0, \\ 20z_1 + 19z_3 - 20z_4 + 16z_5 + 4z_6 - 15z_7 - 19z_8 = 0, \\ -5z_1 - 5z_3 + 5z_4 - 4z_5 - 1z_6 + 4z_7 + 5z_8 = 0, \\ 6z_1 - z_2 + 5z_3 - 7z_4 + 3z_5 + 2z_6 - 2z_7 - 5z_8 = 0, \\ -3z_1 - 3z_3 + 3z_4 - 2z_5 - z_6 + 2z_7 + 3z_8 = 0, \\ -23z_1 + z_2 - 21z_3 + 24z_4 - 15z_5 - 6z_6 + 13z_7 + 20z_8 = 0, \\ 21z_1 - z_2 + 19z_3 - 22z_4 + 15z_5 + 5z_6 - 13z_7 - 19z_8 = 0, \\ -25z_1 + z_2 - 23z_3 + 26z_4 - 18z_5 - 6z_6 + 16z_7 + 23z_8 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B) &= \{(r_1, r_2, r_3, r_1 + r_3, r_3, -r_1 - r_2 - r_3, r_1 + r_2 + 2r_3, -r_1 - r_2 - r_3) \\ &\qquad \mid r_1, r_2, r_3 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,1,0,-1,1,-1), (0,1,0,0,0,-1,1,-1), \\ &\qquad (0,0,1,1,1,-1,2,-1)\}. \end{aligned}$$

Kernel of B².
We calculate the kernel of B^2 and we have to solve the matrix equation

$$\begin{pmatrix} 2 & 0 & 2 & -2 & 5 & 0 & -5 & -5 \\ 5 & 0 & 5 & -5 & 9 & 0 & -9 & -9 \\ -1 & 0 & -1 & 1 & -2 & 0 & 2 & 2 \\ 1 & 0 & 1 & -1 & 3 & 0 & -3 & -3 \\ -1 & 0 & -1 & 1 & -2 & 0 & 2 & 2 \\ -5 & 0 & -5 & 5 & -11 & 0 & 11 & 11 \\ 5 & 0 & 5 & -5 & 10 & 0 & -10 & -10 \\ -6 & 0 & -6 & 6 & -12 & 0 & 12 & 12 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} 2z_1 + 2z_3 - 2z_4 + 5z_5 - 5z_7 - 5z_8 = 0, \\ 5z_1 + 5z_3 - 5z_4 + 9z_5 - 9z_7 - 9z_8 = 0, \\ -z_1 - z_3 + z_4 - 2z_5 + 2z_7 + 2z_8 = 0, \\ z_1 + z_3 - z_4 + 3z_5 - 3z_7 - 3z_8 = 0, \\ -z_1 - z_3 + z_4 - 2z_5 + 2z_7 + 2z_8 = 0, \\ -5z_1 - 5z_3 + 5z_4 - 11z_5 + 11z_7 + 11z_8 = 0, \\ 5z_1 + 5z_3 - 5z_4 + 10z_5 - 10z_7 - 10z_8 = 0, \\ -6z_1 - 6z_3 + 6z_4 - 12z_5 + 12z_7 + 12z_8 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B^2) &= \{(r_1, r_2, r_3, r_1 + r_3, r_5, r_6, r_7, r_5 - r_7) \\ &\qquad\qquad \mid r_1, r_2, r_3, r_5, r_6, r_7 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,1,0,0,0,0), (0,1,0,0,0,0,0,0), \\ &\qquad (0,0,1,1,0,0,0,0), (0,0,0,0,1,0,0,1), (0,0,0,0,0,1,0,0), \\ &\qquad (0,0,0,0,0,0,1,-1)\}. \end{aligned}$$

Kernel of B^3.

We calculate the kernel of B^3 and we have to solve the matrix equation

$$\begin{pmatrix} 3 & 0 & 3 & -3 & 3 & 0 & -3 & -3 \\ 4 & 0 & 4 & -4 & 4 & 0 & -4 & -4 \\ -1 & 0 & -1 & 1 & -1 & 0 & 1 & 1 \\ 2 & 0 & 2 & -2 & 2 & 0 & -2 & -2 \\ -1 & 0 & -1 & 1 & -1 & 0 & 1 & 1 \\ -6 & 0 & -6 & 6 & -6 & 0 & 6 & 6 \\ 5 & 0 & 5 & -5 & 5 & 0 & -5 & -5 \\ -6 & 0 & -6 & 6 & -6 & 0 & 6 & 6 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} 3z_1 + 3z_3 - 3z_4 + 3z_5 - 3z_7 - 3z_8 = 0, \\ 4z_1 + 4z_3 - 4z_4 + 4z_5 - 4z_7 - 4z_8 = 0, \\ -z_1 - z_3 + z_4 - z_5 + z_7 + z_8 = 0, \\ 2z_1 + 2z_3 - 2z_4 + 2z_5 - 2z_7 - 2z_8 = 0, \\ -z_1 - z_3 + z_4 - z_5 + z_7 + z_8 = 0, \\ -6z_1 - 6z_3 + 6z_4 - 6z_5 + 6z_7 + 6z_8 = 0, \\ 5z_1 + 5z_3 - 5z_4 + 5z_5 - 5z_7 - 5z_8 = 0, \\ -6z_1 - 6z_3 + 6z_4 - 6z_5 + 6z_7 + 6z_8 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B^3) &= \{(r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_1 + r_3 - r_4 + r_5 - r_7) \\ &\qquad \mid r_1, r_2, r_3, r_4, r_5, r_6, r_7 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,0,0,0,1), (0,1,0,0,0,0,0,0), \\ &\qquad (0,0,1,0,0,0,0,1), (0,0,0,1,0,0,0,-1), \\ &\qquad (0,0,0,0,1,0,0,1), (0,0,0,0,0,1,0,0), \\ &\qquad (0,0,0,0,0,0,1,-1)\}. \end{aligned}$$

Kernel of B⁴.
We want to calculate the kernel of B^4 and we observe first that

$$B^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So the kernel is $\mathbf{K}^8$.

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	3	$3 = \dim(\ker(B))$
$\ker(B^2)$	6	$3 = \dim(\ker(B^2)) - \dim(\ker(B))$
$\ker(B^3)$	7	$1 = \dim(\ker(B^3)) - \dim(\ker(B^2))$
$\ker(B^4)$	8	$1 = \dim(\ker(B^4)) - \dim(\ker(B^3))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(B))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column and fourth line is 1 and this gives the information that there will be a Jordan chain of length 4. The first number 3 in the last column is the dimension of the kernel of B. After we have calculated this chain, the last column will be from top to bottom $\{2, 2, 0, 0\}$. Then we will first search for a Jordan chain of length 2. The last column will be from top to bottom $\{1, 1, 0, 0\}$. Then we will search for another Jordan chain of length 2. The last column will be $\{0, 0, 0, 0\}$. We have found a basis of 8 linearly independent Jordan chain vectors.

After that, there are no linearly independent vectors to be found. We have indeed at this moment already vectors which form a base for this vector space.

4. Calculation of the Jordan chains.

The first Jordan chain.
We have to choose a vector $\mathbf{w_4}$ that is in the kernel $\ker(B^4)$ but not in the kernel $\ker(B^3)$.

We look for a starting $\mathbf{w_4}$. We see from the information table that there is a generating vector $\mathbf{w_4}$ for a chain of length 4.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^4)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^4)$.

2. The generating vector may not be in the $\ker(B^3)$ because the length of the chain must be exactly 4. So it has to be independent from all vectors in $\ker(B^3)$. It is sufficient that it is linearly independent from a basis of $\ker(B^3)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 4. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^4 together with the vectors in $\ker(B^3)$ and also the vectors, if any, of exactly height 4 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^4) = \mathbf{K}^8.$$

So the generating vector has the following generic form

$$(a, b, c, d, e, f, g, h).$$

The kernel of B³.
We remember that the kernel $\ker(B^3)$ is

$$\begin{aligned}\ker(B^3) = \operatorname{span}\{&(1,0,0,0,0,0,0,1),(0,1,0,0,0,0,0,0),\\ &(0,0,1,0,0,0,0,1),(0,0,0,1,0,0,0,-1),\\ &(0,0,0,0,1,0,0,1),(0,0,0,0,0,1,0,0),\\ &(0,0,0,0,0,0,1,-1)\}.\end{aligned}$$

Vectors chosen in previous Jordan chains.
We have previously not chosen a vector of exactly height 4 in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ a & b & c & d & e & f & g & h \end{pmatrix}.$$

If we impose the condition $-a - c + d - e + g + h \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $-a - c + d - e + g + h \neq 0$. We can choose $a = 1$, $b = 0$, $c = 0$, $d = 0$, $e = 0$, $f = 0$, $g = 0$, $h = 0$.

We can choose a valid generating vector of the chain as follows

$$\mathbf{w_4} = (1, 0, 0, 0, 0, 0, 0, 0).$$

We know that

$$\{\mathbf{w_1} = B^3\,\mathbf{w_4}, \mathbf{w_2} = B^2\,\mathbf{w_4}, \mathbf{w_3} = B\,\mathbf{w_4}, \mathbf{w_4}\}$$

is a Jordan chain of 4 linearly independent vectors and because $B^4\,\mathbf{w_4} = \mathbf{0}$, we know that the length of the chain is exactly 4.

We calculate $\mathbf{w_3}$.

$$\begin{aligned}
\mathbf{w_3} &= B\,\mathbf{w_4} \\
&= \begin{pmatrix} 10 & -1 & 9 & -11 & 6 & 3 & -5 & -9 \\ 20 & 0 & 19 & -20 & 16 & 4 & -15 & -19 \\ -5 & 0 & -5 & 5 & -4 & -1 & 4 & 5 \\ 6 & -1 & 5 & -7 & 3 & 2 & -2 & -5 \\ -3 & 0 & -3 & 3 & -2 & -1 & 2 & 3 \\ -23 & 1 & -21 & 24 & -15 & -6 & 13 & 20 \\ 21 & -1 & 19 & -22 & 15 & 5 & -13 & -19 \\ -25 & 1 & -23 & 26 & -18 & -6 & 16 & 23 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 10 \\ 20 \\ -5 \\ 6 \\ -3 \\ -23 \\ 21 \\ -25 \end{pmatrix}.
\end{aligned}$$

We calculate $\mathbf{w_2}$.

$$\begin{aligned}
\mathbf{w_2} &= B^2\,\mathbf{w_4} \\
&= \begin{pmatrix} 2 & 0 & 2 & -2 & 5 & 0 & -5 & -5 \\ 5 & 0 & 5 & -5 & 9 & 0 & -9 & -9 \\ -1 & 0 & -1 & 1 & -2 & 0 & 2 & 2 \\ 1 & 0 & 1 & -1 & 3 & 0 & -3 & -3 \\ -1 & 0 & -1 & 1 & -2 & 0 & 2 & 2 \\ -5 & 0 & -5 & 5 & -11 & 0 & 11 & 11 \\ 5 & 0 & 5 & -5 & 10 & 0 & -10 & -10 \\ -6 & 0 & -6 & 6 & -12 & 0 & 12 & 12 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ 5 \\ -1 \\ 1 \\ -1 \\ -5 \\ 5 \\ -6 \end{pmatrix}.
\end{aligned}$$

We calculate $\mathbf{w_1}$.

$$
\begin{aligned}
\mathbf{w_1} &= B^3\,\mathbf{w_4} \\
&= \begin{pmatrix} 3 & 0 & 3 & -3 & 3 & 0 & -3 & -3 \\ 4 & 0 & 4 & -4 & 4 & 0 & -4 & -4 \\ -1 & 0 & -1 & 1 & -1 & 0 & 1 & 1 \\ 2 & 0 & 2 & -2 & 2 & 0 & -2 & -2 \\ -1 & 0 & -1 & 1 & -1 & 0 & 1 & 1 \\ -6 & 0 & -6 & 6 & -6 & 0 & 6 & 6 \\ 5 & 0 & 5 & -5 & 5 & 0 & -5 & -5 \\ -6 & 0 & -6 & 6 & -6 & 0 & 6 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 3 \\ 4 \\ -1 \\ 2 \\ -1 \\ -6 \\ 5 \\ -6 \end{pmatrix}.
\end{aligned}
$$

We have now found the first Jordan chain. It has length 4.

$$
\{\mathbf{w_1} = (3,4,-1,2,-1,-6,5,-6), \mathbf{w_2} = (2,5,-1,1,-1,-5,5,-6),
\mathbf{w_3} = (10,20,-5,6,-3,-23,21,-25), \mathbf{w_4} = (1,0,0,0,0,0,0,0)\}.
$$

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	3	$\mathbf{w_1}$	2
$\ker(B^2)$	6	$\mathbf{w_2}$	2
$\ker(B^3)$	7	$\mathbf{w_3}$	0
$\ker(B^4)$	8	$\mathbf{w_4}$	0

with

$$\mathbf{w_1} = (3, 4, -1, 2, -1, -6, 5, -6)$$

$$\mathbf{w_2} = (2, 5, -1, 1, -1, -5, 5, -6)$$

$$\mathbf{w_3} = (10, 20, -5, 6, -3, -23, 21, -25)$$

$$\mathbf{w_4} = (1, 0, 0, 0, 0, 0, 0, 0)$$

We see from this table that we can build 2 Jordan chains of length 2.

The second Jordan chain.
We look for a starting $\mathbf{w_6}$. We see from the information table that there is a generating vector $\mathbf{w_6}$ for a chain of length 2.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^2)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^2)$.

2. The generating vector may not be in the $\ker(B)$ because the length of the chain must be exactly 2. So it has to be independent from all vectors in $\ker(B)$. It is sufficient that it is linearly independent from a basis of $\ker(B)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 2. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^2 together with the vectors in $\ker(B)$ and also the vectors, if any, of exactly height 2 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^2) = \operatorname{span}\{(1,0,0,1,0,0,0,0),(0,1,0,0,0,0,0,0),(0,0,1,1,0,0,0,0),\\ (0,0,0,0,1,0,0,1),(0,0,0,0,0,1,0,0),\\ (0,0,0,0,0,0,1,-1)\}.$$

So the generating vector has the following generic form

$$\begin{aligned} & a\,(1,0,0,1,0,0,0,0) + b\,(0,1,0,0,0,0,0,0) \\ & \quad + c\,(0,0,1,1,0,0,0,0) + d\,(0,0,0,0,1,0,0,1) \\ & \quad + e\,(0,0,0,0,0,1,0,0) + f\,(0,0,0,0,0,0,1,-1) \\ & = (a,b,c,a+c,d,e,f,d-f). \end{aligned}$$

The kernel of B.
We remember that the kernel $\ker(B)$ is

$$\ker(B) = \operatorname{span}\{(1,0,0,1,0,-1,1,-1),(0,1,0,0,0,-1,1,-1),\\ (0,0,1,1,1,-1,2,-1)\}.$$

Vectors chosen in previous Jordan chains.
We have previously chosen the vector $\mathbf{w_2} = (2,5,-1,1,-1,-5,5,-6)$ of exactly height 2 in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & -1 & 2 & -1 \\ 2 & 5 & -1 & 1 & -1 & -5 & 5 & -6 \\ a & b & c & a+c & c & e & f & c-f \end{pmatrix}.$$

If we impose the condition $a + b + 2\,c - f \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition that $a + b + 2c - f \neq 0$. We can choose $a = 0$, $b = 1$, $c = 0$, $d = 0$, $e = 0$, $f = 0$.

We can choose a valid generating vector of the chain as follows

$$\mathbf{w_6} = (0, 1, 0, 0, 0, 0, 0, 0).$$

We calculate now $\mathbf{w_5}$.

$$\begin{aligned} \mathbf{w_5} &= B\,\mathbf{w_6} \\ &= \begin{pmatrix} 10 & -1 & 9 & -11 & 6 & 3 & -5 & -9 \\ 20 & 0 & 19 & -20 & 16 & 4 & -15 & -19 \\ -5 & 0 & -5 & 5 & -4 & -1 & 4 & 5 \\ 6 & -1 & 5 & -7 & 3 & 2 & -2 & -5 \\ -3 & 0 & -3 & 3 & -2 & -1 & 2 & 3 \\ -23 & 1 & -21 & 24 & -15 & -6 & 13 & 20 \\ 21 & -1 & 19 & -22 & 15 & 5 & -13 & -19 \\ -25 & 1 & -23 & 26 & -18 & -6 & 16 & 23 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}. \end{aligned}$$

So we have that $\mathbf{w_5} = (-1, 0, 0, -1, 0, 1, -1, 1)$.

We have now found a second Jordan chain. It has length 2.

$$\{\mathbf{w}_5 = (-1, 0, 0, -1, 0, 1, -1, 1), \mathbf{w}_6 = (0, 1, 0, 0, 0, 0, 0, 0)\}.$$

We look now at our information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(B)$	3	$\mathbf{w}_1$	$\mathbf{w}_5$	1
$\ker(B^2)$	6	$\mathbf{w}_2$	$\mathbf{w}_6$	1
$\ker(B^3)$	7	$\mathbf{w}_3$		0
$\ker(B^4)$	8	$\mathbf{w}_4$		0

with

$\mathbf{w}_1 = (3, 4, -1, 2, -1, -6, 5, -6)$
$\mathbf{w}_2 = (2, 5, -1, 1, -1, -5, 5, -6)$
$\mathbf{w}_3 = (10, 20, -5, 6, -3, -23, 21, -25)$
$\mathbf{w}_4 = (1, 0, 0, 0, 0, 0, 0, 0)$
$\mathbf{w}_5 = (-1, 0, 0, -1, 0, 1, -1, 1)$
$\mathbf{w}_6 = (0, 1, 0, 0, 0, 0, 0, 0)$

The third Jordan chain.
We look for a starting $\mathbf{w}_8$. We see from the information table that there is a generating vector $\mathbf{w}_8$ for a chain of length 2.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^2)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^2)$.

2. The generating vector may not be in the ker(B) because the length of the chain must be exactly 2. So it has to be independent from all vectors in ker(B). It is sufficient that it is linearly independent from a basis of ker(B).

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 2. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^2 together with the vectors in ker(B) and also the vectors, if any, of exactly height 2 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\begin{aligned}\ker(B^2) = \operatorname{span}\{&(1,0,0,1,0,0,0,0),(0,1,0,0,0,0,0,0),(0,0,1,1,0,0,0,0),\\&(0,0,0,0,1,0,0,1),(0,0,0,0,0,1,0,0),\\&(0,0,0,0,0,0,1,-1)\}.\end{aligned}$$

So the generating vector has the following generic form

$$\begin{aligned}&a\,(1,0,0,1,0,0,0,0)+b\,(0,1,0,0,0,0,0,0)\\&\qquad+c\,(0,0,1,1,0,0,0,0)+d\,(0,0,0,0,1,0,0,1)\\&\qquad+e\,(0,0,0,0,0,1,0,0)+f\,(0,0,0,0,0,0,1,-1)\\&=(a,b,c,a+c,d,e,f,d-f).\end{aligned}$$

The kernel of B.
We remember that the kernel ker(B) is

$$\begin{aligned}\ker(B) = \operatorname{span}\{&(1,0,0,1,0,-1,1,-1),(0,1,0,0,0,-1,1,-1),\\&(0,0,1,1,1,-1,2,-1)\}.\end{aligned}$$

Vectors chosen in previous Jordan chains.

We have previously chosen the vector $\mathbf{w}_2 = (2, 5, -1, 1, -1, -5, 5, -6)$ and $\mathbf{w}_6 = (0, 1, 0, 0, 0, 0, 0, 0)$ of exactly height 2 in Jordan chains.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & -1 & 2 & -1 \\ 2 & 5 & -1 & 1 & -1 & -5 & 5 & -6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & b & c & a+c & d & e & f & d-f \end{pmatrix}.$$

If we impose the condition $c - d \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition that $c - d \neq 0$. We can choose $a = 0$, $b = 0$, $c = 1$, $d = 0$, $e = 0$, $f = 0$.

We can choose a valid generating vector of the chain as follows

$$\mathbf{w}_8 = (0, 0, 1, 1, 0, 0, 0, 0).$$

We calculate now $\mathbf{w}_7$.

$$\mathbf{w}_7 = B\,\mathbf{w}_8$$

$$= \begin{pmatrix} 10 & -1 & 9 & -11 & 6 & 3 & -5 & -9 \\ 20 & 0 & 19 & -20 & 16 & 4 & -15 & -19 \\ -5 & 0 & -5 & 5 & -4 & -1 & 4 & 5 \\ 6 & -1 & 5 & -7 & 3 & 2 & -2 & -5 \\ -3 & 0 & -3 & 3 & -2 & -1 & 2 & 3 \\ -23 & 1 & -21 & 24 & -15 & -6 & 13 & 20 \\ 21 & -1 & 19 & -22 & 15 & 5 & -13 & -19 \\ -25 & 1 & -23 & 26 & -18 & -6 & 16 & 23 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ -1 \\ 0 \\ -2 \\ 0 \\ 3 \\ -3 \\ 3 \end{pmatrix}.$$

So we have that $\mathbf{w}_7 = (-2, -1, 0, -2, 0, 3, -3, 3)$.

We have now found the third Jordan chain. It has length 2.

$$\{\mathbf{w}_7 = (-2, -1, 0, -2, 0, 3, -3, 3), \mathbf{w}_8 = (0, 0, 1, 1, 0, 0, 0, 0)\}.$$

We look now at our information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	chain 3	remaining dim
$\ker(B)$	3	$\mathbf{w}_1$	$\mathbf{w}_5$	$\mathbf{w}_7$	0
$\ker(B^2)$	6	$\mathbf{w}_2$	$\mathbf{w}_6$	$\mathbf{w}_8$	0
$\ker(B^3)$	7	$\mathbf{w}_3$			0
$\ker(B^4)$	8	$\mathbf{w}_4$			0

with

$$\mathbf{w_1} = (3, 4, -1, 2, -1, -6, 5, -6)$$
$$\mathbf{w_2} = (2, 5, -1, 1, -1, -5, 5, -6)$$
$$\mathbf{w_3} = (10, 20, -5, 6, -3, -23, 21, -25)$$
$$\mathbf{w_4} = (1, 0, 0, 0, 0, 0, 0, 0)$$
$$\mathbf{w_5} = (-1, 0, 0, -1, 0, 1, -1, 1)$$
$$\mathbf{w_6} = (0, 1, 0, 0, 0, 0, 0, 0)$$
$$\mathbf{w_7} = (-2, -1, 0, -2, 0, 3, -3, 3)$$
$$\mathbf{w_8} = (0, 0, 1, 1, 0, 0, 0, 0)$$

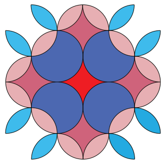
5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} 3 & 2 & 10 & 1 & -1 & 0 & -2 & 0 \\ 4 & 5 & 20 & 0 & 0 & 1 & -1 & 0 \\ -1 & -1 & -5 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 6 & 0 & -1 & 0 & -2 & 1 \\ -1 & -1 & -3 & 0 & 0 & 0 & 0 & 0 \\ -6 & -5 & -23 & 0 & 1 & 0 & 3 & 0 \\ 5 & 5 & 21 & 0 & -1 & 0 & -3 & 0 \\ -6 & -6 & -25 & 0 & 1 & 0 & 3 & 0 \end{pmatrix}.$$

We check this solution.

$$
\begin{aligned}
A &= P^{-1} B P \\
&= \begin{pmatrix}
0 & 0 & 0 & 0 & -2 & -1 & 1 & 2 \\
0 & 0 & 0 & 0 & -2 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\
1 & 0 & 1 & -1 & 1 & 0 & -1 & -1 \\
0 & 0 & 3 & -3 & 1 & -3 & 2 & 3 \\
0 & 1 & -1 & 1 & -2 & 0 & 3 & 4 \\
0 & 0 & -1 & 1 & 0 & 1 & -3 & -3 \\
0 & 0 & 1 & 0 & 1 & 0 & -2 & -2
\end{pmatrix} \\
&\quad \times \begin{pmatrix}
10 & -1 & 9 & -11 & 6 & 3 & -5 & -9 \\
20 & 0 & 19 & -20 & 16 & 4 & -15 & -19 \\
-5 & 0 & -5 & 5 & -4 & -1 & 4 & 5 \\
6 & -1 & 5 & -7 & 3 & 2 & -2 & -5 \\
-3 & 0 & -3 & 3 & -2 & -1 & 2 & 3 \\
-23 & 1 & -21 & 24 & -15 & -6 & 13 & 20 \\
21 & -1 & 19 & -22 & 15 & 5 & -13 & -19 \\
-25 & 1 & -23 & 26 & -18 & -6 & 16 & 23
\end{pmatrix} \\
&\quad \times \begin{pmatrix}
3 & 2 & 10 & 1 & -1 & 0 & -2 & 0 \\
4 & 5 & 20 & 0 & 0 & 1 & -1 & 0 \\
-1 & -1 & -5 & 0 & 0 & 0 & 0 & 1 \\
2 & 1 & 6 & 0 & -1 & 0 & -2 & 1 \\
-1 & -1 & -3 & 0 & 0 & 0 & 0 & 0 \\
-6 & -5 & -23 & 0 & 1 & 0 & 3 & 0 \\
5 & 5 & 21 & 0 & -1 & 0 & -3 & 0 \\
-6 & -6 & -25 & 0 & 1 & 0 & 3 & 0
\end{pmatrix} \\
&= \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
&= \begin{pmatrix}
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 & 0 & 0 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 & 0 & 0 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0}
\end{pmatrix}.
\end{aligned}
$$



17 exercise. (3×3); $(J_3(0))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B so that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 3 & 2 & -1 \\ -3 & -2 & 1 \\ 2 & 1 & -1 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial $p_{\text{C-H}}(\lambda)$.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_3| = -\lambda^3.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 3. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field **K**. It is thus possible to put the matrix B in Jordan normal form over the field **K**.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 2. The solution will be completely independent from this section.

We want to investigate the endomorphism A associated with this Jordan block.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right).$$

Remark that this is already an elementary Jordan block.

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right),$$

$$A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right).$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the third power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 3, is called the height of nilpotency of the matrix A.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A) = \operatorname{span}\{\mathbf{e}_1\}; \\ \ker(A^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}; \\ \ker(A^3) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \mathbf{K}^3. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	1	$1 = \dim(\ker(A))$
$\ker(A^2)$	2	$1 = \dim(\ker(A^2)) - \dim(\ker(A))$
$\ker(A^3)$	3	$1 = \dim(\ker(A^3)) - \dim(\ker(A^2))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the third power onwards.

$$\ker(A) \subsetneq \ker(A^2) \subsetneq \ker(A^3) = \ker(A^4) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 + n_3 = 1 = \dim(\ker(A)), \\ \qquad\; n_2 + n_3 = 1 = \dim(\ker(A^2)) - \dim(\ker(A)), \\ \qquad\qquad\; n_3 = 1 = \dim(\ker(A^3)) - \dim(\ker(A^2)). \end{cases}$$

Solving this system, we have $n_1 = 0$, $n_2 = 0$, $n_3 = 1$.

A consequence from this fact is that the numbers in the last column are descending.

We see that

$$\begin{cases} A\,\mathbf{e_3} = \mathbf{e_2}, \\ A\,\mathbf{e_2} = \mathbf{e_1}, \\ A\,\mathbf{e_1} = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices A^i that

$$\begin{cases} A\,\mathbf{e_3} = \mathbf{e_2}, \\ A^2\,\mathbf{e_3} = \mathbf{e_1}, \\ A^3\,\mathbf{e_3} = \mathbf{0}. \end{cases}$$

We have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e_1} = A^2\,\mathbf{e_3}, \mathbf{e_2} = A\,\mathbf{e_3}, \mathbf{e_3}\}.$$

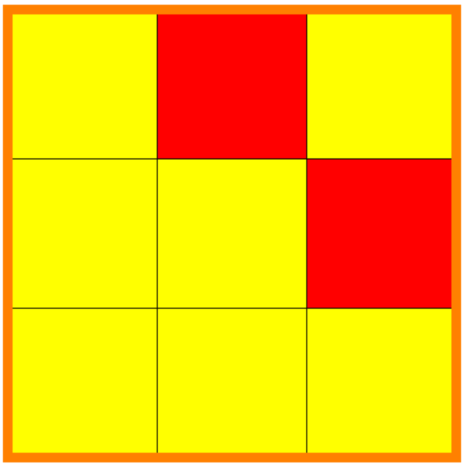
After we have found the first Jordan chain of length 3, we have then the following table.

Keeping track of chains and dimensions.

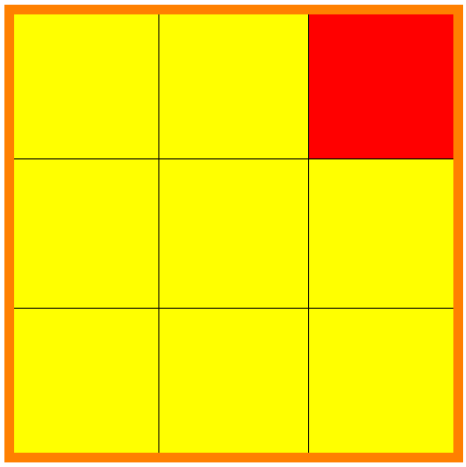
	dim	chain 1	remaining dim
$\ker(A)$	1	$\mathbf{e_1}$	0
$\ker(A^2)$	2	$\mathbf{e_2}$	0
$\ker(A^2)$	3	$\mathbf{e_2}$	0

Because the last column consists now of 0's, we are done with looking for Jordan chains. We have found at this point a basis of V consisting entirely of vectors in Jordan chains.

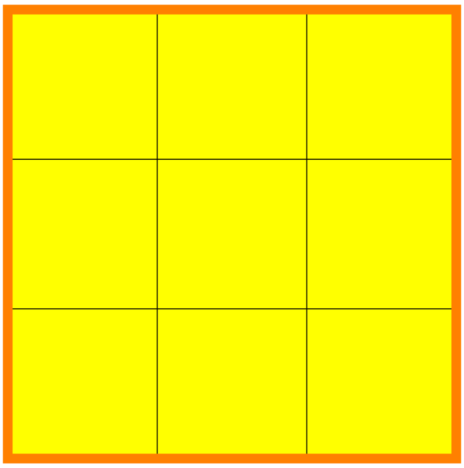
Let us try to visualise the situation



(a) Visualisation of A.



(b) Visualisation of A^2.



(c) Visualisation of A^3.

Figure 18

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up

into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

The powers of the matrix B.
Let us calculate the powers of the matrix B.

$$B = \begin{pmatrix} 3 & 2 & -1 \\ -3 & -2 & 1 \\ 2 & 1 & -1 \end{pmatrix}; \qquad B^2 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix};$$

$$B^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} 3 & 2 & -1 \\ -3 & -2 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} 3z_1 + 2z_2 - z_3 = 0, \\ -3z_1 - 2z_2 + z_3 = 0, \\ 2z_1 + z_2 - z_3 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\ker(B) = \{(r_1, -r_1, r_1) \mid r_1 \in \mathbf{K}\} = \operatorname{span}\{(1, -1, 1)\}.$$

Kernel of B^2.
We calculate the kernel of B^2 and we have to solve the matrix equation

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} z_1 + z_2 = 0, \\ -z_1 - z_2 = 0, \\ z_1 + z_2 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\ker(B^2) = \{(r_1, -r_1, r_3) \mid r_1, r_3 \in \mathbf{K}\} = \operatorname{span}\{(1, -1, 0), (0, 0, 1)\}.$$

Kernel of $\mathbf{B^3}$.
We have

$$\ker(B^3) = \mathbf{K}^3.$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	1	$1 = \dim(\ker(B))$
$\ker(B^2)$	2	$1 = \dim(\ker(B^2)) - \dim(\ker(B))$
$\ker(B^3)$	3	$1 = \dim(\ker(B^3)) - \dim(\ker(B^2))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(B))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column and third line is 1 and this gives the information that there will be one Jordan chain of length 3. The first number 1 in the last column and first line is the dimension of the kernel of B. This says that there will be one Jordan chain in the result.

After we have calculated this chain, the last column will be from top to bottom $\{0, 0, 0\}$. There is no chain left to be found. We already have a basis consisting of vectors in Jordan chains.

4. Calculation of the Jordan chains.

The first Jordan chain.
We look for a starting $\mathbf{w}_3$. We see from the information table that there is a generating vector $\mathbf{w}_3$ for a chain of length 3.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^3)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^3)$.
2. The generating vector may not be in the $\ker(B^2)$ because the length of the chain must be exactly 3. So it has to be independent from all vectors in $\ker(B^2)$. It is sufficient that it is linearly independent from a basis of $\ker(B^2)$.
3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 3. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.
4. We summarise: the generating vector in B^3 together with the vectors in $\ker(B^2)$ and also the vectors, if any, of exactly height 3 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^3) = \mathbf{K}^3.$$

So the generating vector has the following generic form

$$(a, b, c).$$

The kernel of B^2.
We remember that the kernel $\ker(B^2)$ is

$$\ker(B^2) = \text{span}\{(1, -1, 0), (0, 0, 1)\}.$$

Vectors chosen in previous Jordan chains.
We have previously not chosen a vector of exactly height 3 in Jordan chains.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix}.$$

If we impose the condition $a + b \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition that $a + b \neq 0$. We can choose $a = 1$, $b = 0$, $c = 0$.

We can choose a valid generating vector of the chain as follows

$$\mathbf{w_3} = (1, 0, 0).$$

We calculate then $B\,(1, 0, 0)$, $B^2\,(1, 0, 0)$, and we know that the vectors

$$\{\mathbf{w_1} = B^2\,\mathbf{w_3}, \mathbf{w_2} = B\,\mathbf{w_3}, \mathbf{w_3}\}$$

are a Jordan chain of 3 linearly independent vectors and because $B^3\,\mathbf{w_3} = \mathbf{0}$, we know that the length of the chain is exactly 3.

We calculate $\mathbf{w_2}$.

$$\mathbf{w_2} = B\,\mathbf{w_3} = \begin{pmatrix} -2 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}.$$

We calculate $\mathbf{w_1}$.

$$\mathbf{w_1} = B^2\,\mathbf{w_3} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

We have now found the Jordan chain

$$\{\mathbf{w_1} = (1, -1, 1), \mathbf{w_2} = (3, -3, 2), \mathbf{w_3} = (1, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	1	$\mathbf{w_1}$	0
$\ker(B^2)$	2	$\mathbf{w_2}$	0
$\ker(B^3)$	3	$\mathbf{w_3}$	0

with

$$\mathbf{w_1} = (1, -1, 1)$$
$$\mathbf{w_2} = (3, -3, 2)$$
$$\mathbf{w_3} = (1, 0, 0)$$

We have only 0's in the last column, so we are done searching for Jordan chains. The last column consists of only 0's. We have found at this point a basis of V consisting entirely of vectors in Jordan chains.

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} 1 & 3 & 1 \\ -1 & -3 & 0 \\ 1 & 2 & 0 \end{pmatrix}.$$

We check this solution.

$$\begin{aligned} A &= P^{-1} B P \\ &= \begin{pmatrix} 0 & 2 & 3 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 2 & -1 \\ -3 & -2 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ -1 & -3 & 0 \\ 1 & 2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \left(\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \right). \end{aligned}$$

18 exercise. (5×5); $(J_3(0), J_2(0))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B so that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 2 & 1 & 1 & -2 & -1 \\ -2 & -1 & 0 & 1 & 2 \\ -3 & -1 & -1 & 2 & 2 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial $p_{\text{C-H}}(\lambda)$.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_5| = -\lambda^5.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 5. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field $\mathbf{K}$. It is thus possible to put the matrix B in Jordan normal form over the field $\mathbf{K}$.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to

this subsection 2. The solution will be completely independent from this section.

We want to investigate the endomorphism A.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We show this in a way that emphasises the position of the elementary Jordan blocks.

$$A = \left(\begin{array}{ccccc} \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \end{array}\right).$$

We compute also the powers of A.

$$A = \left(\begin{array}{ccccc} \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \end{array}\right); \quad A^2 = \left(\begin{array}{ccccc} \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \end{array}\right);$$
$$A^3 = \left(\begin{array}{ccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \end{array}\right).$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the third power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 3, is called the height of nilpotency of the matrix A.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_4\}. \\ \ker(A^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5\}. \\ \ker(A^3) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\} = \mathbf{K}^5. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	2	$2 = \dim(\ker(A))$
$\ker(A^2)$	4	$2 = \dim(\ker(A^2)) - \dim(\ker(A))$
$\ker(A^3)$	5	$1 = \dim(\ker(A^3)) - \dim(\ker(A^2))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.
2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.
3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.
4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.
5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the third power onwards.

$$\ker(A) \subsetneq \ker(A^2) \subsetneq \ker(A^3) = \ker(A^4) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 + n_3 = 2 = \dim(\ker(A)), \\ \qquad n_2 + n_3 = 2 = \dim(\ker(A^2)) - \dim(\ker(A)), \\ \qquad\qquad n_3 = 1 = \dim(\ker(A^3)) - \dim(\ker(A^2)). \end{cases}$$

Solving this system, we have $n_1 = 0$, $n_2 = 1$, $n_3 = 1$.

A consequence from this fact is that the numbers in the last column are descending.

We remark by looking at the matrices A^i that

$$\begin{cases} A\,\mathbf{e_3} = \mathbf{e_2}, \\ A\,\mathbf{e_2} = \mathbf{e_1}, \\ A\,\mathbf{e_1} = \mathbf{0}, \end{cases}$$

or

$$\begin{cases} \mathbf{e_1} = A\,\mathbf{e_3}, \\ \mathbf{e_2} = A^2\,\mathbf{e_3}, \\ A^3\,\mathbf{e_3} = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain with length three of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e_1} = A^2\,\mathbf{e_3}, \mathbf{e_2} = A\,\mathbf{e_3}, \mathbf{e_3}\}.$$

After we have found the first Jordan chain of length 3, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A)$	2	$\mathbf{e}_1$	1
$\ker(A^2)$	4	$\mathbf{e}_2$	1
$\ker(A^3)$	5	$\mathbf{e}_3$	0

Because the last column consists now of at least one number that is not 0, we are **not** done with looking for Jordan chains.

We remark by looking at the matrices A^i that

$$\begin{cases} A\,\mathbf{e}_5 = \mathbf{e}_4, \\ A\,\mathbf{e}_4 = \mathbf{0}, \end{cases}$$

or

$$\begin{cases} A\,\mathbf{e}_5 = \mathbf{e}_4, \\ A^2\,\mathbf{e}_5 = \mathbf{0}. \end{cases}$$

We have found a second Jordan chain. It has length two. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_4 = A\,\mathbf{e}_5, \mathbf{e}_5\}.$$

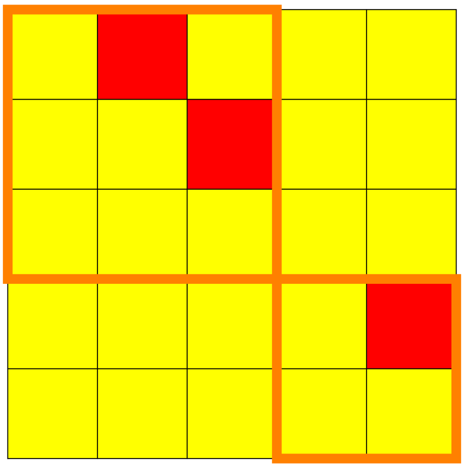
Let us note this in the new information table.

Keeping track of chains and dimensions.

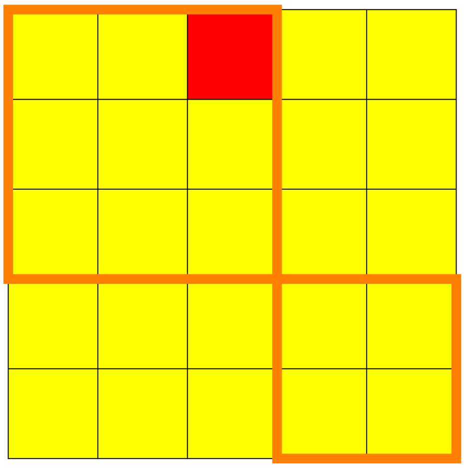
	dim	chain 1	chain 2	remaining dim
$\ker(A)$	2	$\mathbf{e}_1$	$\mathbf{e}_4$	0
$\ker(A^2)$	4	$\mathbf{e}_2$	$\mathbf{e}_5$	0
$\ker(A^3)$	5	$\mathbf{e}_3$		0

The last column consists of only 0's and this ends the search for Jordan chains. We have indeed found 5 linearly independent vectors and these form a basis for $\mathbf{K}^5$.

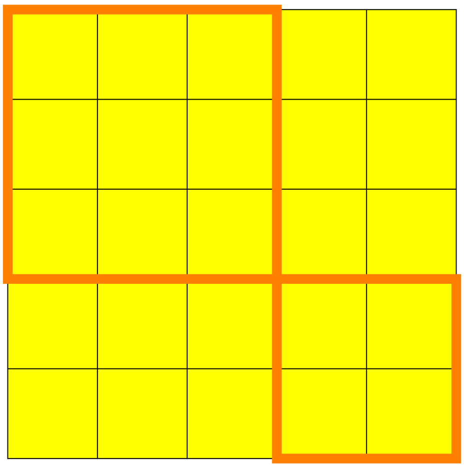
Let us express our findings in a way that emphasises the position of the elementary Jordan blocks and their relation with the powers of the matrix A.



(a) Visualisation of A.



(b) Visualisation of A^2.



(c) Visualisation of A^3.

Figure 19

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red,

and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

The powers of the matrix B.
Let us calculate the powers of the matrix B.

$$B = \begin{pmatrix} 2 & 1 & 1 & -2 & -1 \\ -2 & -1 & 0 & 1 & 2 \\ -3 & -1 & -1 & 2 & 2 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & 0 \end{pmatrix}; \quad B^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$B^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} 2 & 1 & 1 & -2 & -1 \\ -2 & -1 & 0 & 1 & 2 \\ -3 & -1 & -1 & 2 & 2 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} 2z_1 + z_2 + z_3 - 2z_4 - z_5 = 0, \\ -2z_1 - z_2 + z_4 + 2z_5 = 0, \\ -3z_1 - z_2 - z_3 + 2z_4 + 2z_5 = 0, \\ -z_1 + z_5 = 0, \\ z_1 + z_3 - z_4 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B) &= \{(r_1, r_2, -r_1 + r_2, r_2, r_1) \mid r_1, r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,-1,0,1), (0,1,1,1,0)\}. \end{aligned}$$

Kernel of B^2.
We calculate the kernel of B^2 and we have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} -z_1 - z_2 + z_4 + z_5 = 0, \\ -z_1 - z_2 + z_4 + z_5 = 0, \\ -z_1 - z_2 + z_4 + z_5 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B^2) &= \{(r_1, r_2, r_3, r_4, r_1 + r_2 - r_4) \mid r_1, r_2, r_3, r_4 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,1), (0,1,0,0,1), (0,0,1,0,0), (0,0,0,1,-1)\}. \end{aligned}$$

Kernel of B^3.
We calculate the kernel of B^3 and we have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is the vector space $\mathbf{K}^5$.

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	2	$2 = \dim(\ker(B))$
$\ker(B^2)$	4	$2 = \dim(\ker(B^2)) - \dim(\ker(B))$
$\ker(B^3)$	5	$1 = \dim(\ker(B^3)) - \dim(\ker(B^2))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(B))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The “remaining dim” give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column and third line is 1 and this gives the information that there will be one Jordan chain of length 3. The first number 2 in the last column is the dimension of the kernel of B. This number gives the information that there will be two Jordan chains. After we have calculated the chain with length 3, the last column will be from top to bottom $\{1, 1, 0\}$. There is still one two chain of independent vectors left to be found. After this calculation, the last column will be from top to bottom $\{0, 0, 0\}$. We have at this moment 5 linearly independent vectors which form a basis for this vector space.

4. Calculation of the Jordan chains.

The first Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ satisfying

$$\begin{cases} B\,\mathbf{w}_3 = \mathbf{w}_2, \\ B^2\,\mathbf{w}_3 = \mathbf{w}_1, \\ B^3\,\mathbf{w}_3 = \mathbf{0}. \end{cases}$$

where $\mathbf{w}_3$ is in the vector space $\ker(B^3)$ but not in $\ker(B^2)$.

We look for a starting $\mathbf{w}_3$. We see from the information table that there is a generating vector $\mathbf{w}_3$ for a chain of length 3.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^3)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^3)$.

2. The generating vector may not be in the $\ker(B^2)$ because the length of the chain must be exactly 3. So it has to be independent from all vectors in $\ker(B^2)$. It is sufficient that it is linearly independent from a basis of $\ker(B^2)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 3. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^3 together with the vectors in $\ker(B^2)$ and also the vectors, if any, of exactly height 3 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^3) = \mathbf{K}^5.$$

So the generating vector has the following generic form

$$(a, b, c, d, e).$$

The kernel of $\mathbf{B}^2$.
We remember that the kernel $\ker(B^2)$ is

$$\ker(B^2) = \text{span}\{(1,0,0,0,1),(0,1,0,0,1),(0,0,1,0,0),(0,0,0,1,-1)\}.$$

Vectors chosen in previous Jordan chains.
We have previously not chosen a vector of exactly height 3 in Jordan chains.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ a & b & c & d & e \end{pmatrix}.$$

If we impose the condition $a + b - d - e \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition that $a + b - d - e \neq 0$. We can choose $a = 1, b = 0, c = 0, d = 0, e = 0$.

We can choose a valid generating vector of the chain as follows

$$\mathbf{w_3} = (1, 0, 0, 0, 0).$$

We calculate then $B^i(1, 0, 0, 0, 0)$ from $i = 1$ to $i = 2$, and we know that the vectors

$$\{\mathbf{w_1} = B^2\,\mathbf{w_3}, \mathbf{w_2} = B\,\mathbf{w_3}, \mathbf{w_3}\}$$

are a Jordan chain of 3 linearly independent vectors and because $B^3\,\mathbf{w_3} = \mathbf{0}$, we know that the length of the chain is exactly 3.

We calculate $\mathbf{w_2}$.

$$\mathbf{w_2} = B\,\mathbf{w_3} = \begin{pmatrix} 2 & 1 & 1 & -2 & -1 \\ -2 & -1 & 0 & 1 & 2 \\ -3 & -1 & -1 & 2 & 2 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -3 \\ -1 \\ 1 \end{pmatrix}.$$

We calculate $\mathbf{w_1}$.

$$\mathbf{w_1} = B^2\mathbf{w_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \\ 0 \end{pmatrix}.$$

We have found our first Jordan chain. It has length 3.

$$\{\mathbf{w_1} = (0, -1, -1, -1, 0), \mathbf{w_2} = (2, -2, -3, -1, 1), \mathbf{w_3} = (1, 0, 0, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	2	$\mathbf{w_1}$	1
$\ker(B^2)$	4	$\mathbf{w_2}$	1
$\ker(B^3)$	5	$\mathbf{w_3}$	0

with

$$\mathbf{w_1} = (0, -1, -1, -1, 0)$$
$$\mathbf{w_2} = (2, -2, -3, -1, 1)$$
$$\mathbf{w_3} = (1, 0, 0, 0, 0)$$

We see at the number 1 in the second row of the last column that we have still to find a vector $\mathbf{w_5}$ satisfying two criteria.

We look for a starting $\mathbf{w_5}$. We see from the information table that there is a generating vector $\mathbf{w_5}$ for a chain of length 2.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^2)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^2)$.

2. The generating vector may not be in the $\ker(B)$ because the length of the chain must be exactly 2. So it has to be independent from all vectors in $\ker(B)$. It is sufficient that it is linearly independent from a basis of $\ker(B)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 2. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^2 together with the vectors in $\ker(B)$ and also the vectors, if any, of exactly height 2 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^2) = \operatorname{span}\{(1,0,0,0,1),(0,1,0,0,1),(0,0,1,0,0),(0,0,0,1,-1)\}.$$

So the generating vector has the following generic form

$$\begin{aligned} &a\,(1,0,0,0,1) + b\,(0,1,0,0,1) + c\,(0,0,1,0,0) + d\,(0,0,0,1,-1) \\ &\quad = (a,b,c,d,a+b-d). \end{aligned}$$

The kernel of B.
We remember that the kernel of B is

$$\ker(B) = \operatorname{span}\{(1,0,-1,0,1),(0,1,1,1,0)\}.$$

Vectors chosen in previous Jordan chains.
We have previously chosen the vector $\mathbf{w_2} = (2,-2,-3,-1,1)$ of height 2 in a Jordan chain.

Condition of linear independency of the vectors.

We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 2 & -2 & -3 & -1 & 1 \\ a & b & c & d & a+b-d \end{pmatrix}.$$

If we impose the condition $a + c - d \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

We see that these vectors are independent if we impose the condition that $a + c - d \neq 0$. We can choose $a = 1$, $b = 0$, $c = 0$, $d = 0$, $e = 0$.

We can choose a valid generating vector of the chain as follows

$$\mathbf{w}_5 = (1, 0, 0, 0, 1).$$

We calculate now $B(1, 0, 0, 0, 1)$, and we know that the vectors

$$\{B\,\mathbf{w}_5, \mathbf{w}_5\}$$

form a Jordan chain of two linearly independent vectors.

We calculate $\mathbf{w}_4$.

$$\mathbf{w}_4 = B\,\mathbf{w}_5 = \begin{pmatrix} 2 & 1 & 1 & -2 & -1 \\ -2 & -1 & 0 & 1 & 2 \\ -3 & -1 & -1 & 2 & 2 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

We have found our second Jordan chain. It has length 2.

$$\{\mathbf{w}_4 = (1, 0, -1, 0, 1), \mathbf{w}_5 = (1, 0, 0, 0, 1)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(B)$	2	$\mathbf{w_1}$	$\mathbf{w_4}$	0
$\ker(B^2)$	4	$\mathbf{w_2}$	$\mathbf{w_5}$	0
$\ker(B^3)$	5	$\mathbf{w_3}$		0

with

$$\mathbf{w_1} = (0, -1, -1, -1, 0)$$
$$\mathbf{w_2} = (2, -2, -3, -1, 1)$$
$$\mathbf{w_3} = (1, 0, 0, 0, 0)$$
$$\mathbf{w_4} = (1, 0, -1, 0, 1)$$
$$\mathbf{w_5} = (1, 0, 0, 0, 1)$$

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns. We have now

$$P = \begin{pmatrix} 0 & 2 & 1 & 1 & 1 \\ -1 & -2 & 0 & 0 & 0 \\ -1 & -3 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

We check our calculations.

$$
\begin{aligned}
A &= P^{-1}\,B\,P \\
&= \begin{pmatrix} 0 & 1 & 0 & -2 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 2 & 1 & 1 & -2 & -1 \\ -2 & -1 & 0 & 1 & 2 \\ -3 & -1 & -1 & 2 & 2 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & 0 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 & 2 & 1 & 1 & 1 \\ -1 & -2 & 0 & 0 & 0 \\ -1 & -3 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \left(\begin{array}{ccccc} \multicolumn{3}{c}{\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & & & \multicolumn{2}{c}{\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}} \end{array}\right).
\end{aligned}
$$

19 exercise. (4×4); $(J_4(0))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B so that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} -1 & 1 & -1 & 0 \\ 7 & -4 & 9 & 3 \\ 3 & -2 & 4 & 1 \\ 3 & -2 & 3 & 1 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial $p_{\text{C-H}}(\lambda)$.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_4| = \lambda^4.$$

The eigenvalue $\lambda = 0$ has algebraic multiplicity 4. The characteristic polynomial $p_{\text{C-H}}(\lambda)$ factors completely in linear polynomials over the field $\mathbf{K}$. It is thus possible to put the matrix B in Jordan normal form over the field $\mathbf{K}$.

2. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and continue with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 2. The solution will be completely independent from this section.

We want to investigate the endomorphism A.

We compute also the powers of A.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left(\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}\right);$$

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left(\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}\right);$$

$$A^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left(\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}\right);$$

$$A^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left(\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}\right).$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the fourth power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 4, is called the height of nilpotency of the matrix A.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A) = \operatorname{span}\{\mathbf{e}_1\}; \\ \ker(A^2) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}; \\ \ker(A^3) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}; \\ \ker(A^4) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\} = \mathbf{K}^4. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	1	$1 = \dim(\ker(A))$
$\ker(A^2)$	2	$1 = \dim(\ker(A^2)) - \dim(\ker(A))$
$\ker(A^3)$	3	$1 = \dim(\ker(A^3)) - \dim(\ker(A^2))$
$\ker(A^4)$	4	$1 = \dim(\ker(A^4)) - \dim(\ker(A^3))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.
2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.
3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.
4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.
5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

We see also that there is equality in the inclusion of sets from the fourth power onwards.

$$\ker(A) \subsetneq \ker(A^2) \subsetneq \ker(A^3) \subsetneq \ker(A^4) = \ker(A^5) = \cdots .$$

The dimensions of the kernels increase until they stabilise.

We mention also the following fact. If n_i is the number of linearly independent Jordan chains of length exactly i, then we have the following set of equations:

$$\begin{cases} n_1 + n_2 + n_3 + n_4 = 1 = \dim(\ker(A)), \\ n_2 + n_3 + n_4 = 1 = \dim(\ker(A^2)) - \dim(\ker(A)), \\ n_3 + n_4 = 1 = \dim(\ker(A^3)) - \dim(\ker(A^2)), \\ n_4 = 1 = \dim(\ker(A^4)) - \dim(\ker(A^3)). \end{cases}$$

Solving this system, we have $n_1 = 0$, $n_2 = 0$, $n_3 = 0$, $n_4 = 1$.

A consequence from this fact is that the numbers in the last column are descending.

We remark by looking at the matrices A^i that

$$\begin{cases} A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A\,\mathbf{e}_3 = \mathbf{e}_2, \\ A\,\mathbf{e}_2 = \mathbf{e}_1, \\ A\,\mathbf{e}_1 = \mathbf{0}. \end{cases}$$

$$\begin{cases} A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A^2\,\mathbf{e}_4 = \mathbf{e}_2, \\ A^3\,\mathbf{e}_4 = \mathbf{e}_1, \\ A^4\,\mathbf{e}_4 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = A^3\,\mathbf{e}_4, \mathbf{e}_2 = A^2\,\mathbf{e}_4, \mathbf{e}_3 = A\,\mathbf{e}_4, \mathbf{e}_4\}.$$

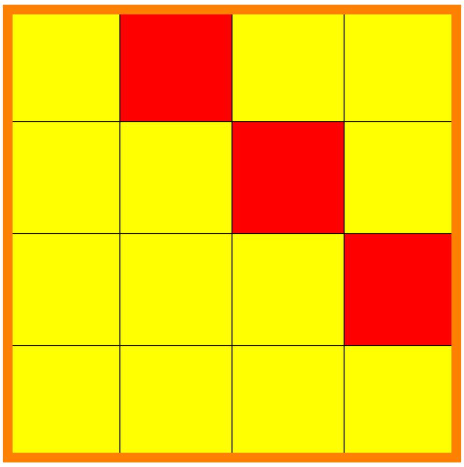
After we have found the first Jordan chain of length 4, we have then the following table.

Keeping track of chains and dimensions.

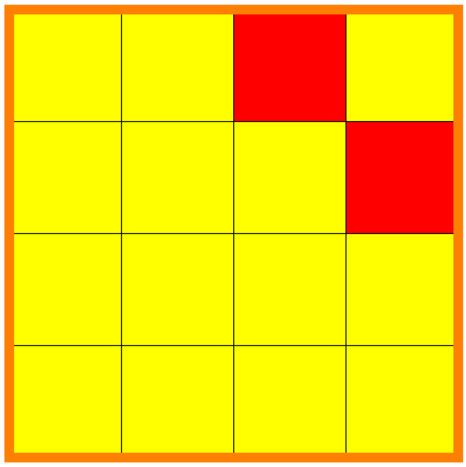
	dim	chain 1	remaining dim
ker(A)	1	$\mathbf{e}_1$	0
ker(A^2)	1	$\mathbf{e}_2$	0
ker(A^3)	1	$\mathbf{e}_3$	0
ker(A^4)	1	$\mathbf{e}_4$	0

Because the last column consists now of 0's, we are done with looking for Jordan chains. We have found at this point a basis of V consisting entirely of vectors in Jordan chains.

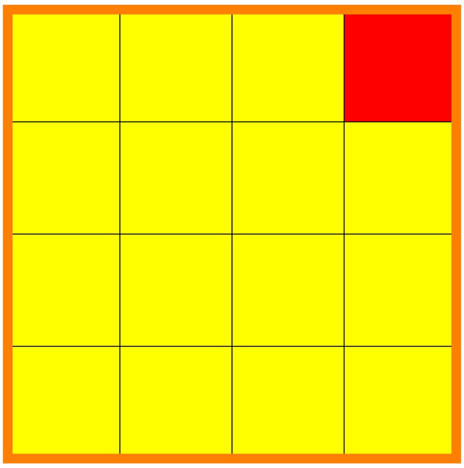
Let us express our findings in a way that emphasises the position of the elementary Jordan blocks and their relation with the powers of the matrix A.



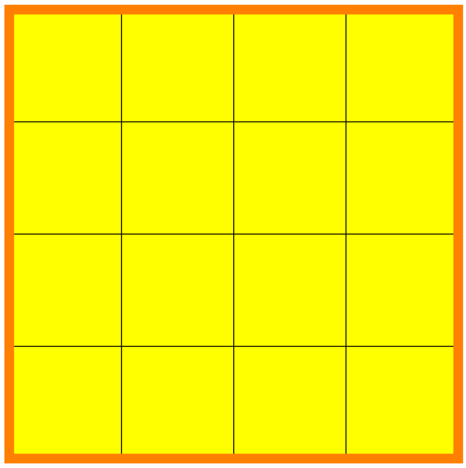
(a) Visualisation of A.



(b) Visualisation of A^2.



(c) Visualisation of A^3.



(d) Visualisation of A^4.

Figure 20

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.

2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.

3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up

into their respective elementary Jordan blocks until they ultimately completely disappear.

4. We can visually read from these matrices the height of nilpotency.

5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.

6. The orange lines delimit the original position of the elementary Jordan blocks.

3. Calculation of the kernels of B^i.

Let us calculate the powers of the matrix B.

$$B = \begin{pmatrix} -1 & 1 & -1 & 0 \\ 7 & -4 & 9 & 3 \\ 3 & -2 & 4 & 1 \\ 3 & -2 & 3 & 1 \end{pmatrix}; \qquad B^2 = \begin{pmatrix} 5 & -3 & 6 & 2 \\ 1 & -1 & 2 & 0 \\ -2 & 1 & -2 & -1 \\ -5 & 3 & -6 & -2 \end{pmatrix};$$

$$B^3 = \begin{pmatrix} -2 & 1 & -2 & -1 \\ -2 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 2 & -1 & 2 & 1 \end{pmatrix}; \quad B^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Kernel of B.
We calculate the kernel of B and we have to solve the matrix equation

$$\begin{pmatrix} -1 & 1 & -1 & 0 \\ 7 & -4 & 9 & 3 \\ 3 & -2 & 4 & 1 \\ 3 & -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} -z_1 + z_2 - z_3 = 0, \\ 7z_1 - 4z_2 + 9z_3 + 3z_4 = 0, \\ 3z_1 - 2z_2 + 4z_3 + z_4 = 0, \\ 3z_1 - 2z_2 + 3z_3 + z_4 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B) &= \{(r_1, r_1, 0, -r_1) \mid r_1 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, 1, 0, -1)\}. \end{aligned}$$

Kernel of B^2.
We calculate the kernel of B^2 and we have to solve the matrix equation

$$B^2 = \begin{pmatrix} 5 & -3 & 6 & 2 \\ 1 & -1 & 2 & 0 \\ -2 & 1 & -2 & -1 \\ -5 & 3 & -6 & -2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} 5z_1 - 3z_2 + 6z_3 + 2z_4 = 0, \\ z_1 - z_2 + 2z_3 = 0, \\ -2z_1 + z_2 - 2z_3 - z_4 = 0, \\ -5z_1 + 3z_2 - 6z_3 - 2z_4 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B^2) &= \{(r_1, r_2, -r_1/2 + r_2/2, -r_1) \mid r_1, r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, 0, -1/2, -1), (0, 1, 1/2, 0)\}. \end{aligned}$$

Kernel of B^3.
We calculate the kernel of B^3 and we have to solve the matrix equation

$$\begin{pmatrix} -2 & 1 & -2 & -1 \\ -2 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 2 & -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in the following system of linear equations

$$\begin{cases} -2z_1 + z_2 - 2z_3 - z_4 = 0, \\ -2z_1 + z_2 - 2z_3 - z_4 = 0, \\ 2z_1 - z_2 + 2z_3 + z_4 = 0. \end{cases}$$

We solve this system and find the solution set which is a subspace

$$\begin{aligned} \ker(B^3) &= \{(r_1, r_2, r_3, -2\,r_1 + r_2 - 2\,r_3) \mid r_1, r_2, r_3 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, 0, 0, -2), (0, 1, 0, 1), (0, 0, 1, -2)\}. \end{aligned}$$

Kernel of B^4.
We want to calculate the kernel of B^4 and we observe first that

$$B^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The kernel is $\mathbf{K}^4$.

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B)$	1	$1 = \dim(\ker(B))$
$\ker(B^2)$	2	$1 = \dim(\ker(B^2)) - \dim(\ker(B))$
$\ker(B^3)$	3	$1 = \dim(\ker(B^3)) - \dim(\ker(B^2))$
$\ker(B^4)$	4	$1 = \dim(\ker(B^4)) - \dim(\ker(B^3))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix B.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(B))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The “remaining dim” give the number of linearly independent vectors in Jordan chains that still have to be determined.

The last number in the last column and fourth line is 1 and this gives the information that there will be one Jordan chain of length 4. The first number 1 in the last column is the dimension of the kernel of B. This gives the information that there will be one Jordan chain. After we have calculated this chain, the last column will be $\{0, 0, 0, 0\}$. There are no linearly independent vectors left to be found. We have indeed at this moment already 4 linearly independent vectors which form a base for this vector space.

4. Calculation of the Jordan chains.

The first Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ satisfying

$$\begin{cases} B\,\mathbf{w}_4 = \mathbf{w}_3, \\ B\,\mathbf{w}_3 = \mathbf{w}_2, \\ B\,\mathbf{w}_2 = \mathbf{w}_1, \\ B\,\mathbf{w}_1 = \mathbf{0} \end{cases}$$

or

$$\begin{cases} B\,\mathbf{w}_4 = \mathbf{w}_3, \\ B^2\,\mathbf{w}_3 = \mathbf{w}_2, \\ B^3\,\mathbf{w}_2 = \mathbf{w}_1, \\ B^4\,\mathbf{w}_1 = \mathbf{0} \end{cases}$$

where $\mathbf{w}_4$ is in the vector space $\ker(B^4)$ but not in $\ker(B^3)$.

We look for a starting $\mathbf{w}_4$. We see from the information table that there is a generating vector $\mathbf{w}_4$ for a chain of length 4.

The generating vector has to satisfy the following conditions.

1. It has to be in the kernel $\ker(B^4)$. So we can describe such a vector as a linear combination of vectors of a basis of the subspace $\ker(B^4)$.

2. The generating vector may not be in the $\ker(B^3)$ because the length of the chain must be exactly 4. So it has to be independent from all vectors in $\ker(B^3)$. It is sufficient that it is linearly independent from a basis of $\ker(B^3)$.

3. This generating vector has to be linearly independent from all vectors already chosen in previous Jordan chains which have exactly height 4. It can be the case that no previous chains are already chosen or there are no vectors of that height previously chosen in which case this is no condition at all.

4. We summarise: the generating vector in B^4 together with the vectors in $\ker(B^3)$ and also the vectors, if any, of exactly height 4 chosen in previous Jordan chains must be a linearly independent set of vectors.

Generic form of the generating vector.
We remember that

$$\ker(B^4) = \mathbf{K}^4.$$

So the generating vector has the following generic form

$$(a, b, c, d).$$

The kernel of B³.
We remember that the kernel $\ker(B^3)$ is

$$\ker(B^3) = \operatorname{span}\{(1, 0, 0, -2), (0, 1, 0, 1), (0, 0, 1, -2)\}.$$

Vectors chosen in previous Jordan chains.
We have previously not chosen a vector of height 4 in a Jordan chain.

Condition of linear independency of the vectors.
We have to take care that the generic vector is linearly independent of all the vectors mentioned before. We collect for this purpose all these vectors in the rows of the matrix H and row reduce then this matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ a & b & c & d \end{pmatrix}.$$

If we impose the condition $2\,a - b + 2\,c + d \neq 0$, then we can row reduce this matrix to the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition that $2\,a - b + 2\,c + d \neq 0$. We can choose $a = 1$, $b = 0$, $c = 0$ and $d = 0$.

We can choose a valid generating vector of the chain as follows

$$\mathbf{w_4} = (1, 0, 0, 0).$$

We calculate $\mathbf{w_3}$.

$$\mathbf{w_3} = B\,\mathbf{w_4} = \begin{pmatrix} -1 & 1 & -1 & 0 \\ 7 & -4 & 9 & 3 \\ 3 & -2 & 4 & 1 \\ 3 & -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ 3 \\ 3 \end{pmatrix}.$$

We calculate $\mathbf{w_2}$.

$$\mathbf{w_2} = B^2\,\mathbf{w_4} = \begin{pmatrix} 5 & -3 & 6 & 2 \\ 1 & -1 & 2 & 0 \\ -2 & 1 & -2 & -1 \\ -5 & 3 & -6 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -2 \\ -5 \end{pmatrix}.$$

We calculate $\mathbf{w_1}$.

$$\mathbf{w_1} = B^3\,\mathbf{w_4} = \begin{pmatrix} -2 & 1 & -2 & -1 \\ -2 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 2 & -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 0 \\ 2 \end{pmatrix}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B)$	1	$\mathbf{w_1}$	0
$\ker(B^2)$	2	$\mathbf{w_2}$	0
$\ker(B^3)$	3	$\mathbf{w_3}$	0
$\ker(B^4)$	4	$\mathbf{w_4}$	0

with

$$\mathbf{w_1} = (-2, -2, 0, 2)$$

$$\mathbf{w_2} = (5, 1, -2, -5)$$

$$\mathbf{w_3} = (-1, 7, 3, 3)$$

$$\mathbf{w_4} = (1, 0, 0, 0)$$

5. Result and check of the result.

We construct the matrix P by using the coordinates of the vectors $\mathbf{w_i}$ that we found in the Jordan chains as its columns.

$$P = \begin{pmatrix} -2 & 5 & -1 & 1 \\ -2 & 1 & 7 & 0 \\ 0 & -2 & 3 & 0 \\ 2 & -5 & 3 & 0 \end{pmatrix}.$$

We check this solution.

$$
\begin{aligned}
A &= P^{-1} B P \\
&= \begin{pmatrix} 0 & 9/16 & -19/8 & 17/16 \\ 0 & 3/8 & -5/4 & 3/8 \\ 0 & 1/4 & -1/2 & 1/4 \\ 1 & -1/2 & 1 & 1/2 \end{pmatrix} \\
&\quad \times \begin{pmatrix} -1 & 1 & -1 & 0 \\ 7 & -4 & 9 & 3 \\ 3 & -2 & 4 & 1 \\ 3 & -2 & 3 & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} -2 & 5 & -1 & 1 \\ -2 & 1 & 7 & 0 \\ 0 & -2 & 3 & 0 \\ 2 & -5 & 3 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \left(\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \right).
\end{aligned}
$$

20 exercise. (14×14)

Find the kernels of A^i and Jordan chains of the matrix A.

$$A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Solution.

We want to investigate the endomorphism A associated with this Jordan block.

Before trying to find a solution of this exercise, let us try to see if this matrix is a Jordan matrix. We write the matrix by emphasising the elementary Jordan blocks.

$$A = \begin{pmatrix} \begin{pmatrix} 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0 \end{pmatrix} & \begin{matrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{matrix} \\ \begin{matrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{matrix} & \begin{matrix} \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} & \begin{matrix} 0&0\\0&0\\0&0\\0&0 \end{matrix} \\ \begin{matrix} 0&0&0&0\\0&0&0&0 \end{matrix} & \begin{pmatrix} 0&1\\0&0 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

The powers of the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0 \end{pmatrix} & \begin{matrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{matrix} \\ \begin{matrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{matrix} & \begin{matrix} \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} & \begin{matrix} 0&0\\0&0\\0&0\\0&0 \end{matrix} \\ \begin{matrix} 0&0&0&0\\0&0&0&0 \end{matrix} & \begin{pmatrix} 0&1\\0&0 \end{pmatrix} \end{matrix} \end{pmatrix},$$

$$A^2 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$A^3 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$A^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

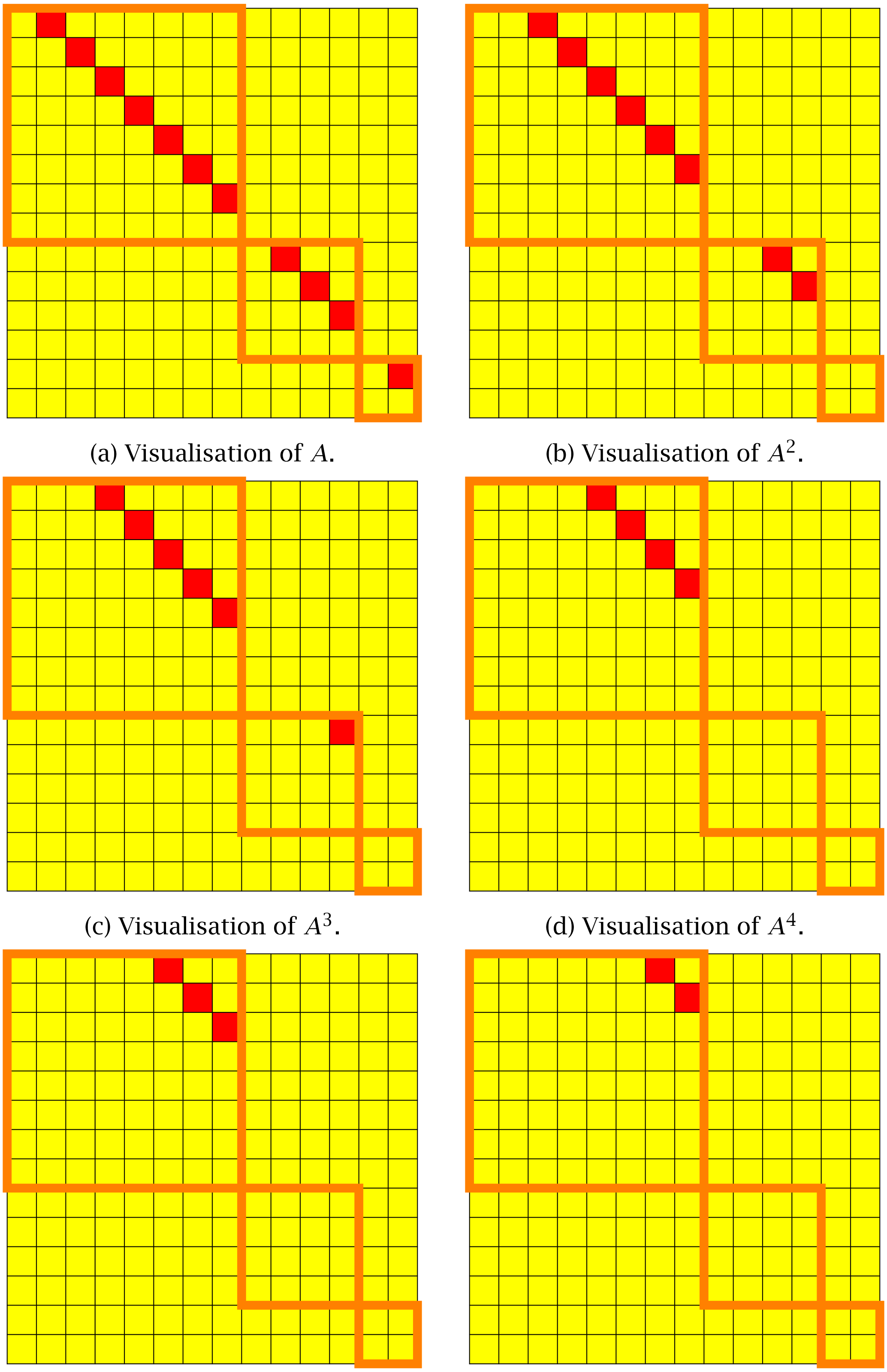
$$A^6 = \begin{pmatrix} \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} & \begin{matrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{matrix} \\ \begin{matrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{matrix} & \begin{matrix} \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} & \begin{matrix} 0&0\\ 0&0\\ 0&0\\ 0&0 \end{matrix} \\ \begin{matrix} 0&0&0&0\\ 0&0&0&0 \end{matrix} & \begin{pmatrix} 0&0\\ 0&0 \end{pmatrix} \end{matrix} \end{pmatrix},$$

$$A^7 = \begin{pmatrix} \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} & \begin{matrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{matrix} \\ \begin{matrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{matrix} & \begin{matrix} \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} & \begin{matrix} 0&0\\ 0&0\\ 0&0\\ 0&0 \end{matrix} \\ \begin{matrix} 0&0&0&0\\ 0&0&0&0 \end{matrix} & \begin{pmatrix} 0&0\\ 0&0 \end{pmatrix} \end{matrix} \end{pmatrix},$$

$$A^8 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{matrix} \end{pmatrix}.$$

We observe that when increasing the exponents of the matrix A the original superdiagonals of 1's in the elementary Jordan blocks of the matrix A are going upwards in their respective Jordan blocks in the powers of the matrix A until they finally disappear when taking the eighth power of A. The smallest exponent that makes the power of the matrix A the zero matrix, in this case 8, is called the height of nilpotency of the matrix A.

Let us try to visualise these matrices in order to get an easier view of the situation.



(a) Visualisation of A.

(b) Visualisation of A^2.

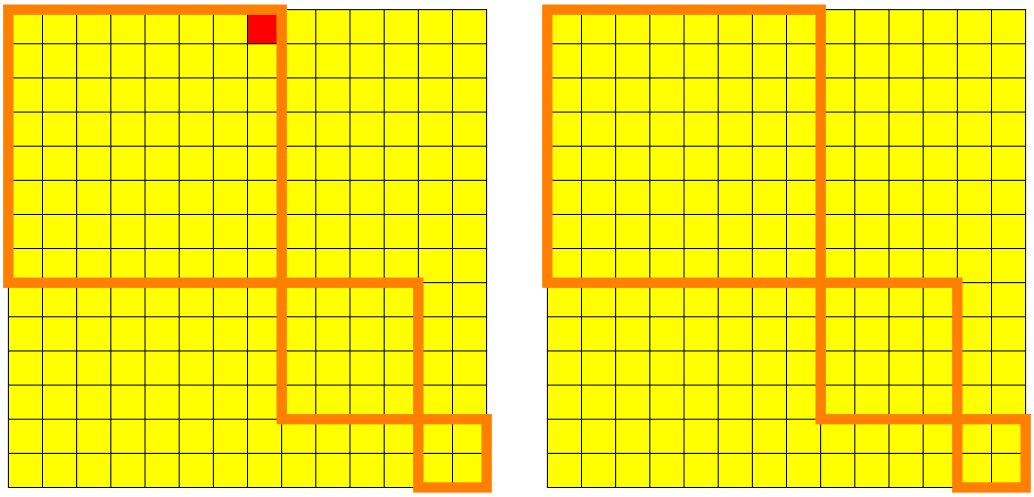
(c) Visualisation of A^3.

(d) Visualisation of A^4.

(e) Visualisation of A^5.

(f) Visualisation of A^6.

Figure 21. The comments follow after the next figure.



(a) Visualisation of A^7. (b) Visualisation of A^8.

Figure 22

We give here some comments about the figure.

1. We see in these figures a visualisation of the matrices A^i with the squares or cells on which there are 1's, drawn with the colour red, and the squares or cells on which there are 0's, drawn with the colour yellow.
2. We can immediately observe the kernels of the A^i and the dynamic creation of the Jordan chains.
3. We see also when increasing the exponents how the superdiagonals in the elementary Jordan blocks go steadily higher and higher up into their respective elementary Jordan blocks until they ultimately completely disappear.
4. We can visually read from these matrices the height of nilpotency.
5. We can read from these matrices the invariant subspaces with respect to the endomorphism A.
6. The orange lines delimit the original position of the elementary Jordan blocks.

Kernels of $\mathbf{A^i}$.

We can immediately observe that the kernel of A has dimension 3. The dimension for the kernel of an endomorphism with the matrix in this echelon form matrix is the number of 0 columns or alternatively the number of 0 rows.

Remark also that empty columns stay empty columns reflecting the fact that the kernel of A^i is a subset of the kernel of A^{i+1}. We can see almost without calculation that

$$\ker(A) = \text{span}\{\mathbf{e}_1, \mathbf{e}_9, \mathbf{e}_{13}\}.$$

We look at the matrix A^2. It is interesting to observe how the kernels change. We can see almost without calculation that the superdiagonal of the 2×2 matrix has completely disappeared. The two other superdiagonals shift to the top in their respective elementary matrices. We can immediately observe that the kernel has dimension 6. Other supplementary vectors start to appear in this kernel.

$$\ker(A^2) = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_9, \mathbf{e}_{10}, \mathbf{e}_{13}, \mathbf{e}_{14}\}.$$

We look at the matrix A^3. We can immediately observe that the kernel has dimension 8. Other supplementary vectors start to appear in this kernel.

$$\ker(A^3) = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_9, \mathbf{e}_{10}, \mathbf{e}_{11}, \mathbf{e}_{13}, \mathbf{e}_{14}\}$$

We look at the matrix A^4. We can immediately observe that the kernel has dimension 10. Other supplementary vectors start to appear in this kernel.

$$\ker(A^4) = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_9, \mathbf{e}_{10}, \mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{14}\}.$$

We look at the matrix A^5. We can immediately observe that the kernel has dimension 11. Other supplementary vectors start to appear in this kernel.

$$\ker(A^5) = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_9, \mathbf{e}_{10}, \mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{14}\}.$$

We look at the matrix A^6. We can immediately observe that the kernel has dimension 12. Other supplementary vectors start to appear in this kernel.

$$\ker(A^6) = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_9, \mathbf{e}_{10}, \mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{14}\}.$$

We look at the matrix A^7. We can immediately observe that the kernel has dimension 13. Other supplementary vectors start to appear in this kernel.

$$\ker(A^7) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_9, \mathbf{e}_{10}, \mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{14}\}.$$

We look at the matrix A^8. We can immediately observe that the kernel has dimension 14 and this is the vector space $\mathbf{K}^{14}$.

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A)$	3	$3 = \dim(\ker(A))$
$\ker(A^2)$	6	$3 = \dim(\ker(A^2)) - \dim(\ker(A))$
$\ker(A^3)$	8	$2 = \dim(\ker(A^3)) - \dim(\ker(A^2))$
$\ker(A^4)$	10	$2 = \dim(\ker(A^4)) - \dim(\ker(A^3))$
$\ker(A^5)$	11	$1 = \dim(\ker(A^5)) - \dim(\ker(A^4))$
$\ker(A^6)$	12	$1 = \dim(\ker(A^6)) - \dim(\ker(A^5))$
$\ker(A^7)$	13	$1 = \dim(\ker(A^7)) - \dim(\ker(A^6))$
$\ker(A^8)$	14	$1 = \dim(\ker(A^8)) - \dim(\ker(A^7))$

We give some explanation about this information table.

1. We find in the first column the dimensions of the kernels of the consecutive powers of the matrix A.

2. We have in the second column in every row i but the first the consecutive differences of the kernel dimensions.

3. In the first row, we have $\dim(\ker(A))$. This number is equal to the number of elementary Jordan blocks and is usually called the geometric multiplicity of the eigenvalue $\lambda = 0$.

4. The last number of the second column that is not equal to zero, is the number of elementary Jordan chains having the following properties. These elementary Jordan chains consist out of vectors so that their union consists entirely of linearly independent vectors. The lengths of these chains are all exactly equal to the number of the row in which this last non zero number occurs.

5. The "remaining dim" give the number of linearly independent vectors in Jordan chains that still have to be determined.

By looking at the first elementary 8×8 Jordan block in the matrix A, we see that we have the following mappings

$$\begin{cases} A\,\mathbf{e}_1 = \mathbf{0}, \\ A\,\mathbf{e}_2 = \mathbf{e}_1, \\ A\,\mathbf{e}_3 = \mathbf{e}_2, \\ A\,\mathbf{e}_4 = \mathbf{e}_3, \\ A\,\mathbf{e}_5 = \mathbf{e}_4, \\ A\,\mathbf{e}_6 = \mathbf{e}_5, \\ A\,\mathbf{e}_7 = \mathbf{e}_6, \\ A\,\mathbf{e}_8 = \mathbf{e}_7. \end{cases}$$

This can be rewritten as follows

$$\begin{cases} A^8\,\mathbf{e}_8 = \mathbf{0}, \\ A^7\,\mathbf{e}_8 = \mathbf{e}_1, \\ A^6\,\mathbf{e}_8 = \mathbf{e}_2, \\ A^5\,\mathbf{e}_8 = \mathbf{e}_3, \\ A^4\,\mathbf{e}_8 = \mathbf{e}_4, \\ A^3\,\mathbf{e}_8 = \mathbf{e}_5, \\ A^2\,\mathbf{e}_8 = \mathbf{e}_6, \\ A\,\mathbf{e}_8 = \mathbf{e}_7. \end{cases}$$

Because $A^8\,\mathbf{e}_8 = \mathbf{0}$, we conclude that we have found a first Jordan chain. It has length 8.

$$\{\mathbf{e}_1 = A^7\,\mathbf{e}_8, \mathbf{e}_2 = A^6\,\mathbf{e}_8, \mathbf{e}_3 = A^5\,\mathbf{e}_8, \mathbf{e}_4 = A^4\,\mathbf{e}_8, \\ \mathbf{e}_5 = A^3\,\mathbf{e}_8, \mathbf{e}_6 = A^2\,\mathbf{e}_8, \mathbf{e}_7 = A\,\mathbf{e}_8, \mathbf{e}_8\}.$$

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A)$	3	$\mathbf{e}_1$	2
$\ker(A^2)$	6	$\mathbf{e}_2$	2
$\ker(A^3)$	8	$\mathbf{e}_3$	1
$\ker(A^4)$	10	$\mathbf{e}_4$	1
$\ker(A^5)$	11	$\mathbf{e}_5$	0
$\ker(A^6)$	12	$\mathbf{e}_6$	0
$\ker(A^7)$	13	$\mathbf{e}_7$	0
$\ker(A^8)$	14	$\mathbf{e}_8$	0

The last 1 in the last column and the fourth line indicates that there is still one chain to be found of length 4. The second elementary Jordan block in the matrix A says also that there is a chain of length 4. We can see from the matrix A that we have the following mappings.

$$\begin{cases} A\,\mathbf{e}_9 = \mathbf{0}, \\ A\,\mathbf{e}_{10} = \mathbf{e}_9, \\ A\,\mathbf{e}_{11} = \mathbf{e}_{10}, \\ A\,\mathbf{e}_{12} = \mathbf{e}_{11}, \end{cases}$$

$$\begin{cases} A^4\,\mathbf{e}_{12} = \mathbf{0}, \\ A^3\,\mathbf{e}_{12} = \mathbf{e}_9, \\ A^2\,\mathbf{e}_{12} = \mathbf{e}_{10}, \\ A\,\mathbf{e}_{12} = \mathbf{e}_{11}. \end{cases}$$

Because $A^4\,\mathbf{e}_{12} = \mathbf{0}$, we conclude that we have found a second Jordan chain. It has length 4.

$$\{\mathbf{e}_9 = A^3\,\mathbf{e}_{12}, \mathbf{e}_{10} = A^2\,\mathbf{e}_{12}, \mathbf{e}_{11} = A\,\mathbf{e}_{12}, \mathbf{e}_{12}\}.$$

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(A)$	3	$\mathbf{e}_1$	$\mathbf{e}_9$	1
$\ker(A^2)$	6	$\mathbf{e}_2$	$\mathbf{e}_{10}$	1
$\ker(A^3)$	8	$\mathbf{e}_3$	$\mathbf{e}_{11}$	0
$\ker(A^4)$	10	$\mathbf{e}_4$	$\mathbf{e}_{12}$	0
$\ker(A^5)$	11	$\mathbf{e}_5$		0
$\ker(A^6)$	12	$\mathbf{e}_6$		0
$\ker(A^7)$	13	$\mathbf{e}_7$		0
$\ker(A^8)$	14	$\mathbf{e}_8$		0

The last number 1 in the last column indicates that there is still a chain to be found of exact length 2.

We can see from the matrix A that we have the following map.

$$A\,\mathbf{e}_{14} = \mathbf{e}_{13}.$$

Because $A^2\,\mathbf{e}_{14} = \mathbf{0}$, we conclude that we have found a third Jordan chain. It has length 2.

$$\{\mathbf{e}_{13} = A\,\mathbf{e}_{14}, \mathbf{e}_{14}\}.$$

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	chain 3	remaining dim
$\ker(A)$	3	$\mathbf{e}_1$	$\mathbf{e}_9$	$\mathbf{e}_{13}$	0
$\ker(A^2)$	6	$\mathbf{e}_2$	$\mathbf{e}_{10}$	$\mathbf{e}_{14}$	0
$\ker(A^3)$	8	$\mathbf{e}_3$	$\mathbf{e}_{11}$		0
$\ker(A^4)$	10	$\mathbf{e}_4$	$\mathbf{e}_{12}$		0
$\ker(A^5)$	11	$\mathbf{e}_5$			0
$\ker(A^6)$	12	$\mathbf{e}_6$			0
$\ker(A^7)$	13	$\mathbf{e}_7$			0
$\ker(A^8)$	14	$\mathbf{e}_8$			0

Part 3
General matrices

We stated before that the large block Jordan matrices associated with the eigenvalue $\lambda = 0$ were essential in the investigation of the general case. The general case are the Jordan matrices with large blocks associated with arbitrary eigenvalues. In the previous chapter, we only considered large blocks associated with eigenvalue $\lambda = 0$.

Let us take a look at some examples. We will see that these exercises can be reduced to the type of exercises we have handled before in part two. We will however to take care of some more bookkeeping.

21 example. (4×4); $(\mathbf{J_2}(\lambda = 3), \mathbf{J_2}(\lambda = 2))$.

We will make in this example some remarks of procedures that have to dealt with which were not used to when working with matrices that have only zero eigenvalues as in part 2. Let us take a look at the matrix A that is already in Jordan normal Form.

$$A = \begin{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix}.$$

This matrix consists of two elementary Jordan blocks. The first is a block with eigenvalue $\lambda = 3$ and the second is a block with eigenvalue $\lambda = 2$.

It is crucial is to remark that vectors of the subspace $\text{span}\{\mathbf{e}_1, \mathbf{e}_2\}$ are mapped into the same space. We say that the subspace is invariant with respect to the endomorphism A.

The subspace $\text{span}\{\mathbf{e}_3, \mathbf{e}_4\}$ is also mapped into itself.

Let us see what happens to the powers of this matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix};$$

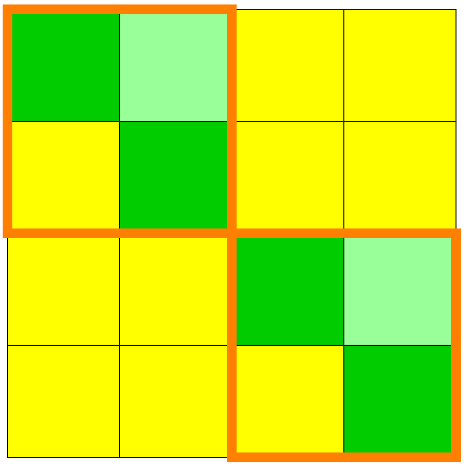
$$A^2 = \begin{pmatrix} \begin{pmatrix} 9 & 6 \\ 0 & 9 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix} \end{pmatrix};$$

$$A^3 = \begin{pmatrix} \begin{pmatrix} 27 & 27 \\ 0 & 27 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 8 & 12 \\ 0 & 8 \end{pmatrix} \end{pmatrix};$$

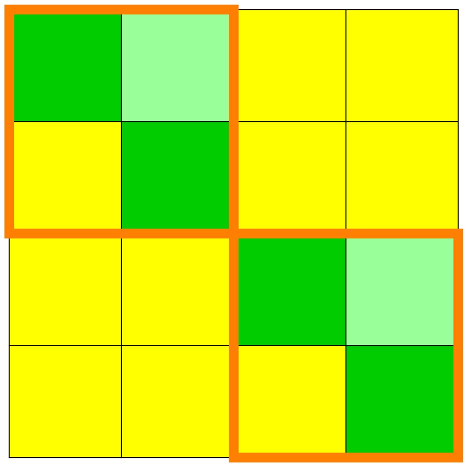
$$A^4 = \begin{pmatrix} \begin{pmatrix} 81 & 108 \\ 0 & 81 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 16 & 32 \\ 0 & 16 \end{pmatrix} \end{pmatrix}.$$

Let us compare this behaviour with the behaviour of the powers of the matrices that were handled in the previous part 2. In part 2 all the eigenvalues were zero. In contrast they are not zero in this example. We do not have in this case that the superdiagonals of the Jordan blocks go to the top and slowly disappear when taking higher powers. We have on the contrary that the diagonals of the elementary blocks associated with a nonzero λ do not change into zero's.

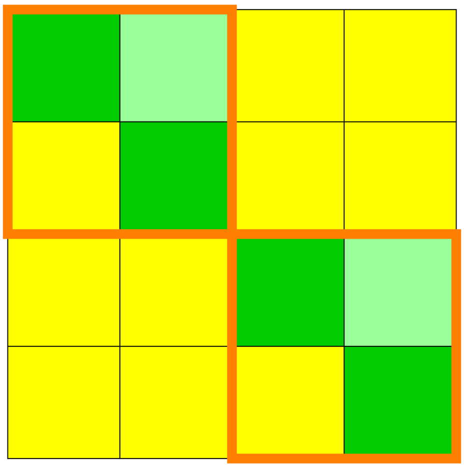
Let us visualise this situation. We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The exact value of the numbers in the green cells are of no importance to us in this context. They are however very important in the applications and there are very nice formulas for these numbers. The yellow cells are representing the number zero and the red cells represent the number 1 as before in part two.



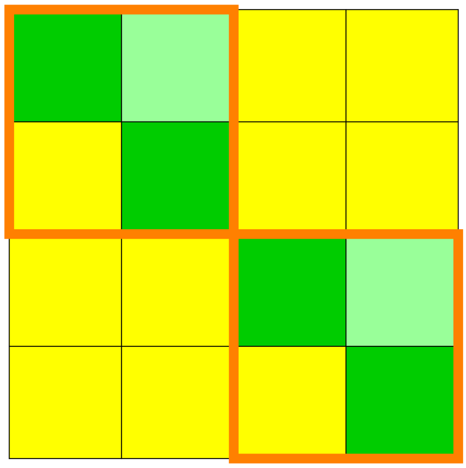
(a) Visualisation of A.



(b) Visualisation of A^2.



(c) Visualisation of A^3.



(d) Visualisation of A^4.

Figure 23

We give here some comments about this figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix.

The situation changes drastically by using the following trick. We take the first eigenvalue $\lambda = 3$ and subtract from the matrix A the matrix λI_4 where I_4 is the notation for the 4×4 identity matrix.

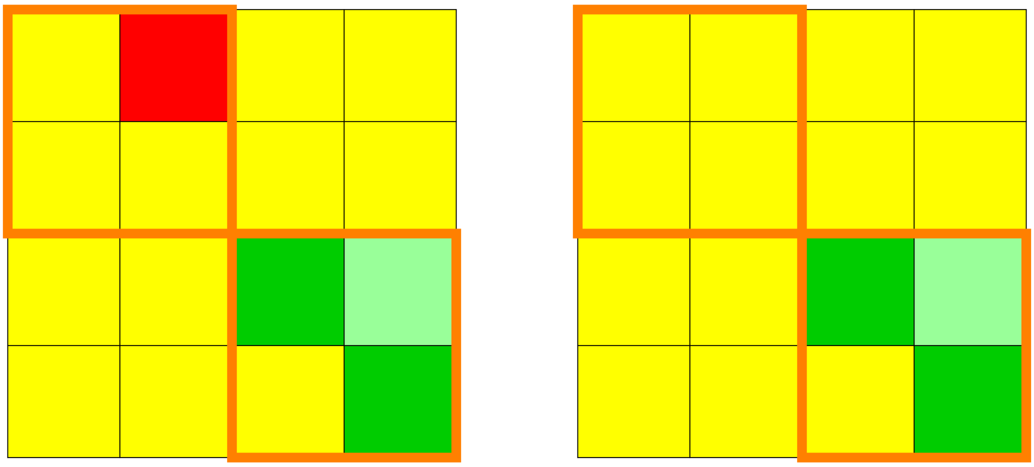
$$A - \lambda I_4 = \begin{pmatrix} \begin{pmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{2} \end{pmatrix} \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{-1} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} \end{pmatrix}.$$

We see here that the top left submatrix in the colour green is a matrix that we immediately recognise from the previous chapter with a diagonal consisting entirely out of 0's and a superdiagonal consisting of 1's.

We visualise now this new matrix $A - 3 I_4$ and its second power $(A - \lambda I_4)^2$. We observe that

$$(A - \lambda I_4)^2 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \end{pmatrix}.$$



(a) Visualisation of $(A - 3\,I_4)$.

(b) Visualisation of $(A - 3\,I_4)^2$.

Figure 24

We give here some comments about the preceding figure.

We visualise this matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells represent the number 1. We see in the first elementary matrix at the top left the same change as in the preceding chapter. The superdiagonal is going to the top in this block to disappear at the second power of $A - 3\,I_4$. The top left matrix block is recognised as one of the type we studied in the previous part. It is a nilpotent matrix block.

An analogous procedure happens with the submatrix at the bottom right. We repeat this completely.

The situation changes drastically if we take the second eigenvalue $\lambda = 2$ and subtract from the matrix A the matrix $\lambda\,I_4$ where I_4 stands for the 4×4 identity matrix.

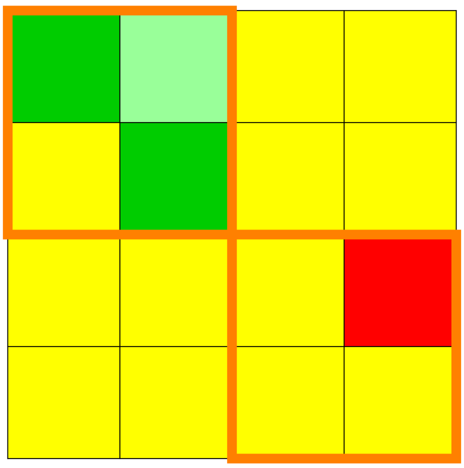
$$A - \lambda I_4 = \begin{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

We observe that the bottom right block in the colour red is a matrix that we recognise from the previous part with a diagonal consisting entirely out of 0's and a superdiagonal consisting of 1's.

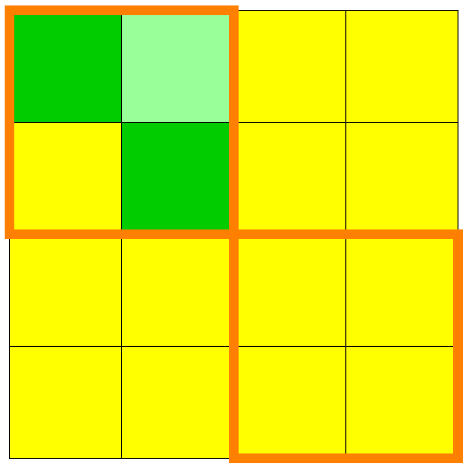
We observe also that

$$(A - \lambda I_4)^2 = \begin{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

We visualise now this new matrix $A - 2I_4$ and its second power $(A - 2I_4)^2$.



(a) Visualisation of $(A - 2I_4)$.



(b) Visualisation of $(A - 2I_4)^2$.

Figure 25

We give some comments about this figure. We visualise this matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells represent the number 1. The non zero diagonal in the top left submatrix is right there to stay nonzero even when taking higher powers.
We see however in the elementary submatrix right bottom the same behaviour as in the preceding part on nilpotent matrices.The superdiagonal in the right bottom submatrix is going to the top in the submatrix to disappear at taking the second power of $A - 2\,I_4$.

The conclusion we draw from this analysis is that $A - 3\,I_4$ is a nilpotent matrix when we restrict the matrix to operate on the space span$\{\mathbf{e}_1, \mathbf{e}_2\}$ and that $A - 2\,I_4$ is a nilpotent matrix when we restrict the matrix to operate on the space span$\{\mathbf{e}_3, \mathbf{e}_4\}$. So one could exploit this and the machinery we learned in the previous exercises in the previous part on nilpotent matrices to build a solution strategy for the matrix A.

We work first with the eigenvalue $\lambda = 3$. We remark that we have for the matrix A

$$\begin{cases} A\,\mathbf{e}_1 = 3\,\mathbf{e}_1, \\ A\,\mathbf{e}_2 = 3\,\mathbf{e}_2 + \mathbf{e}_1. \end{cases}$$

This is equivalent with the following facts for the matrix $A - 3\,I_4$

$$\begin{cases} (A - 3\,I_4)\,\mathbf{e}_1 = \mathbf{0}, \\ (A - 3\,I_4)\,\mathbf{e}_2 = \mathbf{e}_1. \end{cases}$$

So we have the Jordan chain for the matrix $A - 3\,I_4$.

$$\mathbf{e}_1 = \{(A - 3\,I_4)\,\mathbf{e}_2, \mathbf{e}_2\}.$$

We work successively with the eigenvalue $\lambda = 2$. We remark that we have for the matrix A

$$\begin{cases} A\,\mathbf{e}_3 = 2\,\mathbf{e}_3, \\ A\,\mathbf{e}_4 = 2\,\mathbf{e}_4 + \mathbf{e}_3. \end{cases}$$

This is equivalent with the following facts for the matrix $A - 2\,I_4$

$$\begin{cases} (A - 2\,I_4)\,\mathbf{e}_3 = \mathbf{0}, \\ (A - 2\,I_4)\,\mathbf{e}_4 = \mathbf{e}_3. \end{cases}$$

So we have the Jordan chain for the matrix $A - 2\,I_4$.

$$\{(A - 2\,I_4)\,\mathbf{e}_4, \mathbf{e}_4\}.$$

22 example. (12×12); $(J_3(4), J_3(-3), J_2(-3), J_4(7))$.

Let us start with the matrix A.

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}.$$

We see here a matrix consisting of elementary Jordan blocks and no diagonal element is 0. In order to apply the theory of the preceding chapter, we will work again with adapted versions of this matrix A.

1. Investigation of the first eigenvalue.

Let us start with the first elementary Jordan block top left associated with the eigenvalue $\lambda_1 = 4$. We construct the matrix $A - \lambda_1\,I_{12} = A - 4\,I_{12}$. This forces the elementary submatrix top left to be a nilpotent matrix and we know a method to deal with that case.

$$A-\lambda_1 I_{12} = \begin{pmatrix} \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 7 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix} \end{pmatrix}$$

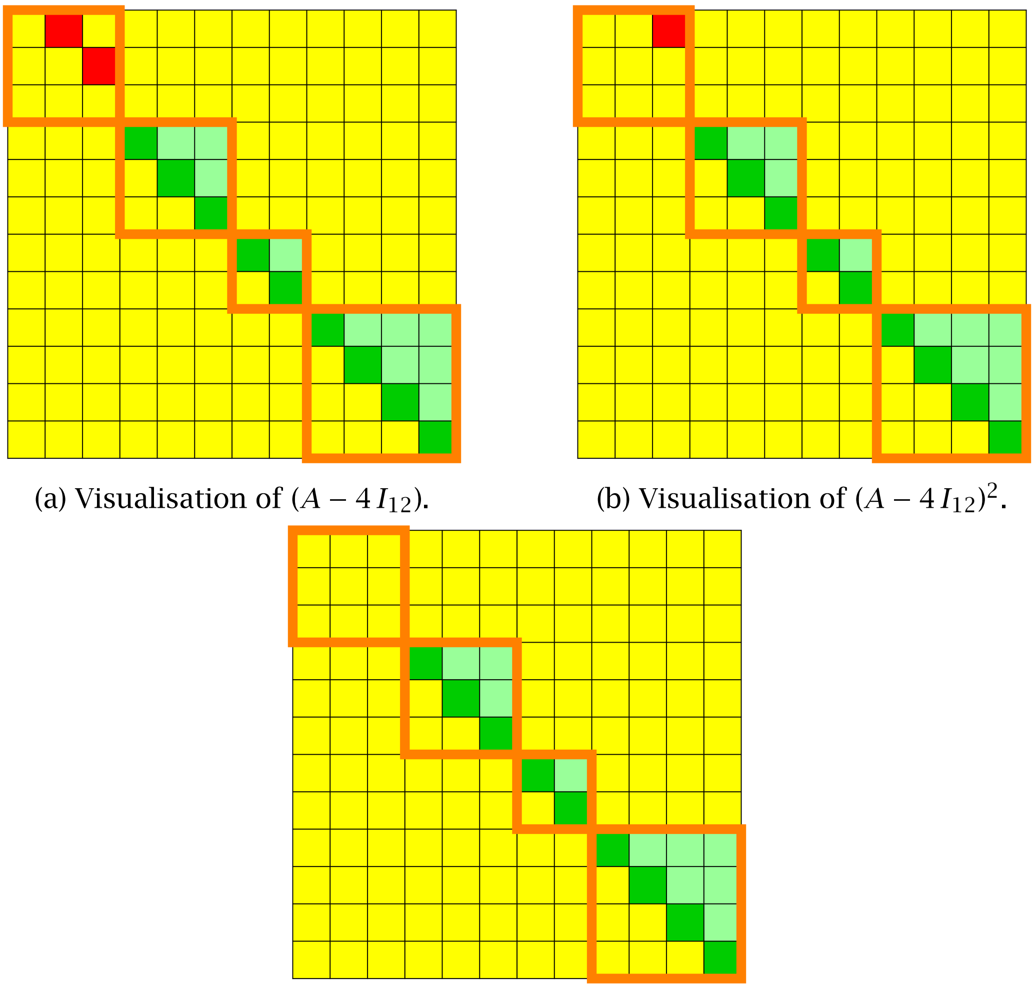
$$-4 \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} -7 & 1 & 0 \\ 0 & -7 & 1 \\ 0 & 0 & -7 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} -7 & 1 \\ 0 & -7 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \end{pmatrix}.$$

We see that the matrices A and $A - \lambda I_{12}$ are invariant for the subspace $\text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

We immediately observe that the top left elementary block is a matrix that we immediately recognise from the previous chapter with a diagonal consisting entirely out of 0's and a superdiagonal consisting of 1's. We force that top left submatrix to be a singular Jordan elementary block by applying this trick.

We visualise now this new matrix $A - 4I_{12}$ and its powers $(A - 4I_{12})^i$.



(a) Visualisation of $(A - 4I_{12})$.

(b) Visualisation of $(A - 4I_{12})^2$.

(c) Visualisation of $(A - 4I_{12})^3$.

Figure 26

We give here some comments about the preceding figure. We visualise this matrix $A - 4\,I_{12}$ and its powers by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells represent the number 1. We see in the elementary matrix left above the same change as in the preceding chapter. The superdiagonal is going to the top in the block to disappear at the third power of $A - 4\,I_{12}$. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix.

We see how the kernels of the matrices $(A - 4\,I_{12})^i$ grow. We have

$$\begin{cases} \ker(A - 4\,I_{12}) = \operatorname{span}\{\mathbf{e}_1\}, \\ \ker(A - 4\,I_{12})^2 = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}, \\ \ker(A - 4\,I_{12})^3 = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}. \end{cases}$$

We can see the following equalities from the matrix A.

$$\begin{cases} A\,\mathbf{e}_1 = 4\,\mathbf{e}_1, \\ A\,\mathbf{e}_2 = 4\,\mathbf{e}_2 + \mathbf{e}_1, \\ A\,\mathbf{e}_3 = 4\,\mathbf{e}_3 + \mathbf{e}_2, \end{cases}$$

We conclude and can also visually see that we have

$$\begin{cases} (A - 4\,I_{12})\,\mathbf{e}_1 = \mathbf{0}, \\ (A - 4\,I_{12})\,\mathbf{e}_2 = \mathbf{e}_1, \\ (A - 4\,I_{12})\,\mathbf{e}_3 = \mathbf{e}_2. \end{cases}$$

or

$$\begin{cases} (A - 4\,I_{12})^3\,\mathbf{e}_3 = \mathbf{0}, \\ (A - 4\,I_{12})^2\,\mathbf{e}_3 = \mathbf{e}_1, \\ \ (A - 4\,I_{12})\,\mathbf{e}_3 = \mathbf{e}_2. \end{cases}$$

So we have the Jordan chain for the matrix $A - 4\,I_{12}$.

$$\{\mathbf{e}_1 = (A - 4\,I_{12})^2\,\mathbf{e}_3, \mathbf{e}_2 = (A - 4\,I_{12})\,\mathbf{e}_3, \mathbf{e}_3\}.$$

2. Investigation of the second eigenvalue.

We see that the matrix A is invariant on the subspace

$$\text{span}\{\mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8\}.$$

Let us take a look at the second and third elementary Jordan blocks in the middle of the matrix A associated with the eigenvalue $\lambda_2 = -3$. We construct the matrix $A - \lambda_2 I_{12} = A + 3 I_{12}$. This forces the two submatrices in the middle of A to become nilpotent matrices on the subspace just mentioned and we are used to work and to deal with these nilpotent matrices.

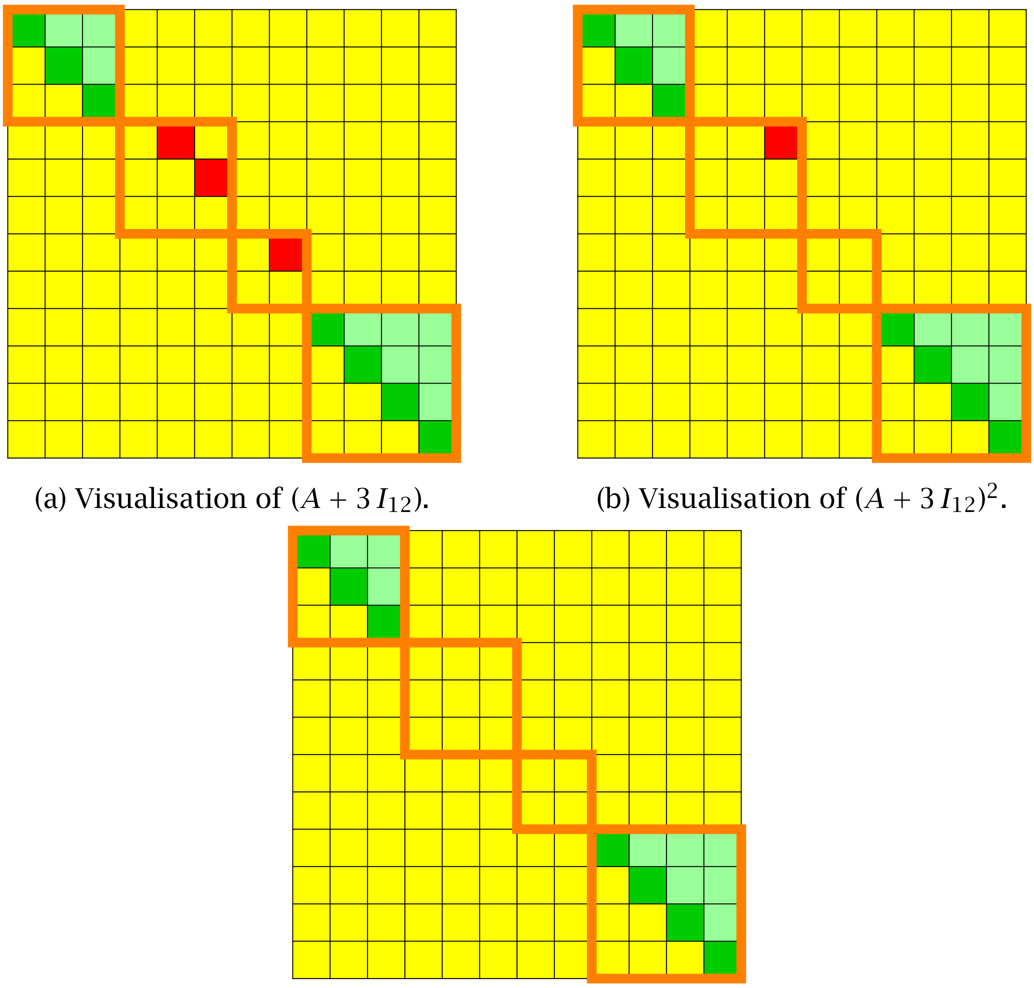
$$A-\lambda_2 I_{12}=\begin{pmatrix}
4&1&0&0&0&0&0&0&0&0&0&0\\
0&4&1&0&0&0&0&0&0&0&0&0\\
0&0&4&0&0&0&0&0&0&0&0&0\\
0&0&0&-3&1&0&0&0&0&0&0&0\\
0&0&0&0&-3&1&0&0&0&0&0&0\\
0&0&0&0&0&-3&0&0&0&0&0&0\\
0&0&0&0&0&0&-3&1&0&0&0&0\\
0&0&0&0&0&0&0&-3&0&0&0&0\\
0&0&0&0&0&0&0&0&7&1&0&0\\
0&0&0&0&0&0&0&0&0&7&1&0\\
0&0&0&0&0&0&0&0&0&0&7&1\\
0&0&0&0&0&0&0&0&0&0&0&7
\end{pmatrix}$$

$$+3\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}$$

$$=\begin{pmatrix}
7&1&0&0&0&0&0&0&0&0&0&0\\
0&7&1&0&0&0&0&0&0&0&0&0\\
0&0&7&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&10&1&0&0\\
0&0&0&0&0&0&0&0&0&10&1&0\\
0&0&0&0&0&0&0&0&0&0&10&1\\
0&0&0&0&0&0&0&0&0&0&0&10
\end{pmatrix}.$$

We immediately observe that the middle blocks are blocks that we immediately recognise from the previous chapter with a diagonal consisting entirely out of 0's and a superdiagonal consisting of 1's. They form together a large Jordan block associated to the eigenvalue $\lambda_2 = -3$.

We visualise now this new matrix $A + 3\,I_{12}$ and its powers $(A + 3\,I_{12})^i$.



(a) Visualisation of $(A + 3\,I_{12})$.

(b) Visualisation of $(A + 3\,I_{12})^2$.

(c) Visualisation of $(A + 3\,I_{12})^3$.

Figure 27

We give here some comments about the preceding figure.

We visualise this matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells represent the number 1. We see in the elementary Jordan blocks in the middle the same change as in the preceding chapter. The super-diagonals are going to the top in their respective blocks to disappear at the third power of $A + 3\,I_{12}$. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix.

We see how the kernels of the matrices $(A + 3\,I_{12})^i$ grow. We have

$$\begin{aligned}\ker(A + 3\,I_{12}) &= \operatorname{span}\{\mathbf{e}_4, \mathbf{e}_7\},\\ \ker(A + 3\,I_{12})^2 &= \operatorname{span}\{\mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_7, \mathbf{e}_8\},\\ \ker(A + 3\,I_{12})^3 &= \operatorname{span}\{\mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8\}.\end{aligned}$$

We can see the following equalities from the matrix A.

$$\begin{cases} A\,\mathbf{e}_4 = -3\,\mathbf{e}_4,\\ A\,\mathbf{e}_5 = -3\,\mathbf{e}_5 + \mathbf{e}_4,\\ A\,\mathbf{e}_6 = -3\,\mathbf{e}_6 + \mathbf{e}_5.\end{cases}$$

We have also

$$\begin{cases} A\,\mathbf{e}_7 = -3\,\mathbf{e}_7,\\ A\,\mathbf{e}_8 = -3\,\mathbf{e}_8 + \mathbf{e}_7.\end{cases}$$

We conclude and can also visually see that the matrix $A + 3\,I_{12}$ satisfies

$$\begin{cases} (A + 3\,I_{12})\,\mathbf{e}_4 = \mathbf{0},\\ (A + 3\,I_{12})\,\mathbf{e}_5 = \mathbf{e}_4,\\ (A + 3\,I_{12})\,\mathbf{e}_6 = \mathbf{e}_5\end{cases}$$

or

$$\begin{cases} (A + 3\,I_{12})^3\,\mathbf{e}_6 = \mathbf{0},\\ (A + 3\,I_{12})^2\,\mathbf{e}_6 = \mathbf{e}_4,\\ (A + 3\,I_{12})\,\mathbf{e}_6 = \mathbf{e}_5\end{cases}$$

and

$$\begin{cases} (A + 3\,I_{12})\,\mathbf{e}_7 = \mathbf{0},\\ (A + 3\,I_{12})\,\mathbf{e}_8 = \mathbf{e}_7\end{cases}$$

or

$$\begin{cases}(A + 3\,I_{12})^2\,\mathbf{e_8} = \mathbf{0},\\(A + 3\,I_{12})\,\mathbf{e_8} = \mathbf{e_7}.\end{cases}$$

So we have two Jordan chains for the matrix $A + 3\,I_{12}$.

$$\{\mathbf{e_4} = (A + 3\,I_{12})^2\,\mathbf{e_6}, \mathbf{e_5} = (A + 3\,I_{12})\,\mathbf{e_6}, \mathbf{e_6}\}$$

and

$$\{\mathbf{e_7} = (A + 3\,I_{12})\,\mathbf{e_8}, \mathbf{e_8}\}.$$

3. Investigation of the third eigenvalue.

We see that the matrix A is invariant on the subspace

$$\text{span}\{\mathbf{e_9}, \mathbf{e_{10}}, \mathbf{e_{11}}, \mathbf{e_{12}}\}.$$

Let us take a look to the bottom right elementary Jordan block in the matrix A associated with the eigenvalue $\lambda_3 = 7$. We construct the matrix $A - \lambda_3\,I_{12} = A - 7\,I_{12}$. This forces that submatrix at the right bottom in A to become a nilpotent matrix on the subspace $\text{span}\{\mathbf{e_9}, \mathbf{e_{10}}, \mathbf{e_{11}}, \mathbf{e_{12}}\}$ and we are used to work with these nilpotent matrices in the preceding chapter.

$A - \lambda_2 I_{12}$

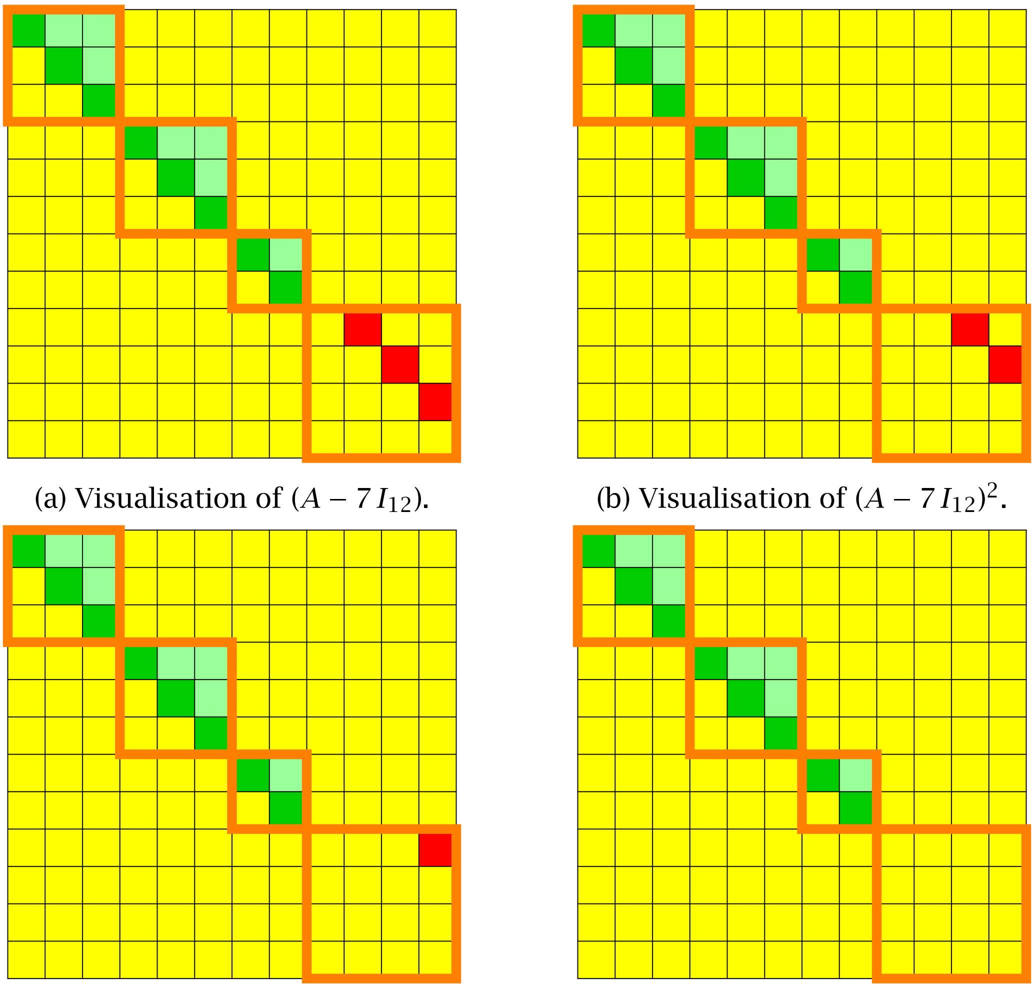
$$= \begin{pmatrix}
4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7
\end{pmatrix}$$

$$- 7 \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$= \begin{pmatrix}
-3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -10 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

We immediately observe that the bottom right elementary Jordan submatrix is a matrix that we recognise from the previous chapter to have a diagonal consisting entirely out of 0's and a superdiagonal consisting of 1's. This was an elementary Jordan block associated with the eigenvalue $\lambda_3 = 7$. It has now only 0's in its diagonal.

We visualise now this new matrix $A - 7I_{12}$ and its powers $(A - 7I_{12})^i$.



(a) Visualisation of $(A - 7I_{12})$.

(b) Visualisation of $(A - 7I_{12})^2$.

(c) Visualisation of $(A - 7I_{12})^3$.

(d) Visualisation of $(A - 7I_{12})^4$.

Figure 28

We give here some comments about the preceding figure.

We visualise this matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells represent the number 1. We see in the elementary matrix at the right bottom the same dynamic change as in the preceding part 2 on nilpotent matrices. The submatrix at the bottom right is indeed a nilpotent elementary Jordan block. The superdiagonal of this submatrix is going to the top in the submatrix to disappear at the fourth power of $A - 7I_{12}$. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix.

We see how the kernels of the matrices $(A - 7I_{12})^i$ grow. We have

$$\begin{cases} \ker(A - 7I_{12}) = \operatorname{span}\{\mathbf{e}_9\}, \\ \ker(A - 7I_{12})^2 = \operatorname{span}\{\mathbf{e}_9, \mathbf{e}_{10}\}, \\ \ker(A - 7I_{12})^3 = \operatorname{span}\{\mathbf{e}_9, \mathbf{e}_{10}, \mathbf{e}_{11}\}, \\ \ker(A - 7I_{12})^4 = \operatorname{span}\{\mathbf{e}_9, \mathbf{e}_{10}, \mathbf{e}_{11}, \mathbf{e}_{12}\}. \end{cases}$$

We can see the following equalities from the matrix A.

$$\begin{cases} A\,\mathbf{e}_9 = 7\,\mathbf{e}_9, \\ A\,\mathbf{e}_{10} = 7\,\mathbf{e}_{10} + \mathbf{e}_9, \\ A\,\mathbf{e}_{11} = 7\,\mathbf{e}_{11} + \mathbf{e}_{10}, \\ A\,\mathbf{e}_{12} = 7\,\mathbf{e}_{12} + \mathbf{e}_{11}. \end{cases}$$

We conclude and can also visually see that the matrix $A - 7I_{12}$ satisfies

$$\begin{cases} (A - 7I_{12})\,\mathbf{e}_9 = \mathbf{0}, \\ (A - 7I_{12})\,\mathbf{e}_{10} = \mathbf{e}_9, \\ (A - 7I_{12})\,\mathbf{e}_{11} = \mathbf{e}_{10}, \\ (A - 7I_{12})\,\mathbf{e}_{12} = \mathbf{e}_{11}, \end{cases}$$

We have also

$$\begin{cases} (A - 7I_{12})^4\,\mathbf{e}_{12} = \mathbf{0}, \\ (A - 7I_{12})^3\,\mathbf{e}_{12} = \mathbf{e}_9, \\ (A - 7I_{12})^2\,\mathbf{e}_{12} = \mathbf{e}_{10}, \\ (A - 7I_{12})\,\mathbf{e}_{12} = \mathbf{e}_{11}. \end{cases}$$

So we have a Jordan chain for the matrix $A - 7I_{12}$.

$$\{\mathbf{e}_9 = (A - 7I_{12})^3\,\mathbf{e}_{12}, \mathbf{e}_{10} = (A - 7I_{12})^2\,\mathbf{e}_{12}, \mathbf{e}_{11} = (A - 7I_{12})\,\mathbf{e}_{12}, \mathbf{e}_{12}\}.$$

23 exercise. (7×7); $(\mathbf{J}_2(1), \mathbf{J}_2(1), \mathbf{J}_3(-1))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B so that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 0 & 2 & 0 & -2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 4 & 3 & -1 & 0 \\ 1 & 0 & 0 & -4 & -4 & 2 & 0 \\ -1 & 1 & 1 & -3 & -2 & 0 & 1 \\ -1 & 1 & -2 & -2 & -1 & 0 & -1 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_7| = -(\lambda - 1)^4 (\lambda + 1)^3.$$

We see that the characteristic Cayley-Hamilton polynomial $p_{\text{C-H}}(\lambda)$ can be factorised in linear polynomials over the field $\mathbf{K}$. So we can apply the Jordan normalisation machinery.

The eigenvalue $\lambda = 1$ has algebraic multiplicity 4. The eigenvalue $\lambda = -1$ has algebraic multiplicity 3.

2. Investigation of the first eigenvalue.

We work now with the eigenvalue $\lambda = 1$.

3. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 3 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 2. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_7 with the first eigenvalue $\lambda = 1$.

$$A - \lambda I_7 = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix} \end{pmatrix}.$$

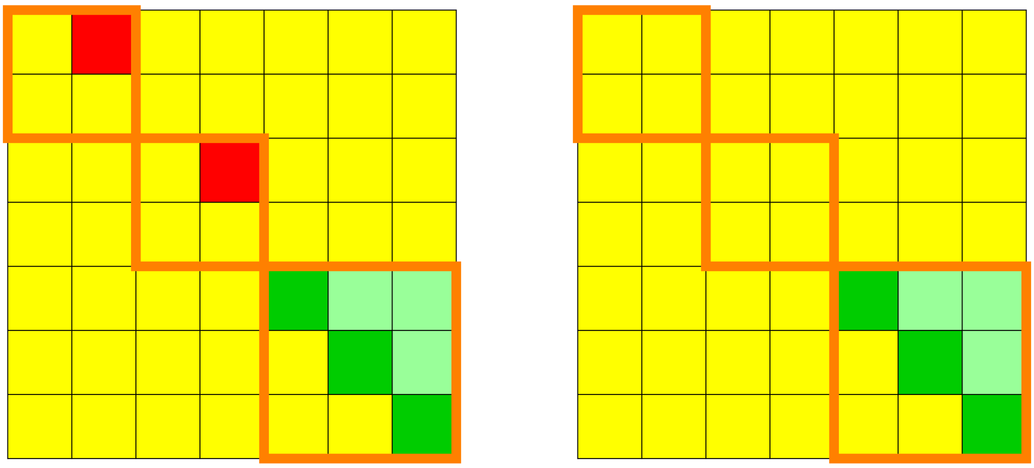
We want to investigate the endomorphism $A - \lambda I_7$ restricted on the invariant subspace $\operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. We see that $A - \lambda I_7$ is a nilpotent operator on this subspace. We have the classic case of a nilpotent operator on a finite dimensional space.

We compute also the powers of $A - \lambda I_7$, $\lambda = 1$.

$$A - \lambda I_7 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix},$$

$$(A - \lambda I_7)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $A - \lambda I_7$. (b) Visualisation of $(A - \lambda I_7)^2$.

Figure 29

We give some comments about the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero.

All higher powers starting from 2 of this matrix have the same look. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix. We see also that the space $\text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is invariant with respect to $A - \lambda I_7$ with height of nilpotency equal to 2.

Let us take a look at the first and second elementary Jordan blocks at the left in this matrix.

We look now at the superdiagonals of the blocks of $A - \lambda I_7$ associated with nilpotent elementary Jordan blocks. They are coloured in red. We observe that these superdiagonals consisting of 1's in the matrix $A - \lambda I_7$ are going upwards in their respective elementary Jordan blocks when increasing the powers of the matrix $A - \lambda I_7$ until they ultimately disappear when taking the second power of $A - \lambda I_7$.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_7) = \text{span}\{\mathbf{e}_1, \mathbf{e}_3\}; \\ \ker(A - \lambda I_7)^2 = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_7)$	2	$2 = \dim(\ker(A - \lambda I_7))$
$\ker(A - \lambda I_7)^2$	4	$2 = \dim(\ker(A - \lambda I_7)^2) - \dim(\ker(A - \lambda I_7))$

We see here a table for the invariant subspace $\text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ with respect to its nilpotent operator $\ker(A - \lambda I_7)$ restricted to this space. We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A - \lambda I_7)^i) - \dim(\ker(A - \lambda I_7)^{i-1})$. In the first row, we have $\dim(\ker(A - \lambda I_7))$.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A - \lambda I_7) \subsetneq \ker(A - \lambda I_7)^2 = \ker(A - \lambda I_7)^3 = \cdots .$$

We remark by looking at the matrices $(A - \lambda I_7)^i$ that we have the following mappings

$$\begin{cases} (A - \lambda I_7)\,\mathbf{e}_2 = \mathbf{e}_1, \\ (A - \lambda I_7)\,\mathbf{e}_1 = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A - \lambda I_7)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases} (A - \lambda I_7)\,\mathbf{e}_4 = \mathbf{e}_3, \\ (A - \lambda I_7)\,\mathbf{e}_3 = \mathbf{0}. \end{cases}$$

One sees that we have two Jordan chains of linearly independent vectors. We write a Jordan chain in reverse order. The first chain is

$$\{\mathbf{e}_1 = (A - \lambda I_7)\,\mathbf{e}_2, \mathbf{e}_2\}.$$

The second chain is

$$\{\mathbf{e}_3 = (A - \lambda I_7)\,\mathbf{e}_4, \mathbf{e}_4\}.$$

After we have found the first and second Jordan chains of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$(A - \lambda I_7)$	2	$\mathbf{e}_1$	$\mathbf{e}_3$	0
$(A - \lambda I_7)^2$	4	$\mathbf{e}_2$	$\mathbf{e}_4$	0

We have now in the last column only 0's and this means that we are done looking for Jordan chains.

4. Kernels of $(\mathbf{B} - \lambda\,\mathbf{I}_7)^{\mathbf{i}}$.

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I}_7)$.
The matrix $B - \lambda I_7$ is

$$B - \lambda I_7 = \begin{pmatrix} -1 & 2 & 0 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 3 & 3 & -1 & 0 \\ 1 & 0 & 0 & -4 & -5 & 2 & 0 \\ -1 & 1 & 1 & -3 & -2 & -1 & 1 \\ -1 & 1 & -2 & -2 & -1 & 0 & -2 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -1 & 2 & 0 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 3 & 3 & -1 & 0 \\ 1 & 0 & 0 & -4 & -5 & 2 & 0 \\ -1 & 1 & 1 & -3 & -2 & -1 & 1 \\ -1 & 1 & -2 & -2 & -1 & 0 & -2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -z_1 + 2z_2 - 2z_4 - z_5 = 0, \\ z_1 - z_2 + 2z_4 + z_5 = 0, \\ -z_2 + 3z_4 + 3z_5 - z_6 = 0, \\ z_1 - 4z_4 - 5z_5 + 2z_6 = 0, \\ -z_1 + z_2 + z_3 - 3z_4 - 2z_5 - z_6 + z_7 = 0, \\ -z_1 + z_2 - 2z_3 - 2z_4 - z_5 - 2z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_7) &= \{(r_1, 0, r_3, -r_1, r_1, 0, -r_3) \mid r_1, r_3 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,-1,1,0,0), (0,0,1,0,0,0,-1)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I_7})^2$.
We remember that $\lambda = 1$.

We calculate the kernel of $(B - \lambda I_7)^2$. The matrix $(B - \lambda I_7)^2$ is

$$(B-\lambda I_7)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & -4 & -1 & 0 & -4 & 4 & -1 \\ -8 & 8 & 2 & 0 & 8 & -8 & 2 \\ 0 & 0 & -3 & 4 & 4 & 0 & -3 \\ 0 & 0 & 4 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & -4 & -1 & 0 & -4 & 4 & -1 \\ -8 & 8 & 2 & 0 & 8 & -8 & 2 \\ 0 & 0 & -3 & 4 & 4 & 0 & -3 \\ 0 & 0 & 4 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 4z_1 - 4z_2 - z_3 - 4z_5 + 4z_6 - z_7 = 0, \\ -8z_1 + 8z_2 + 2z_3 + 8z_5 - 8z_6 + 2z_7 = 0, \\ -3z_3 + 4z_4 + 4z_5 - 3z_7 = 0, \\ 4z_3 + 4z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B-\lambda I_7)^2 &= \{(r_1, r_2, r_3, r_4, -r_4, -r_1 + r_2 - r_4, -r_3) \mid r_1, r_2, r_3, r_4 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,0,-1,0), (0,1,0,0,0,1,0), \\ &\qquad (0,0,1,0,0,0,-1), (0,0,0,1,-1,-1,0)\}. \end{aligned}$$

Kernel of $(\mathbf{B}-\lambda\,\mathbf{I}_7)^3$.
We remember that $\lambda = 1$.

We calculate the kernel of $(B-\lambda I_7)^3$. The matrix $(B-\lambda I_7)^3$ is

$$(B-\lambda I_7)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & 12 & 6 & -4 & 8 & -12 & 6 \\ 24 & -24 & -12 & 8 & -16 & 24 & -12 \\ 4 & -4 & 6 & -4 & -8 & 4 & 6 \\ 0 & 0 & -8 & 0 & 0 & 0 & -8 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & 12 & 6 & -4 & 8 & -12 & 6 \\ 24 & -24 & -12 & 8 & -16 & 24 & -12 \\ 4 & -4 & 6 & -4 & -8 & 4 & 6 \\ 0 & 0 & -8 & 0 & 0 & 0 & -8 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -12z_1 + 12z_2 + 6z_3 - 4z_4 + 8z_5 - 12z_6 + 6z_7 = 0, \\ 24z_1 - 24z_2 - 12z_3 + 8z_4 - 16z_5 + 24z_6 - 12z_7 = 0, \\ 4z_1 - 4z_2 + 6z_3 - 4z_4 - 8z_5 + 4z_6 + 6z_7 = 0, \\ -8z_3 - 8z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B-\lambda I_7)^3 &= \{(r_1,r_2,r_3,r_4,-r_4,-r_1+r_2-r_4,-r_3) \mid r_1,r_2,r_3,r_4 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,0,-1,0),(0,0,1,0,0,0,-1), \\ &\qquad (0,1,0,0,0,1,0),(0,0,0,1,-1,-1,0)\}. \end{aligned}$$

We see here that the inclusion of the kernels stabilises here. The kernels of $(B-\lambda I_7)^2$ and $(B-\lambda I_7)^3$ are equal.

Stabilisation of the kernels.
We remarked that the inclusion of the kernels is starting to stabilise from the second power onwards. We mean by this that we are having equality from the second power onwards in the following chain of inclusion of sets.

$$\ker(B-\lambda I_7) \subsetneq \ker((B-\lambda I_7)^2) = \ker((B-\lambda I_7)^3) = \cdots .$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B-\lambda I_7)$	2	$2 = \dim(\ker(B-\lambda I_7))$
$\ker(B-\lambda I_7)^2$	4	$2 = \dim(\ker(B-\lambda I_7)^2) - \dim(\ker(B-\lambda I_7))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B-\lambda I_7)^i)) - \dim(\ker((B-\lambda I_7)^{i-1}))$. The first number of this last column is $\dim(\ker(B-\lambda I_7))$.

5. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w}_1, \mathbf{w}_2\}$ satisfying

$$\begin{cases} (B-\lambda I_7)\,\mathbf{w}_1 = \mathbf{0}, \\ (B-\lambda I_7)\,\mathbf{w}_2 = \mathbf{w}_1, \end{cases}$$

where $\mathbf{w}_2$ is in the vector space $\ker((B-\lambda I_7)^2)$ but not in $\ker(B-\lambda I_7)$.

We look for a generating vector $\mathbf{w}_2$. This vector must be linearly independent of all vectors in $\ker(B-\lambda I_7)$ and must be independent from vectors of height 2 that were already chosen in $\ker(B-\lambda I_7)^2$. There is at this moment no vector of the second category. We know that

$$\ker((B-\lambda I_7)^2) = \operatorname{span}\{(1,0,0,0,0,-1,0),(0,1,0,0,0,1,0), (0,0,1,0,0,0,-1),(0,0,0,1,-1,-1,0)\}.$$

We remember that $\ker(B-\lambda I_7)$ is

$$\ker(B-\lambda I_7) = \operatorname{span}\{(1,0,0,-1,1,0,0),(0,0,1,0,0,0,-1)\}.$$

We know that a vector in $\ker(B-\lambda I_7)^2$ must be of the generic form

$$a\,(1,0,0,0,0,-1,0)+b\,(0,1,0,0,0,1,0)+c\,(0,0,1,0,0,0,-1)\\ +d\,(0,0,0,1,-1,-1,0)=(a,b,c,d,-d,-a+b-d,-c).$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H=\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & -1\\ a & b & c & d & -d & -a+b-d & -c \end{pmatrix}.$$

We row reduce this matrix H and if we impose $b\neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 1 & 0 & \frac{a+d}{b} & \frac{-a-d}{b} & \frac{-a+b-d}{b} & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

We have that these vectors are linearly independent if $b\neq 0$. We can choose $a=0$, $b=1$, $c=0$, $d=0$.

We can choose the generating vector

$$\mathbf{w}_2=(0,1,0,0,0,1,0).$$

We calculate $\mathbf{w}_1$.

$$\mathbf{w}_1=(B-\lambda I_7)\,\mathbf{w}_2\\ =\begin{pmatrix} -1 & 2 & 0 & -2 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & -1 & 0 & 2 & 1 & 0 & 0\\ 0 & -1 & 0 & 3 & 3 & -1 & 0\\ 1 & 0 & 0 & -4 & -5 & 2 & 0\\ -1 & 1 & 1 & -3 & -2 & -1 & 1\\ -1 & 1 & -2 & -2 & -1 & 0 & -2 \end{pmatrix}\begin{pmatrix}0\\1\\0\\0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}2\\0\\-1\\-2\\2\\0\\1\end{pmatrix}.$$

We have found a first Jordan chain. It has length 2.

$$\{\mathbf{w}_1=(2,0,-1,-2,2,0,1),\mathbf{w}_2=(0,1,0,0,0,1,0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B-\lambda I_7)$	2	$\mathbf{w_1}$	1
$\ker(B-\lambda I_7)^2$	4	$\mathbf{w_2}$	1

with

$$\mathbf{w_1} = (2,0,-1,-2,2,0,1)$$
$$\mathbf{w_2} = (0,1,0,0,0,1,0)$$

Calculation of the second Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w_3},\mathbf{w_4}\}$ satisfying

$$\begin{cases}(B-\lambda I_7)\mathbf{w_3} = \mathbf{0},\\ (B-\lambda I_7)\mathbf{w_4} = \mathbf{w_3},\end{cases}$$

where $\mathbf{w_4}$ is in the vector space $\ker(B-\lambda I_7)^2$ but not in $\ker(B-\lambda I_7)$.

We look for a generating vector $\mathbf{w_4}$. This vector must be linearly independent of all vectors in $\ker(B-\lambda I_7)$ and must be independent from vectors of height 2 that were already chosen in $\ker(B-\lambda I_7)^2$. We know that

$$\ker(B-\lambda I_7)^2 = \text{span}\{(1,0,0,0,0,-1,0),(0,1,0,0,0,1,0),\\ (0,0,1,0,0,0,-1),(0,0,0,1,-1,-1,0)\}.$$

We have at this point chosen in $\ker(B-\lambda I_7)^2$ already the vector $\mathbf{w_2} = (0,1,0,0,0,1,0)$ of height 2.

We remember that $\ker(B-\lambda I_7)$ equals

$$\ker(B-\lambda I_7) = \text{span}\{(1,0,0,-1,1,0,0),(0,0,1,0,0,0,-1)\}.$$

We know that a vector in $\ker(B-\lambda I_7)^2$ must be of the form

$$a\,(1,0,0,0,0,-1,0)+b\,(0,1,0,0,0,1,0)+c\,(0,0,1,0,0,0,-1)\\ +d\,(0,0,0,1,-1,-1,0)=(a,b,c,d,-d,-a+b-d,-c).$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H=\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & -1\\ 0 & 1 & 0 & 0 & 0 & 1 & 0\\ a & b & c & d & -d & -a+b-d & -c \end{pmatrix}.$$

We row reduce this matrix H and if we impose $a+d\neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 1 & -1 & -1 & 0 \end{pmatrix}.$$

We see that these vectors are independent if we choose $a=0$, $b=0$, $c=0$, $d=1$.

We have then the following valid generating vector

$$\mathbf{w}_4=(0,0,0,1,-1,-1,0).$$

We calculate

$$\mathbf{w}_3=(B-\lambda I_7)\,\mathbf{w}_4\\ =\begin{pmatrix} -1 & 2 & 0 & -2 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & -1 & 0 & 2 & 1 & 0 & 0\\ 0 & -1 & 0 & 3 & 3 & -1 & 0\\ 1 & 0 & 0 & -4 & -5 & 2 & 0\\ -1 & 1 & 1 & -3 & -2 & -1 & 1\\ -1 & 1 & -2 & -2 & -1 & 0 & -2 \end{pmatrix}\begin{pmatrix} 0\\ 0\\ 0\\ 1\\ -1\\ -1\\ 0 \end{pmatrix}=\begin{pmatrix} -1\\ 0\\ 1\\ 1\\ -1\\ 0\\ -1 \end{pmatrix}.$$

We have found a second Jordan chain. It has length 2.

$$\{\mathbf{w}_3 = (-1,0,1,1,-1,0,-1), \mathbf{w}_4 = (0,0,0,1,-1,-1,0)\}.$$

Let us take a look at our current information table. We remember that $\lambda = 1$.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(B - \lambda I_7)$	2	$\mathbf{w}_1$	$\mathbf{w}_3$	0
$\ker(B - \lambda I_7)^2$	4	$\mathbf{w}_2$	$\mathbf{w}_4$	0

with

$$\mathbf{w}_1 = (2,0,-1,-2,2,0,1)$$
$$\mathbf{w}_2 = (0,1,0,0,0,1,0)$$
$$\mathbf{w}_3 = (-1,0,1,1,-1,0,-1)$$
$$\mathbf{w}_4 = (0,0,0,1,-1,-1,0)$$

6. Investigation of the second eigenvalue.

We work now with the second eigenvalue $\lambda = -1$.

7. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 8 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 7. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_7 with $\lambda = -1$.

$$A - \lambda I_7 = \begin{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}.$$

We see that the third elementary Jordan block is now the matrix of a nilpotent operator. If we restrict the mapping $A - \lambda I_7$ to the subspace $\mathrm{span}\{\mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}$ which is an invariant subspace with respect to the operator $A - \lambda I_7$, then we have the classic case of a nilpotent operator on a finite dimensional space.

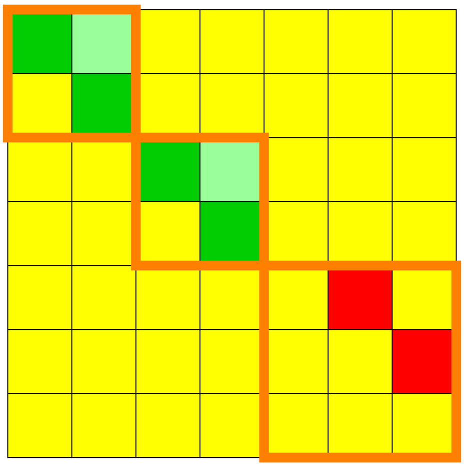
We compute also the powers of $A - \lambda I_7$. We remember that $\lambda = -1$.

$$A - \lambda I_7 = \begin{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix},$$

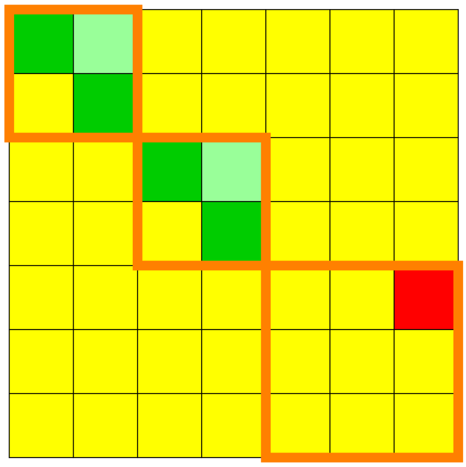
$$(A-\lambda I_7)^2 = \begin{pmatrix} \begin{pmatrix} \mathbf{4} & \mathbf{4} \\ \mathbf{0} & \mathbf{4} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{4} & \mathbf{4} \\ \mathbf{0} & \mathbf{4} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix},$$

$$(A-\lambda I_7)^3 = \begin{pmatrix} \begin{pmatrix} \mathbf{8} & \mathbf{12} \\ \mathbf{0} & \mathbf{8} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{8} & \mathbf{12} \\ \mathbf{0} & \mathbf{8} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{pmatrix}.$$

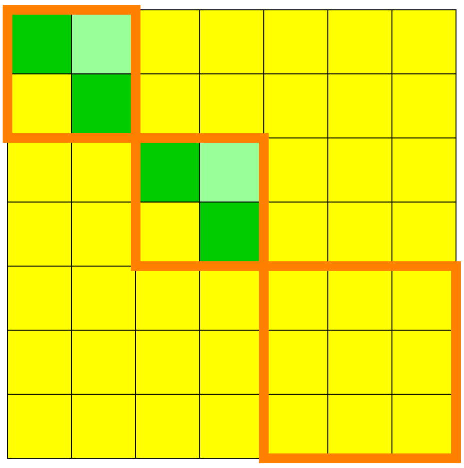
Let us study a visualisation of this situation.



(a) Visualisation of $A - \lambda I_7$.



(b) Visualisation of $(A - \lambda I_7)^2$.



(c) Visualisation of $(A - \lambda I_7)^3$.

Figure 30

We give some remarks on the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. All these green elementary Jordan blocks have the same look however large the exponent of $A - \lambda I_7$. The yellow cells are representing the number zero. The nilpotent part of the matrix is the third elementary Jordan block. It changes by moving the superdiagonal upwards when increasing the exponents of $A - \lambda I_7$. We see that the space $\text{span}\{\mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}$ is invariant with respect to $A - \lambda I_7$ with height of nilpotency equal to 3.

Investigation of the third elementary Jordan chain.
Let us take a look at the third elementary Jordan block in this matrix.

We observe that the original superdiagonal of 1's in third elementary block of the matrix $A - \lambda I_7$ is going upwards in the powers of the matrix $A - \lambda I_7$ until it finally disappears when taking the third power of $A - \lambda I_7$.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_7) = \operatorname{span}\{\mathbf{e}_5\}; \\ \ker(A - \lambda I_7)^2 = \operatorname{span}\{\mathbf{e}_5, \mathbf{e}_6\}; \\ \ker(A - \lambda I_7)^3 = \operatorname{span}\{\mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_7)$	1	$1 = \dim(\ker(A - \lambda I_7))$
$\ker(A - \lambda I_7)^2$	2	$1 = \dim(\ker(A - \lambda I_7)^2) - \dim(\ker(A - \lambda I_7))$
$\ker(A - \lambda I_7)^3$	3	$1 = \dim(\ker(A - \lambda I_7)^3) - \dim(\ker(A - \lambda I_7)^2)$

We see here a table for the invariant subspace $\operatorname{span}\{\mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}$ with respect to the nilpotent operator $\ker(A - \lambda I_7)$ restricted to this space. We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A - \lambda I_7)^i) - \dim(\ker(A - \lambda I_7)^{i-1})$. In the first row, we have $\dim(\ker(A - \lambda I_7))$.

We see also that there is equality in the inclusion of sets from the third power onwards.

$$\ker(A - \lambda I_7) \subsetneq \ker(A - \lambda I_7)^2 \subsetneq \ker(A - \lambda I_7)^3 = \ker(A - \lambda I_7)^4 \cdots .$$

We remark by looking at the matrices $(A - \lambda I_7)^i$ that we have the following mappings

$$\begin{cases}(A-\lambda\, I_7)\,\mathbf{e}_7 = \mathbf{e}_6,\\ (A-\lambda\, I_7)\,\mathbf{e}_6 = \mathbf{e}_5,\\ (A-\lambda\, I_7)\,\mathbf{e}_5 = \mathbf{0}.\end{cases}$$

We remark by looking at the matrices $(A-\lambda\, I_7)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases}(A-\lambda\, I_7)\,\mathbf{e}_7 = \mathbf{e}_6,\\ (A-\lambda\, I_7)^2\,\mathbf{e}_7 = \mathbf{e}_5,\\ (A-\lambda\, I_7)^3\,\mathbf{e}_7 = \mathbf{0}.\end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_5 = (A-\lambda\, I_7)^2\,\mathbf{e}_7, \mathbf{e}_6 = (A-\lambda\, I_7)\,\mathbf{e}_7, \mathbf{e}_7\}.$$

After we have found the first Jordan chain of length 3, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A-\lambda\, I_7)$	1	$\mathbf{e}_5$	0
$\ker(A-\lambda\, I_7)^2$	2	$\mathbf{e}_6$	0
$\ker(A-\lambda\, I_7)^3$	3	$\mathbf{e}_7$	0

We have now in the last column only 0's and this means that we are done looking for Jordan chains.

8. Kernels of $(B-\lambda\, I_7)^i$.

We calculate the kernel of $B-\lambda\, I_7$. We remember that $\lambda = -1$.

The matrix $B-\lambda\, I_7$ is

$$B-\lambda I_7=\begin{pmatrix} 1 & 2 & 0 & -2 & -1 & 0 & 0\\ 0 & 2 & 0 & 0 & 0 & 0 & 0\\ 1 & -1 & 2 & 2 & 1 & 0 & 0\\ 0 & -1 & 0 & 5 & 3 & -1 & 0\\ 1 & 0 & 0 & -4 & -3 & 2 & 0\\ -1 & 1 & 1 & -3 & -2 & 1 & 1\\ -1 & 1 & -2 & -2 & -1 & 0 & 0 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 1 & 2 & 0 & -2 & -1 & 0 & 0\\ 0 & 2 & 0 & 0 & 0 & 0 & 0\\ 1 & -1 & 2 & 2 & 1 & 0 & 0\\ 0 & -1 & 0 & 5 & 3 & -1 & 0\\ 1 & 0 & 0 & -4 & -3 & 2 & 0\\ -1 & 1 & 1 & -3 & -2 & 1 & 1\\ -1 & 1 & -2 & -2 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\\ z_5\\ z_6\\ z_7 \end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} z_1+2z_2-2z_4-z_5=0,\\ 2z_2=0,\\ z_1-z_2+2z_3+2z_4+z_5=0,\\ -z_2+5z_4+3z_5-z_6=0,\\ z_1-4z_4-3z_5+2z_6=0,\\ -z_1+z_2+z_3-3z_4-2z_5+z_6+z_7=0,\\ -z_1+z_2-2z_3-2z_4-z_5=0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B-\lambda I_7) &= \{(0,0,0,r_4,-2r_4,-r_4,0) \mid r_4\in\mathbf{K}\}\\ &= \operatorname{span}\{(0,0,0,1,-2,-1,0)\}. \end{aligned}$$

Kernel of $(\mathbf{B}-\lambda\,\mathbf{I}_7)^2$.
We remember that $\lambda=-1$.

We calculate the kernel of $(B-\lambda I_7)^2$. The matrix $(B-\lambda I_7)^2$ is

$$(B-\lambda I_7)^2 = \begin{pmatrix} 0 & 8 & 0 & -8 & -4 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & -4 & 4 & 8 & 4 & 0 & 0 \\ 4 & -8 & -1 & 16 & 8 & 0 & -1 \\ -4 & 8 & 2 & -16 & -8 & 0 & 2 \\ -4 & 4 & 1 & -8 & -4 & 0 & 1 \\ -4 & 4 & -4 & -8 & -4 & 0 & 0 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & 8 & 0 & -8 & -4 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & -4 & 4 & 8 & 4 & 0 & 0 \\ 4 & -8 & -1 & 16 & 8 & 0 & -1 \\ -4 & 8 & 2 & -16 & -8 & 0 & 2 \\ -4 & 4 & 1 & -8 & -4 & 0 & 1 \\ -4 & 4 & -4 & -8 & -4 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 8z_2 - 8z_4 - 4z_5 = 0, \\ 4z_2 = 0, \\ 4z_1 - 4z_2 + 4z_3 + 8z_4 + 4z_5 = 0, \\ 4z_1 - 8z_2 - z_3 + 16z_4 + 8z_5 - z_7 = 0, \\ -4z_1 + 8z_2 + 2z_3 - 16z_4 - 8z_5 + 2z_7 = 0, \\ -4z_1 + 4z_2 + z_3 - 8z_4 - 4z_5 + z_7 = 0, \\ -4z_1 + 4z_2 - 4z_3 - 8z_4 - 4z_5 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B-\lambda I_7)^2 &= \{(0,0,0,r_4,-2r_4,r_6,0) \mid r_4, r_6 \in \mathbf{K}\} \\ &= \operatorname{span}\{(0,0,0,1,-2,0,0),(0,0,0,0,0,1,0)\}. \end{aligned}$$

Kernel of $(\mathbf{B}-\lambda\,\mathbf{I}_7)^3$.

We calculate the kernel of $(B-\lambda I_7)^3$. The matrix $(B-\lambda I_7)^3$ is

$$(B-\lambda I_7)^3 = \begin{pmatrix} -4 & 24 & 0 & -24 & -12 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 \\ 12 & -12 & 8 & 24 & 12 & 0 & 0 \\ 12 & -24 & 0 & 40 & 20 & 0 & 0 \\ -12 & 24 & 0 & -40 & -20 & 0 & 0 \\ -8 & 8 & 0 & -16 & -8 & 0 & 0 \\ -12 & 12 & -8 & -24 & -12 & 0 & 0 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -4 & 24 & 0 & -24 & -12 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 \\ 12 & -12 & 8 & 24 & 12 & 0 & 0 \\ 12 & -24 & 0 & 40 & 20 & 0 & 0 \\ -12 & 24 & 0 & -40 & -20 & 0 & 0 \\ -8 & 8 & 0 & -16 & -8 & 0 & 0 \\ -12 & 12 & -8 & -24 & -12 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -4z_1 + 24z_2 - 24z_4 - 12z_5 = 0, \\ 8z_2 = 0, \\ 12z_1 - 12z_2 + 8z_3 + 24z_4 + 12z_5 = 0, \\ 12z_1 - 24z_2 + 40z_4 + 20z_5 = 0, \\ -12z_1 + 24z_2 - 40z_4 - 20z_5 = 0, \\ -8z_1 + 8z_2 - 16z_4 - 8z_5 = 0, \\ -12z_1 + 12z_2 - 8z_3 - 24z_4 - 12z_5 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B-\lambda I_7)^3 &= \{(0,0,0,r_4,-2r_4,r_6,r_7) \mid r_4,r_6,r_7 \in \mathbf{K}\} \\ &= \operatorname{span}\{(0,0,0,1,-2,0,0),(0,0,0,0,0,1,0), \\ &\qquad (0,0,0,0,0,0,1)\}. \end{aligned}$$

Kernel of $(\mathbf{B}-\lambda\mathbf{I}_7)^4$.
We remember that $\lambda = -1$.

We calculate the kernel of $(B-\lambda I_7)^4$. The matrix $(B-\lambda I_7)^4$ is

$$(B-\lambda I_7)^4 = \begin{pmatrix} -16 & 64 & 0 & -64 & -32 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 & 0 & 0 \\ 32 & -32 & 16 & 64 & 32 & 0 & 0 \\ 32 & -64 & 0 & 96 & 48 & 0 & 0 \\ -32 & 64 & 0 & -96 & -48 & 0 & 0 \\ -16 & 16 & 0 & -32 & -16 & 0 & 0 \\ -32 & 32 & -16 & -64 & -32 & 0 & 0 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -16 & 64 & 0 & -64 & -32 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 & 0 & 0 \\ 32 & -32 & 16 & 64 & 32 & 0 & 0 \\ 32 & -64 & 0 & 96 & 48 & 0 & 0 \\ -32 & 64 & 0 & -96 & -48 & 0 & 0 \\ -16 & 16 & 0 & -32 & -16 & 0 & 0 \\ -32 & 32 & -16 & -64 & -32 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -16z_1 + 64z_2 - 64z_4 - 32z_5 = 0, \\ 16z_2 = 0, \\ 32z_1 - 32z_2 + 16z_3 + 64z_4 + 32z_5 = 0, \\ 32z_1 - 64z_2 + 96z_4 + 48z_5 = 0, \\ -32z_1 + 64z_2 - 96z_4 - 48z_5 = 0, \\ -16z_1 + 16z_2 - 32z_4 - 16z_5 = 0, \\ -32z_1 + 32z_2 - 16z_3 - 64z_4 - 32z_5 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B-\lambda I_7)^4 &= \{(0,0,0,r_4,-2\,r_4,r_6,r_7) \mid r_4,r_6,r_7 \in \mathbf{K}\} \\ &= \operatorname{span}\{(0,0,0,1,-2,0,0),(0,0,0,0,0,1,0), \\ &\qquad (0,0,0,0,0,0,1)\}. \end{aligned}$$

Stabilisation of the kernels.
We see that inclusion of the kernels is starting to stabilise. We mean by this that we are having equality from the third power onwards in the following chain of inclusion of sets.

$$\ker(B-\lambda I_7) \subsetneq \ker((B-\lambda I_7)^2) \subsetneq \ker((B-\lambda I_7)^3) = \ker((B-\lambda I_7)^4) \cdots .$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B-\lambda I_7)$	1	$1 = \dim(\ker(B-\lambda I_7))$
$\ker(B-\lambda I_7)^2$	2	$1 = \dim(\ker(B-\lambda I_7)^2) - \dim(\ker(B-\lambda I_7))$
$\ker(B-\lambda I_7)^3$	3	$1 = \dim(\ker(B-\lambda I_7)^3) - \dim(\ker(B-\lambda I_7)^2)$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B-\lambda I_7)^i)) - \dim(\ker((B-\lambda I_7)^{i-1}))$. The first number of this last column is $\dim(\ker(B-\lambda I_7))$.

9. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w}_5, \mathbf{w}_6, \mathbf{w}_7\}$ satisfying

$$\begin{cases} (B-\lambda I_7)\mathbf{w}_5 = \mathbf{0}, \\ (B-\lambda I_7)\mathbf{w}_6 = \mathbf{w}_5, \\ (B-\lambda I_7)\mathbf{w}_7 = \mathbf{w}_6, \end{cases}$$

where $\mathbf{w}_7$ is in the vector space $\ker(B-\lambda I_7)^3$ but not in $\ker(B-\lambda I_7)^2$.

We look for a generating vector $\mathbf{w}_7$. This vector must be linearly independent of all vectors in $\ker(B-\lambda I_7)^2$ and must be independent from vectors that were already chosen in $\ker(B-\lambda I_7)^3$ of height 3. We know that

$$\ker((B-\lambda I_7)^3) = \operatorname{span}\{(0,0,0,1,-2,0,0), (0,0,0,0,0,1,0), (0,0,0,0,0,0,1)\}.$$

We have at this point not chosen in $\ker((B-\lambda I_7)^3)$ any vectors of height 3.

We remember that $\ker((B - \lambda I_7)^2)$ equals

$$\ker(B - \lambda I_7)^2 = \operatorname{span}\{(0,0,0,1,-2,0,0),(0,0,0,0,0,1,0)\}.$$

We know that a vector in $\ker((B - \lambda I_7)^3)$ must be of the form

$$\begin{aligned} a\,(0,0,0,1,-2,0,0) + b\,(0,0,0,0,0,1,0) + c\,(0,0,0,0,0,0,1) \\ = (0,0,0,a,-2\,a,b,c). \end{aligned}$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a & -2\,a & b & c \end{pmatrix}.$$

We row reduce this matrix H and if we impose $c \neq 0$, we find

$$\begin{pmatrix} 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent. So we can take $a = 0$, $b = 0$, $c = 1$.

We have the generating vector

$$\mathbf{w}_7 = (0,0,0,0,0,0,1).$$

We start with $\mathbf{w}_7 = (0,0,0,0,0,0,1)$.

We calculate

$$\mathbf{w_6} = (B - \lambda I_7)\mathbf{w_7}$$
$$= \begin{pmatrix} 1 & 2 & 0 & -2 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 5 & 3 & -1 & 0 \\ 1 & 0 & 0 & -4 & -3 & 2 & 0 \\ -1 & 1 & 1 & -3 & -2 & 1 & 1 \\ -1 & 1 & -2 & -2 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

So we have $\mathbf{w_6} = (0, 0, 0, 0, 0, 1, 0)$.

We calculate $\mathbf{w_5}$.

$$\mathbf{w_5} = (B - \lambda I_7)^2\mathbf{w_7}$$
$$= \begin{pmatrix} 0 & 8 & 0 & -8 & -4 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & -4 & 4 & 8 & 4 & 0 & 0 \\ 4 & -8 & -1 & 16 & 8 & 0 & -1 \\ -4 & 8 & 2 & -16 & -8 & 0 & 2 \\ -4 & 4 & 1 & -8 & -4 & 0 & 1 \\ -4 & 4 & -4 & -8 & -4 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

So we have $\mathbf{w_5} = (0, 0, 0, -1, 2, 1, 0)$.

We have our first Jordan chain for this eigenvalue. It has length 3. We have in total found 3 chains.

$$\{\mathbf{w_5} = (0, 0, 0, -1, 2, 1, 0), \mathbf{w_6} = (0, 0, 0, 0, 0, 1, 0), \mathbf{w_7} = (0, 0, 0, 0, 0, 0, 1)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_7)$	1	$\mathbf{w_5}$	0
$\ker(B - \lambda I_7)^2$	2	$\mathbf{w_6}$	0
$\ker(B - \lambda I_7)^3$	3	$\mathbf{w_7}$	0

with

$$\mathbf{w_5} = (0, 0, 0, -1, 2, 1, 0)$$
$$\mathbf{w_6} = (0, 0, 0, 0, 0, 1, 0)$$
$$\mathbf{w_7} = (0, 0, 0, 0, 0, 0, 1)$$

10. Calculation of Jordan chains.

We construct the matrix of base change P with the coordinates of the vectors $\mathbf{w_i}$ found in the Jordan chains in the columns of P.

$$P = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$
\begin{aligned}
A &= P^{-1} B P \\
&= \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 & 2 & 0 & -2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 4 & 3 & -1 & 0 \\ 1 & 0 & 0 & -4 & -4 & 2 & 0 \\ -1 & 1 & 1 & -3 & -2 & 0 & 1 \\ -1 & 1 & -2 & -2 & -1 & 0 & -1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \end{pmatrix}.
\end{aligned}
$$

24 exercise. (7×7); $(J_3(1), J_2(1), J_2(2))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B so that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 1 & 1 & -1 & -2 \\ -2 & -2 & 3 & 1 & -1 & -3 & -1 \\ 1 & 1 & -1 & 1 & 1 & 2 & 1 \\ -1 & 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_7| = -(\lambda - 2)^2 (\lambda - 1)^5.$$

We see that the characteristic Cayley-Hamilton polynomial $p_{\text{C-H}}(\lambda)$ can be factorised in linear polynomials over the field $\mathbf{K}$. So we can apply the Jordan normalisation machinery.

The eigenvalue $\lambda = 1$ has algebraic multiplicity 5. the eigenvalue $\lambda = 2$ has algebraic multiplicity 2.

2. Investigation of the first eigenvalue.

We work now with the eigenvalue $\lambda = 1$.

3. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of

this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 4 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 3. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_7 with eigenvalue $\lambda = 1$.

$$A - \lambda I_7 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We compute also the powers of $A - \lambda I_7$.

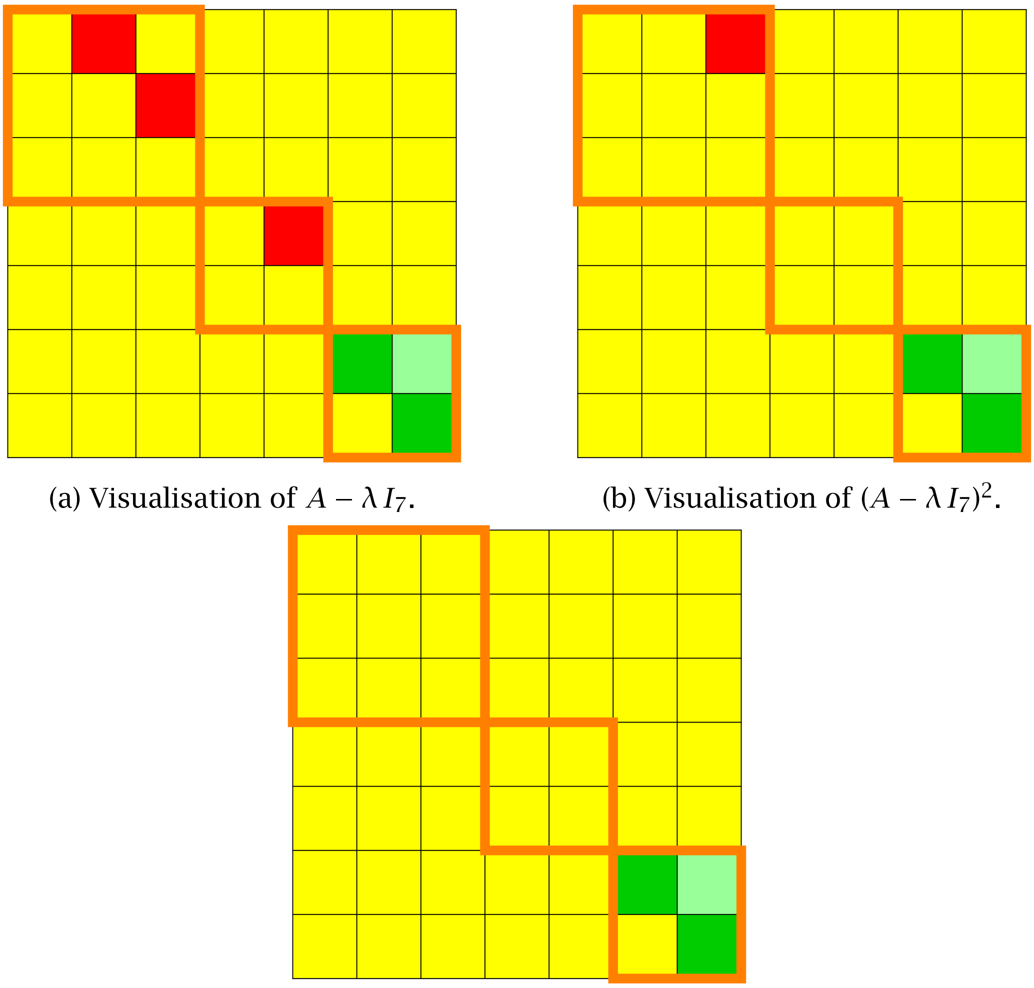
$$A - \lambda I_7 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(A-\lambda I_7)^2 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \end{pmatrix},$$

$$(A-\lambda I_7)^3 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \end{pmatrix}.$$

We see that the superdiagonals in the two first elementary Jordan blocks are going up in their respective Jordan blocks while increasing the exponents until they ultimately disappear when taking the third power.

Let us visualise this situation.



(a) Visualisation of $A - \lambda I_7$.

(b) Visualisation of $(A - \lambda I_7)^2$.

(c) Visualisation of $(A - \lambda I_7)^3$.

Figure 31

We give here some remarks about the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. However large we take the powers of this matrix, the blocks with green cells always keep having the same look. The yellow cells are representing the number zero. The red cells represent the number 1. If we restrict the mapping $A - \lambda I_7$ to the subspace $\text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$ which is an invariant subspace with respect to the operator $A - \lambda I_7$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 3. The nilpotent part of the matrix is represented by the two first elementary Jordan blocks. They change by moving the super-diagonal upwards in their respective Jordan blocks when increasing the exponents of $A - \lambda I_7$. We see that the space $\text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$ is invariant with respect to $A - \lambda I_7$ which has a height of nilpotency equal to 3.

Investigation of the first Jordan chain.
It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_7) = \text{span}\{\mathbf{e}_1, \mathbf{e}_4\}; \\ \ker(A - \lambda I_7)^2 = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5\}; \\ \ker(A - \lambda I_7)^3 = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_7)$	2	$2 = \dim(\ker(A - \lambda I_7))$
$\ker(A - \lambda I_7)^2$	4	$2 = \dim(\ker(A - \lambda I_7)^2) - \dim(\ker(A - \lambda I_7))$
$\ker(A - \lambda I_7)^3$	5	$1 = \dim(\ker(A - \lambda I_7)^3) - \dim(\ker(A - \lambda I_7)^2)$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A - \lambda I_7)^i) - \dim(\ker(A - \lambda I_7)^{i-1})$. In the first row, we have $\dim(\ker(A - \lambda I_7))$.

We see also that there is equality in the inclusion of sets from the third power onwards.

$$\ker(A - \lambda I_7) \subsetneq \ker(A - \lambda I_7)^2 \subsetneq \ker(A - \lambda I_7)^3 = \ker(A - \lambda I_7)^4 = \cdots .$$

We remark by looking at the matrices $(A - \lambda I_7)^i$ that we have the following mappings

$$\begin{cases} (A - \lambda I_7)\,\mathbf{e}_3 = \mathbf{e}_2, \\ (A - \lambda I_7)\,\mathbf{e}_2 = \mathbf{e}_1, \\ (A - \lambda I_7)\,\mathbf{e}_1 = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A - \lambda I_7)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases} (A - \lambda I_7)\,\mathbf{e}_3 = \mathbf{e}_2, \\ (A - \lambda I_7)^2\,\mathbf{e}_3 = \mathbf{e}_1, \\ (A - \lambda I_7)^3\,\mathbf{e}_3 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = (A - \lambda I_7)^2\,\mathbf{e}_3, \mathbf{e}_2 = (A - \lambda I_7)\,\mathbf{e}_3, \mathbf{e}_3\}.$$

After we have found the first Jordan chain of length 3, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A - \lambda I_7)$	2	$\mathbf{e}_1$	1
$\ker(A - \lambda I_7)^2$	4	$\mathbf{e}_2$	1
$\ker(A - \lambda I_7)^3$	5	$\mathbf{e}_3$	0

We have that the last number in the last column is not zero, and this means that there is still a chain left of length 2.

Investigating the second Jordan chain.
We remark by looking at the matrices $(A - \lambda I_7)^i$ that we have the following mappings

$$\begin{cases} (A - \lambda I_7)\,\mathbf{e}_5 = \mathbf{e}_4, \\ (A - \lambda I_7)\,\mathbf{e}_4 = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A - \lambda I_7)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases}(A-\lambda I_7)\,\mathbf{e}_5=\mathbf{e}_4,\\(A-\lambda I_7)^2\,\mathbf{e}_5=\mathbf{0}.\end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_4=(A-\lambda I_7)\,\mathbf{e}_5,\mathbf{e}_5\}.$$

We have found the second Jordan chain. It has length 2. we have then found the following table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(A-\lambda I_7)$	2	$\mathbf{e}_1$	$\mathbf{e}_4$	0
$\ker(A-\lambda I_7)^2$	4	$\mathbf{e}_2$	$\mathbf{e}_5$	0
$\ker(A-\lambda I_7)^3$	5	$\mathbf{e}_3$		0

4. Kernels of $(\mathbf{B}-\lambda\,\mathbf{I}_7)^{\mathbf{i}}$.

Kernel of $(\mathbf{B}-\lambda\,\mathbf{I}_7)$.
We calculate the kernel of $B-\lambda I_7$. We remember that we are still working with the eigenvalue $\lambda=1$.

The matrix $B-\lambda I_7$ is

$$B-\lambda I_7=\begin{pmatrix}1&1&0&0&0&1&0\\-2&-1&1&1&1&-1&-2\\-2&-2&2&1&-1&-3&-1\\1&1&-1&0&1&2&1\\-1&0&1&0&0&-1&-1\\0&-1&0&0&-1&-1&1\\0&0&0&0&0&0&1\end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ -2 & -1 & 1 & 1 & 1 & -1 & -2 \\ -2 & -2 & 2 & 1 & -1 & -3 & -1 \\ 1 & 1 & -1 & 0 & 1 & 2 & 1 \\ -1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} z_1 + z_2 + z_6 = 0, \\ -2z_1 - z_2 + z_3 + z_4 + z_5 - z_6 - 2z_7 = 0, \\ -2z_1 - 2z_2 + 2z_3 + z_4 - z_5 - 3z_6 - z_7 = 0, \\ z_1 + z_2 - z_3 + z_5 + 2z_6 + z_7 = 0, \\ -z_1 + z_3 - z_6 - z_7 = 0, \\ -z_2 - z_5 - z_6 + z_7 = 0, \\ z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_7) &= \{(r_1, r_2, -r_2, r_2, r_1, -r_1 - r_2,) \mid r_1, r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,1,-1,0), (0,1,-1,1,0,-1,0)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda \mathbf{I}_7)^2$.
We remember that $\lambda = 1$.

We calculate the kernel of $(B - \lambda I_7)^2$. The matrix $(B - \lambda I_7)^2$ is

$$(B - \lambda I_7)^2 = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & -1 & -1 \\ -2 & -1 & 1 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & -2 & 2 & 1 & 0 & -3 & -3 \\ 3 & 2 & -2 & -1 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 0 & -1 & -1 \\ -2 & -1 & 1 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & -2 & 2 & 1 & 0 & -3 & -3 \\ 3 & 2 & -2 & -1 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -z_1 - z_2 + z_3 + z_4 - z_6 - z_7 = 0, \\ -2z_1 - z_2 + z_3 - 2z_6 - 2z_7 = 0, \\ z_7 = 0, \\ -3z_1 - 2z_2 + 2z_3 + z_4 - 3z_6 - 3z_7 = 0, \\ 3z_1 + 2z_2 - 2z_3 - z_4 + 3z_6 + 3z_7 = 0, \\ z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_7)^2 &= \{(r_1, r_2, r_3, r_2/2 - r_3/2, r_5, -r_1 - r_2/2 + r_3/2, 0) \\ &\qquad \mid r_1, r_2, r_3, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,0,-1,0), (0,1,0,1/2,0,-1/2,0), \\ &\qquad (0,0,1,-1/2,0,1/2,0), (0,0,0,0,1,0,0)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I}_7)^3$.
We calculate the kernel of $(B - \lambda I_7)^3$.

$$(B - \lambda I_7)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 1 & 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & -1 & 1 & 0 & 0 & -2 & -3 \\ 2 & 1 & -1 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 1 & 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & -1 & 1 & 0 & 0 & -2 & -3 \\ 2 & 1 & -1 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -2z_1 - z_2 + z_3 - 2z_6 - 3z_7 = 0, \\ z_7 = 0, \\ -2z_1 - z_2 + z_3 - 2z_6 - 3z_7 = 0, \\ 2z_1 + z_2 - z_3 + 2z_6 + 3z_7 = 0, \\ z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_7)^3 &= \{(r_1, r_2, r_3, r_4, r_5, -r_1 - r_2/2 + r_3/2, 0) \\ &\qquad \mid r_1, r_2, r_3, r_4, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,0,-1,0), (0,1,0,0,0,-1/2,0), \\ &\qquad (0,0,1,0,0,1/2,0), (0,0,0,1,0,0,0), \\ &\qquad (0,0,0,0,1,0,0)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda \mathbf{I_7})^4$.
We calculate the kernel of $(B - \lambda I_7)^4$. The matrix $(B - \lambda I_7)^4$ is

$$(B - \lambda I_7)^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 1 & 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & -1 & 1 & 0 & 0 & -2 & -4 \\ 2 & 1 & -1 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 1 & 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & -1 & 1 & 0 & 0 & -2 & -4 \\ 2 & 1 & -1 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -2z_1 - z_2 + z_3 - 2z_6 - 4z_7 = 0, \\ z_7 = 0, \\ -2z_1 - z_2 + z_3 - 2z_6 - 4z_7 = 0, \\ 2z_1 + z_2 - z_3 + 2z_6 + 4z_7 = 0, \\ z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_7)^4 &= \{(r_1, r_2, r_3, r_4, r_5, -r_1 - r_2/2 + r_3/2, 0) \\ &\qquad \mid r_1, r_2, r_3, r_4, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,0,-1,0), (0,1,0,0,0,-1/2,0), \\ &\qquad (0,0,1,0,0,1/2,0), (0,0,0,1,0,0,0), \\ &\qquad (0,0,0,0,1,0,0)\}. \end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the third power onwards in the following chain of inclusion of sets.

$$\ker(B - \lambda I_7) \subsetneq \ker((B - \lambda I_7)^2) \subsetneq \ker((B - \lambda I_7)^3) = \ker((B - \lambda I_7)^4) \cdots .$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B - \lambda I_7)$	2	$2 = \dim(\ker(B - \lambda I_7))$
$\ker(B - \lambda I_7)^2$	4	$2 = \dim(\ker(B - \lambda I_7)^2) - \dim(\ker(B - \lambda I_7))$
$\ker(B - \lambda I_7)^3$	5	$1 = \dim(\ker(B - \lambda I_7)^3) - \dim(\ker(B - \lambda I_7)^2)$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B - \lambda I_7)^i)) - \dim(\ker((B - \lambda I_7)^{i-1}))$. The first number of this last column is $\dim(\ker(B - \lambda I_7))$.

5. Calculation of Jordan chains.

Calculation of the first Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ satisfying

$$\begin{cases} (B - \lambda I_7)\,\mathbf{w}_1 = \mathbf{0}, \\ (B - \lambda I_7)\,\mathbf{w}_2 = \mathbf{w}_1, \\ (B - \lambda I_7)\,\mathbf{w}_3 = \mathbf{w}_2, \end{cases}$$

or

$$\begin{cases} (B - \lambda I_7)\,\mathbf{w}_3 = \mathbf{w}_2, \\ (B - \lambda I_7)^2\,\mathbf{w}_3 = \mathbf{w}_1, \\ (B - \lambda I_7)^3\,\mathbf{w}_3 = \mathbf{0}, \end{cases}$$

where $\mathbf{w}_3$ is in the vector space $\ker((B - \lambda I_7)^3)$ but not in $\ker((B - \lambda I_7)^2)$.

We look for a generating vector $\mathbf{w}_3$. This vector must be linearly independent of all vectors in $\ker(B - \lambda I_7)^2$ and must be independent from vectors of height 3 that were already chosen in $\ker(B - \lambda I_7)^3$. We know that

$$\begin{aligned} \ker((B - \lambda I_7)^3) = \operatorname{span}\{&(1,0,0,0,0,-1,0), (0,1,0,0,0,-1/2,0), \\ &(0,0,1,0,0,1/2,0), (0,0,0,1,0,0,0), \\ &(0,0,0,0,1,0,0)\}. \end{aligned}$$

We have at this point chosen in $\ker((B - \lambda I_7)^3)$ no other vector of height 3.

We remember that

$$\ker(B - \lambda I_7)^2 = \operatorname{span}\{(1,0,0,0,0,-1,0),(0,1,0,1/2,0,-1/2,0),\\ (0,0,1,-1/2,0,1/2,0),(0,0,0,0,1,0,0)\}.$$

We know that a vector in $\ker((B - \lambda I_7)^3)$ must be of the form

$$\begin{aligned} a\,(1,0,0,0,0,-1,0) + b\,(0,1,0,1/2,0,-1/2,0) + c\,(0,0,1,-1/2,0,1/2,0)\\ + d\,(0,0,0,1,0,0,0) + e\,(0,0,0,0,1,0,0)\\ = (a,b,c,b/2-c/2+d,e,-a-b/2+c/2,0). \end{aligned}$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ a & b & c & b/2-c/2+d & e & -a-b/2+c/2 & 0 \end{pmatrix}.$$

We row reduce this matrix H and if we impose $d \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $d \neq 0$. So we have that we can choose $a = 0$, $b = 0$, $c = 0$, $d = 1$, $e = 0$.

We can take the generating vector

$$\mathbf{w}_3 = (0,0,0,1,0,0,0).$$

We start with $\mathbf{w_3} = (0, 0, 0, 1, 0, 0, 0)$.

We calculate $\mathbf{w_2}$.

$$\mathbf{w_2} = (B - \lambda I_7)\mathbf{w_3} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ -2 & -1 & 1 & 1 & 1 & -1 & -2 \\ -2 & -2 & 2 & 1 & -1 & -3 & -1 \\ 1 & 1 & -1 & 0 & 1 & 2 & 1 \\ -1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\begin{aligned} \mathbf{w_1} &= (B - \lambda I_7)^2 \mathbf{w_3} \\ &= \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & -1 & -1 \\ -2 & -1 & 1 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & -2 & 2 & 1 & 0 & -3 & -3 \\ 3 & 2 & -2 & -1 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}. \end{aligned}$$

We have found the first Jordan chain. It has length 3.

$$\{\mathbf{w_1} = (1, 0, 0, 0, 1, -1, 0), \mathbf{w_2} = (0, 1, 1, 0, 0, 0, 0), \mathbf{w_3} = (0, 0, 0, 1, 0, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_7)$	2	$\mathbf{w_1}$	1
$\ker(B - \lambda I_7)^2$	4	$\mathbf{w_2}$	1
$\ker(B - \lambda I_7)^3$	5	$\mathbf{w_3}$	0

with

$$\mathbf{w}_1 = (1,0,0,0,1,-1,0)$$
$$\mathbf{w}_2 = (0,1,1,0,0,0,0)$$
$$\mathbf{w}_3 = (0,0,0,1,0,0,0)$$

We have one Jordan chain of length two left to look for.

Calculation of the second Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w}_4, \mathbf{w}_5\}$ satisfying

$$\begin{cases} (B-\lambda I_7)\mathbf{w}_4 = \mathbf{0}, \\ (B-\lambda I_7)\mathbf{w}_5 = \mathbf{w}_4. \end{cases}$$

where $\mathbf{w}_5$ is in the vector space $\ker((B-\lambda I_7)^2)$ but not in $\ker(B-\lambda I_7)$.

We look for a generating vector $\mathbf{w}_5$. This vector must be linearly independent of all vectors in $\ker(B-\lambda I_7)$ and must be independent from vectors of height 2 that were already chosen in $\ker(B-\lambda I_7)^2$. We know that

$$\ker(B-\lambda I_7)^2 = \operatorname{span}\{(1,0,0,0,0,-1,0),(0,1,0,1/2,0,-1/2,0), (0,0,1,-1/2,0,1/2,0),(0,0,0,0,1,0,0)\}.$$

We have at this point chosen in $\ker((B-\lambda I_7)^2)$ already the vector $\mathbf{w}_2 = (0,1,1,0,0,0,0)$ of height 2.

We remember that

$$\ker(B-\lambda I_7) = \operatorname{span}\{(1,0,0,0,1,-1,0),(0,1,-1,1,0,-1,0)\}.$$

We know that a vector in $\ker((B-\lambda I_7)^2)$ must be of the form

$$\begin{aligned} &a\,(1,0,0,0,0,-1,0) + b\,(0,1,0,1/2,0,-1/2,0) \\ &\qquad + c\,(0,0,1,-(1/2),0,1/2,0) + d\,(0,0,0,0,1,0,0) \\ &= (a,b,c,b/2-c/2,d,-a-b/2+c/2,0). \end{aligned}$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ a & b & c & b/2-c/2 & d & -a-b/2+c/2 & 0 \end{pmatrix}.$$

We row reduce this matrix H and if we impose $a-d \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a-d \neq 0$. So we can choose $a=0$, $b=0$, $c=0$, $d=1$.

We have the generating vector

$$\mathbf{w}_5 = (0,0,0,0,1,0,0).$$

We start with $\mathbf{w}_5 = (0,0,0,0,1,0,0)$.

We calculate $\mathbf{w}_4$.

$$\mathbf{w}_4 = (B-\lambda I_7)\mathbf{w}_5 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ -2 & -1 & 1 & 1 & 1 & -1 & -2 \\ -2 & -2 & 2 & 1 & -1 & -3 & -1 \\ 1 & 1 & -1 & 0 & 1 & 2 & 1 \\ -1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

We have found the second Jordan chain. It has length 2.

$$\{\mathbf{w}_4 = (0,1,-1,1,0,-1,0), \mathbf{w}_5 = (0,0,0,0,1,0,0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(B - \lambda I_7)$	1	$\mathbf{w_1}$	$\mathbf{w_4}$	0
$\ker(B - \lambda I_7)^2$	2	$\mathbf{w_2}$	$\mathbf{w_5}$	0
$\ker(B - \lambda I_7)^3$	3	$\mathbf{w_3}$		0

with

$\mathbf{w_1} = (1, 0, 0, 0, 1, -1, 0)$
$\mathbf{w_2} = (0, 1, 1, 0, 0, 0, 0)$
$\mathbf{w_3} = (0, 0, 0, 1, 0, 0, 0)$
$\mathbf{w_4} = (0, 1, -1, 1, 0, -1, 0)$
$\mathbf{w_5} = (0, 0, 0, 0, 1, 0, 0)$

6. Investigation of the second eigenvalue.

We work now with the second eigenvalue $\lambda = 2$.

7. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 8 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 7. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_7.

$$A - \lambda I_7 = \begin{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

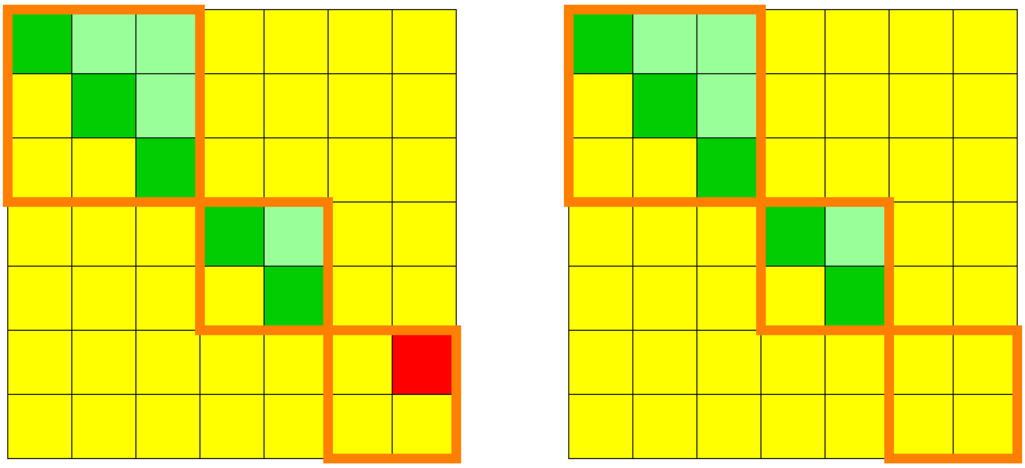
We see that the subspace $\text{span}\{\mathbf{e}_6, \mathbf{e}_7\}$ is invariant with respect to the endomorphism $A - \lambda I_7$. This morphism is nilpotent on this subspace.

We compute the powers of $A - \lambda I_7$.

$$A - \lambda I_7 = \begin{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{matrix} \end{pmatrix},$$

$$(A - \lambda I_7)^2 = \begin{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $(A - \lambda I_7)$. (b) Visualisation of $(A - \lambda I_7)^2$.

Figure 32

We give some comments about this preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The cells which represent 0 are coloured in yellow and those which represent 1 are coloured in red. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix. If we restrict the mapping $A - \lambda I_7$ to the subspace $\text{span}\{\mathbf{e}_6, \mathbf{e}_7\}$ which is an invariant subspace with respect to the operator $A - \lambda I_7$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 2. Let us take a look at the third elementary Jordan block in this matrix.

We observe that the original superdiagonal of 1's in the third block of the matrix $A - \lambda I_7$ is going upwards in the elementary Jordan blocks associated with the nilpotent part of the transformation $A - \lambda I_7$ when increasing the powers of the matrix $A - \lambda I_7$. It finally disappears when taking the second power of $A - \lambda I_7$.

Investigation of the Jordan chain.
It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A-\lambda I_7) = \operatorname{span}\{\mathbf{e}_6\}; \\ \ker(A-\lambda I_7)^2 = \operatorname{span}\{\mathbf{e}_6, \mathbf{e}_7\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A-\lambda I_7)$	1	$1 = \dim(\ker(A-\lambda I_7))$
$\ker(A-\lambda I_7)^2$	2	$1 = \dim(\ker(A-\lambda I_7)^2) - \dim(\ker(A-\lambda I_7))$

We see here a table for the invariant subspace $\operatorname{span}\{\mathbf{e}_6, \mathbf{e}_7\}$ with respect to its nilpotent operator $\ker(A-\lambda I_7)$ restricted to this space. We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A-\lambda I_7)^i) - \dim(\ker(A-\lambda I_7)^{i-1})$. In the first row, we have $\dim(\ker(A-\lambda I_7))$.

We see also that there is equality in the inclusion of sets from the second power onwards

$$\ker(A-\lambda I_7) \subsetneq \ker(A-\lambda I_7)^2 = \ker(A-\lambda I_7)^3 = \cdots .$$

We remark by looking at the matrices $(A-\lambda I_7)^i$ that we have the following mappings

$$\begin{cases} (A-\lambda I_7)\,\mathbf{e}_7 = \mathbf{e}_6, \\ (A-\lambda I_7)\,\mathbf{e}_6 = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A-\lambda I_7)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases} (A-\lambda I_7)\,\mathbf{e}_7 = \mathbf{e}_6, \\ (A-\lambda I_7)^2\,\mathbf{e}_7 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_6 = (A - \lambda I_7)\,\mathbf{e}_7, \mathbf{e}_7\}.$$

After we have found the Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A - \lambda I_7)$	1	$\mathbf{e}_6$	0
$\ker(A - \lambda I_7)^2$	2	$\mathbf{e}_7$	0

We have now in the last column only 0's and this means that we are done looking for Jordan chains.

8. Kernels of $(\mathbf{B} - \lambda\,\mathbf{I}_7)^{\mathbf{i}}$.

We remember that $\lambda = 2$.

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I}_7)$.
We calculate the kernel of $B - \lambda I_7$. The matrix $B - \lambda I_7$ is

$$B - \lambda I_7 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -2 & -2 & 1 & 1 & 1 & -1 & -2 \\ -2 & -2 & 1 & 1 & -1 & -3 & -1 \\ 1 & 1 & -1 & -1 & 1 & 2 & 1 \\ -1 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -2 & -2 & 1 & 1 & 1 & -1 & -2 \\ -2 & -2 & 1 & 1 & -1 & -3 & -1 \\ 1 & 1 & -1 & -1 & 1 & 2 & 1 \\ -1 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} z_2 + z_6 = 0, \\ -2z_1 - 2z_2 + z_3 + z_4 + z_5 - z_6 - 2z_7 = 0, \\ -2z_1 - 2z_2 + z_3 + z_4 - z_5 - 3z_6 - z_7 = 0, \\ z_1 + z_2 - z_3 - z_4 + z_5 + 2z_6 + z_7 = 0, \\ -z_1 + z_3 - z_5 - z_6 - z_7 = 0, \\ -z_2 - z_5 - 2z_6 + z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_7) &= \{(0, r_2, 0, 0, r_2, -r_2, 0) \mid r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(0, 1, 0, 0, 1, -1, 0)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I_7})^2$.

We calculate the kernel of $(B - \lambda I_7)^2$. The matrix $(B - \lambda I_7)^2$ is

$$(B - \lambda I_7)^2 = \begin{pmatrix} -2 & -3 & 1 & 1 & 0 & -3 & -1 \\ 2 & 2 & -1 & -2 & -2 & 0 & 2 \\ 4 & 4 & -3 & -2 & 2 & 6 & 2 \\ -2 & -2 & 2 & 1 & -2 & -4 & -1 \\ -1 & -2 & 0 & 1 & 1 & -1 & -1 \\ 3 & 4 & -2 & -1 & 2 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -2 & -3 & 1 & 1 & 0 & -3 & -1 \\ 2 & 2 & -1 & -2 & -2 & 0 & 2 \\ 4 & 4 & -3 & -2 & 2 & 6 & 2 \\ -2 & -2 & 2 & 1 & -2 & -4 & -1 \\ -1 & -2 & 0 & 1 & 1 & -1 & -1 \\ 3 & 4 & -2 & -1 & 2 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -2z_1 - 3z_2 + z_3 + z_4 - 3z_6 - z_7 = 0, \\ 2z_1 + 2z_2 - z_3 - 2z_4 - 2z_5 + 2z_7 = 0, \\ 4z_1 + 4z_2 - 3z_3 - 2z_4 + 2z_5 + 6z_6 + 2z_7 = 0, \\ -2z_1 - 2z_2 + 2z_3 + z_4 - 2z_5 - 4z_6 - z_7 = 0, \\ -z_1 - 2z_2 + z_4 + z_5 - z_6 - z_7 = 0, \\ 3z_1 + 4z_2 - 2z_3 - z_4 + 2z_5 + 6z_6 + z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_7)^2 &= \{(0, r_2, 0, r_4, r_2, -r_2, r_4) \mid r_2, r_4 \in \mathbf{K}\} \\ &= \operatorname{span}\{(0,1,0,0,1,-1,0), (0,0,0,1,0,0,1)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I_7})^3$.
We calculate the kernel of $(B - \lambda I_7)^3$.

$$(B - \lambda I_7)^3 = \begin{pmatrix} 5 & 6 & -3 & -3 & 0 & 6 & 3 \\ -2 & -2 & 1 & 3 & 3 & 1 & -3 \\ -6 & -6 & 5 & 3 & -3 & -9 & -3 \\ 3 & 3 & -3 & -1 & 3 & 6 & 1 \\ 4 & 5 & -2 & -3 & -1 & 4 & 3 \\ -7 & -8 & 5 & 3 & -3 & -11 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 5 & 6 & -3 & -3 & 0 & 6 & 3 \\ -2 & -2 & 1 & 3 & 3 & 1 & -3 \\ -6 & -6 & 5 & 3 & -3 & -9 & -3 \\ 3 & 3 & -3 & -1 & 3 & 6 & 1 \\ 4 & 5 & -2 & -3 & -1 & 4 & 3 \\ -7 & -8 & 5 & 3 & -3 & -11 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 5z_1 + 6z_2 - 3z_3 - 3z_4 + 6z_6 + 3z_7 = 0, \\ -2z_1 - 2z_2 + z_3 + 3z_4 + 3z_5 + z_6 - 3z_7 = 0, \\ -6z_1 - 6z_2 + 5z_3 + 3z_4 - 3z_5 - 9z_6 - 3z_7 = 0, \\ 3z_1 + 3z_2 - 3z_3 - z_4 + 3z_5 + 6z_6 + z_7 = 0, \\ 4z_1 + 5z_2 - 2z_3 - 3z_4 - z_5 + 4z_6 + 3z_7 = 0, \\ -7z_1 - 8z_2 + 5z_3 + 3z_4 - 3z_5 - 11z_6 - 3z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B-\lambda I_7)^3 &= \{(0,r_2,0,r_4,r_2,-r_2,r_4)\mid r_2,r_4\in\mathbf{K}\}\\ &= \operatorname{span}\{(0,1,0,0,1,-1,0),(0,0,0,1,0,0,1)\}.\end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality starting from the second power onwards in the following chain of inclusion of sets.

$$\ker(B-\lambda I_7)\subsetneq\ker(B-\lambda I_7)^2=\ker(B-\lambda I_7)^3\cdots.$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B-\lambda I_7)$	1	$1=\dim(\ker(B-\lambda I_7))$
$\ker(B-\lambda I_7)^2$	2	$1=\dim(\ker(B-\lambda I_7)^2)-\dim(\ker(B-\lambda I_7))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker(B-\lambda I_7)^i)-\dim(\ker(B-\lambda I_7)^{i-1})$. The first number of this last column is $\dim(\ker(B-\lambda I_7))$.

9. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w_6},\mathbf{w_7}\}$ satisfying

$$\begin{cases}(B-\lambda I_7)\,\mathbf{w_6}=\mathbf{0},\\(B-\lambda I_7)\,\mathbf{w_7}=\mathbf{w_6},\end{cases}$$

or

$$\begin{cases}(B-\lambda I_7)^2\,\mathbf{w_7}=\mathbf{0},\\(B-\lambda I_7)\,\mathbf{w_7}=\mathbf{w_6},\end{cases}$$

where $\mathbf{w_7}$ is in the vector space $\ker((B-\lambda I_7)^2)$ but not in $\ker(B-\lambda I_7)$.

We look for a generating vector $\mathbf{w_7}$. This vector must be linearly independent of all vectors in $\ker(B - \lambda I_7)$ and must be independent from vectors of height 2 that were already chosen in $\ker(B - \lambda I_7)^2$.

We remember that

$$\ker(B - \lambda I_7) = \text{span}\{(0, 1, 0, 0, 1, -1, 0)\}.$$

We know that

$$\ker((B - \lambda I_7)^2) = \text{span}\{(0, 1, 0, 0, 1, -1, 0), (0, 0, 0, 1, 0, 0, 1)\}.$$

We have at this point not chosen any vector of height 2 in a previous Jordan chain of $\ker(B - \lambda I_7)^2$.

We know that a vector in $\ker((B - \lambda I_7)^2)$ must be of the form

$$a\,(0, 1, 0, 0, 1, -1, 0) + b\,(0, 0, 0, 1, 0, 0, 1) = (0, a, 0, b, a, -a, b).$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & a & 0 & b & a & -a & b \end{pmatrix}.$$

We row reduce this matrix H and if we impose $b \neq 0$ we find

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $b \neq 0$. So we choose $a = 0$ and $b = 1$.

We have the generating vector

$$\mathbf{w_7} = (0, 0, 0, 1, 0, 0, 1).$$

We start with $\mathbf{w_7} = (0, 0, 0, 1, 0, 0, 1)$.

We calculate $\mathbf{w_6}$.

$$\mathbf{w_6} = (B - \lambda I_7)\,\mathbf{w_7}$$
$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -2 & -2 & 1 & 1 & 1 & -1 & -2 \\ -2 & -2 & 1 & 1 & -1 & -3 & -1 \\ 1 & 1 & -1 & -1 & 1 & 2 & 1 \\ -1 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

We have now the first Jordan chain for this eigenvalue. We have found in total 3 chains. The length of this chain is 2.

$$\{\mathbf{w_6} = (0, -1, 0, 0, -1, 1, 0), \mathbf{w_7} = (0, 0, 0, 1, 0, 0, 1)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_7)$	1	$\mathbf{w_6}$	0
$\ker(B - \lambda I_7)^2$	2	$\mathbf{w_7}$	0

with

$$\mathbf{w_6} = (0, -1, 0, 0, -1, 1, 0)$$
$$\mathbf{w_7} = (0, 0, 0, 1, 0, 0, 1)$$

10. Result and check of the result.

We construct the matrix of base change P with the coordinates of the vectors $\mathbf{w_i}$ found in the Jordan chains in the columns of P.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$A = P^{-1}\,B\,P$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 1 & 2 & 0 \\ 2 & 1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 1 & 1 & -1 & -2 \\ -2 & -2 & 3 & 1 & -1 & -3 & -1 \\ 1 & 1 & -1 & 1 & 1 & 2 & 1 \\ -1 & 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

25 exercise. (7×7); $(J_2(2), J_2(2), J_1(2), J_2(1))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & -2 & 1 & 2 & 1 \\ -3 & -1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 1 & -2 & -1 \\ 4 & 1 & -1 & 1 & -1 & -1 & -1 \\ -15 & -4 & 2 & -5 & 4 & 10 & 6 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_7| = -(\lambda - 2)^5 (\lambda - 1)^2.$$

We see that the characteristic Cayley-Hamilton polynomial $p_{\text{C-H}}(\lambda)$ can be factorised in linear polynomials over the field $\mathbf{K}$. So we can apply the Jordan normalisation machinery.

The eigenvalue $\lambda = 2$ has algebraic multiplicity 5. the eigenvalue $\lambda = 1$ has algebraic multiplicity 2.

2. Investigation of the first eigenvalue.

We work now with the eigenvalue $\lambda = 2$.

3. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of

this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 4 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 3. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} \; (2) & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_7.

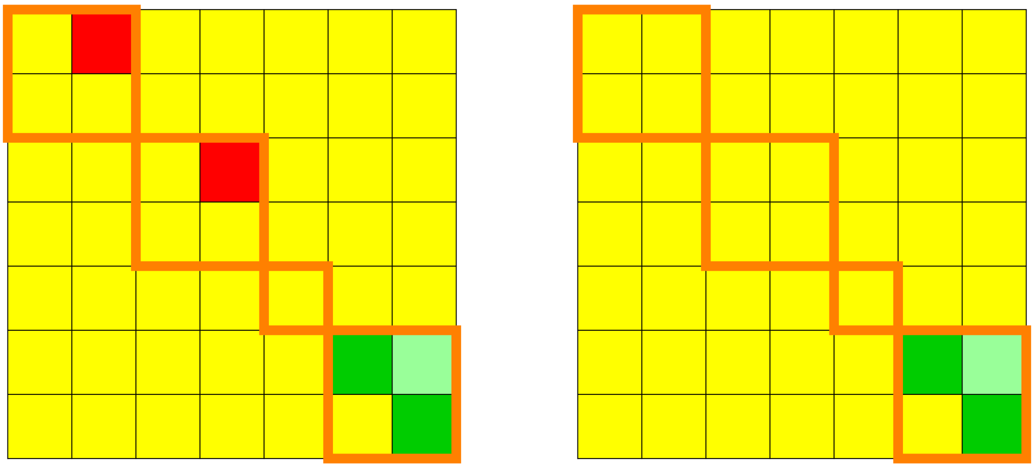
$$A - \lambda I_7 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

We compute also the powers of $A - \lambda I_7$.

$$A - \lambda I_7 = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} \; (0) & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} & \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \end{pmatrix},$$

$$(A - \lambda I_7)^2 = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & (0) & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $A - \lambda I_7$. (b) Visualisation of $(A - \lambda I_7)^2$.

Figure 33

We give some comments about the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells are representing the number one. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix. If we restrict the mapping $A - \lambda I_7$ to the subspace $\text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$ which is an invariant subspace with respect to the operator $A - \lambda I_7$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 2. Let us take a look at the three first elementary Jordan blocks in this matrix. We observe that the original superdiagonals of 1's in the three first blocks of $A - \lambda I_7$ are going upwards in their respective elementary Jordan blocks associated with the nilpotent part of the transformation $A-\lambda I_7$ when increasing the powers of the matrix $A-\lambda I_7$. They finally disappear when taking the second power of $A - \lambda I_7$.

Investigation of the first Jordan chain.

We can also see that the space $\text{span}\{\mathbf{e}_1, \mathbf{e}_2\}$ is invariant with respect to the transformation $A - \lambda I_7$.

The same can be said about the spaces $\text{span}\{\mathbf{e}_3, \mathbf{e}_4\}$ and $\text{span}\{\mathbf{e}_5\}$

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_7) = \text{span}\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5\}; \\ \ker(A - \lambda I_7)^2 = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_7)$	3	$3 = \dim(\ker(A - \lambda I_7))$
$\ker(A - \lambda I_7)^2$	5	$2 = \dim(\ker(A - \lambda I_7)^2) - \dim(\ker(A - \lambda I_7))$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions. In the first row, we have $\dim(\ker(A - \lambda I_7))$.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A - \lambda I_7) \subsetneq \ker(A - \lambda I_7)^2 = \ker(A - \lambda I_7)^3 = \cdots .$$

We remark by looking at the matrices $(A - \lambda I_7)^i$ that we have the following mappings

$$\begin{cases}(A - \lambda I_7)\,\mathbf{e}_2 = \mathbf{e}_1,\\ (A - \lambda I_7)\,\mathbf{e}_1 = \mathbf{0}.\end{cases}$$

We remark by looking at the matrices $(A - \lambda I_7)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases}(A - \lambda I_7)\,\mathbf{e}_2 = \mathbf{e}_1,\\ (A - \lambda I_7)^2\,\mathbf{e}_2 = \mathbf{0}.\end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = (A - \lambda I_7)\,\mathbf{e}_2, \mathbf{e}_2\}.$$

After we have found the first Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A - \lambda I_7)$	3	$\mathbf{e}_1$	2
$\ker(A - \lambda I_7)^2$	5	$\mathbf{e}_2$	1

We have that the last number in the last column is not zero, and this means that there is still a chain left of length 2.

Investigating the second Jordan chain.
We remark by looking at the matrices $(A - \lambda I_7)^i$ that we have the following mappings

$$\begin{cases}(A - \lambda I_7)\,\mathbf{e}_4 = \mathbf{e}_3,\\ (A - \lambda I_7)\,\mathbf{e}_3 = \mathbf{0}.\end{cases}$$

We remark by looking at the matrices $(A - \lambda I_7)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases}(A - \lambda I_7)\,\mathbf{e}_4 = \mathbf{e}_3,\\ (A - \lambda I_7)^2\,\mathbf{e}_4 = \mathbf{0}.\end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_3 = (A - \lambda I_7)\,\mathbf{e}_4, \mathbf{e}_4\}.$$

After we have found the second Jordan chain of length 4, we have then the following table.

Keeping track of chains and dimensions.

	dim	first chain	second chain	remaining dim
$\ker(A - \lambda I_7)$	3	$\mathbf{e}_1$	$\mathbf{e}_3$	1
$\ker(A - \lambda I_7)^2$	5	$\mathbf{e}_2$	$\mathbf{e}_4$	0

Investigating the third Jordan chain.
We remark by looking at the matrices $(A - \lambda I_7)^i$ that we have the following mappings

$$(A - \lambda I_7)\,\mathbf{e}_5 = \mathbf{0}.$$

One sees that we have a Jordan chain of one linearly independent vector.

$$\{\mathbf{e}_5\}.$$

After we have found the third Jordan chain with length 1, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	chain 3	remaining dim
$\ker(A - \lambda I_7)$	3	$\mathbf{e}_1$	$\mathbf{e}_3$	$\mathbf{e}_5$	0
$\ker(A - \lambda I_7)^2$	5	$\mathbf{e}_2$	$\mathbf{e}_4$		0

4. Kernels of $(\mathbf{B} - \lambda\,\mathbf{I}_7)^{\mathbf{i}}$.

We remember that we are still working with the eigenvalue $\lambda = 2$. We calculate the kernel of $B - \lambda I_7$.

The matrix $B - \lambda I_7$ is

$$B - \lambda I_7 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & -2 & 1 & 2 & 1 \\ -3 & -1 & 0 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & -1 & -2 & -1 \\ 4 & 1 & -1 & 1 & -1 & -3 & -1 \\ -15 & -4 & 2 & -5 & 4 & 10 & 4 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & -2 & 1 & 2 & 1 \\ -3 & -1 & 0 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & -1 & -2 & -1 \\ 4 & 1 & -1 & 1 & -1 & -3 & -1 \\ -15 & -4 & 2 & -5 & 4 & 10 & 4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -z_1 = 0, \\ -2z_1 - z_2 - 2z_4 + z_5 + 2z_6 + z_7 = 0, \\ -3z_1 - z_2 - z_4 + z_5 + 2z_6 + z_7 = 0, \\ 3z_1 + z_2 + z_4 - z_5 - 2z_6 - z_7 = 0, \\ 4z_1 + z_2 - z_3 + z_4 - z_5 - 3z_6 - z_7 = 0, \\ -15z_1 - 4z_2 + 2z_3 - 5z_4 + 4z_5 + 10z_6 + 4z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_7) &= \{(0, r_2, r_3, 0, r_5, -r_3, r_2 + 2r_3 - r_5) \mid r_2, r_3, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(0,1,0,0,0,0,1), (0,0,1,0,0,-1,2), \\ &\qquad (0,0,0,0,1,0,-1)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda \mathbf{I_7})^2$.

We calculate the kernel of $(B - \lambda I_7)^2$.

The matrix $(B - \lambda I_7)^2$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 & 0 \\ 9 & 0 & -2 & 0 & 0 & -2 & 0 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 & 0 \\ 9 & 0 & -2 & 0 & 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} z_1 = 0, \\ -z_1 = 0, \\ -3z_1 + z_3 + z_6 = 0, \\ 9z_1 - 2z_3 - 2z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_7)^2 &= \{(0, r_2, r_3, r_4, r_5, -r_3, r_7) \mid r_2, r_3, r_4, r_5, r_7 \in \mathbf{K}\} \\ &= \operatorname{span}\{(0,1,0,0,0,0,0), (0,0,1,0,0,-1,0), \\ &\qquad (0,0,0,1,0,0,0), (0,0,0,0,1,0,0), \\ &\qquad (0,0,0,0,0,0,1)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\, \mathbf{I}_7)^3$.
We calculate the kernel of $(B - \lambda I_7)^3$.

The matrix $(B - \lambda I_7)^3$ is

$$(B-\lambda I_7)^3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & -1 & 0 & 0 & -1 & 0 \\ -11 & 0 & 2 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & -1 & 0 & 0 & -1 & 0 \\ -11 & 0 & 2 & 0 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -\; z_1 = 0, \\ -\; z_1 = 0, \\ z_1 = 0, \\ 4\,z_1 - z_3 - z_6 = 0, \\ -11 z_1 + 2 z_3 + 2 z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B-\lambda I_7)^3 &= \{(0, r_2, r_3, r_4, r_5, -r_3, r_7) \mid r_2, r_3, r_4, r_5, r_7 \in \mathbf{K}\} \\ &= \operatorname{span}\{(0,1,0,0,0,0,0), (0,0,1,0,0,-1,0), \\ &\qquad (0,0,0,1,0,0,0), (0,0,0,0,1,0,0), \\ &\qquad (0,0,0,0,0,0,1)\}. \end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the second power onwards in the following chain of inclusion of sets.

$$\ker(B-\lambda I_7) \subsetneq \ker((B-\lambda I_7)^2) = \ker((B-\lambda I_7)^3) = \cdots.$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B - \lambda I_7)$	3	$3 = \dim(\ker(B - \lambda I_7))$
$\ker(B - \lambda I_7)^2$	5	$2 = \dim(\ker(B - \lambda I_7)^2) - \dim(\ker(B - \lambda I_7))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B - \lambda I_7)^i)) - \dim(\ker((B - \lambda I_7)^{i-1}))$. The first number of this last column is $\dim(\ker(B - \lambda I_7))$.

5. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w_1}, \mathbf{w_2}\}$ satisfying

$$\begin{cases} (B - \lambda I_7)\mathbf{w_1} = \mathbf{0}, \\ (B - \lambda I_7)\mathbf{w_2} = \mathbf{w_1}. \end{cases}$$

where $\mathbf{w_2}$ is in the vector space $\ker((B - \lambda I_7)^2)$ but not in $\ker(B - \lambda I_7)$.

We look for a generating vector $\mathbf{w_2}$. This vector must be linearly independent of all vectors in $\ker(B - \lambda I_7)$ and must be independent from vectors that were already chosen in $\ker(B - \lambda I_7)^2$. We know that

$$\begin{aligned} \ker((B - \lambda I_7)^2) = \operatorname{span}\{&(0,1,0,0,0,0,0), (0,0,1,0,0,-1,0), \\ &(0,0,0,1,0,0,0), (0,0,0,0,1,0,0), \\ &(0,0,0,0,0,0,1)\}. \end{aligned}$$

We have at this point that no vector is already chosen in $\ker(B - \lambda I_7)^2$. We know that a vector in $\ker(B - \lambda I_7)^2$ must be of the form

$$\begin{aligned} a\,(0,1,0,0,0,0,0) + b\,(0,0,1,0,0,-1,0) + c\,(0,0,0,1,0,0,0) \\ + d\,(0,0,0,0,1,0,0) + e\,(0,0,0,0,0,0,1) \\ = (0,a,b,c,d,-b,e). \end{aligned}$$

We remember that

$$\ker(B - \lambda I_7) = \text{span}\{(0,1,0,0,0,0,1),(0,0,1,0,0,-1,2),\\ (0,0,0,0,1,0,-1)\}.$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & a & b & c & d & -b & e \end{pmatrix}.$$

We row reduce this matrix H and if we impose $c \neq 0$, we find

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & (-a-2b+d+e)/c \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $c \neq 0$. So we can take $a = 0$, $b = 0$, $c = 1$, $d = 0$, $e = 0$.

So we have the generating vector

$$\mathbf{w_2} = (0,0,0,1,0,0,0).$$

We start with $\mathbf{w_2} = (0,0,0,1,0,0,0)$.

We calculate $\mathbf{w_1}$.

$$\mathbf{w_1} = (B - \lambda I_7)\mathbf{w_2}$$
$$= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & -2 & 1 & 2 & 1 \\ -3 & -1 & 0 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & -1 & -2 & -1 \\ 4 & 1 & -1 & 1 & -1 & -3 & -1 \\ -15 & -4 & 2 & -5 & 4 & 10 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -1 \\ 0 \\ 1 \\ 1 \\ -5 \end{pmatrix}.$$

We have found the first Jordan chain. It has length 2.

$$\{\mathbf{w_1} = (0, -2, -1, 0, 1, 1, -5), \mathbf{w_2} = (0, 0, 0, 1, 0, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_7)$	3	$\mathbf{w_1}$	2
$\ker((B - \lambda I_7)^2)$	5	$\mathbf{w_2}$	1

with

$$\mathbf{w_1} = (0, -2, -1, 0, 1, 1, -5)$$
$$\mathbf{w_2} = (0, 0, 0, 1, 0, 0, 0)$$

Calculation of the second Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w_3}, \mathbf{w_4}\}$ satisfying

$$\begin{cases} (B - \lambda I_7)\,\mathbf{w_3} = \mathbf{0}, \\ (B - \lambda I_7)\,\mathbf{w_4} = \mathbf{w_3} \end{cases}$$

where $\mathbf{w_4}$ is in the vector space $\ker((B - \lambda I_7)^2)$ but not in $\ker(B - \lambda I_7)$.

We look for a generating vector $\mathbf{w_4}$. This vector must be linearly independent of all vectors in $\ker(B - \lambda I_7)$ and must be independent from vectors of height 2 that were already chosen in $\ker(B - \lambda I_7)^2$. We know that

$$\begin{aligned} \ker((B - \lambda I_7)^2) = \operatorname{span}\{&(0, 1, 0, 0, 0, 0, 0), (0, 0, 1, 0, 0, -1, 0), \\ &(0, 0, 0, 1, 0, 0, 0), (0, 0, 0, 0, 1, 0, 0), \\ &(0, 0, 0, 0, 0, 0, 1)\}. \end{aligned}$$

We have at this point chosen in $\ker((B - \lambda I_7)^2)$ already the vector $\mathbf{w_2} = (0, 0, 0, 1, 0, 0, 0)$ which is of height 2.

We remember that

$$\ker(B - \lambda I_7) = \operatorname{span}\{(0,1,0,0,0,0,1),(0,0,1,0,0,-1,2),\\ (0,0,0,0,1,0,-1)\}.$$

We know that a vector in $\ker((B - \lambda I_7)^2)$ must be of the form

$$\begin{aligned} &a\,(0,1,0,0,0,0,0) + b\,(0,0,1,0,0,-1,0) + c\,(0,0,0,1,0,0,0)\\ &\quad + d\,(0,0,0,0,1,0,0) + e\,(0,0,0,0,0,0,1)\\ &= (0,a,b,c,d,-b,e). \end{aligned}$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a & b & c & d & -b & e \end{pmatrix}.$$

We row reduce this matrix H and if we impose $a + 2b - d - e \neq 0$, we find

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + 2b - d - e \neq 0$. So we can choose $a = 1$, $b = 0$, $c = 0$, $d = 0$ and $e = 0$.

We have the generating vector

$$\mathbf{w_4} = (0,1,0,0,0,0,0).$$

We start with $\mathbf{w_4} = (0,1,0,0,0,0,0)$.

So we calculate

$$\mathbf{w}_3 = (B - \lambda I_7)\,\mathbf{w}_4$$
$$= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & -2 & 1 & 2 & 1 \\ -3 & -1 & 0 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & -1 & -2 & -1 \\ 4 & 1 & -1 & 1 & -1 & -3 & -1 \\ -15 & -4 & 2 & -5 & 4 & 10 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \\ 1 \\ -4 \end{pmatrix}.$$

We have found the second Jordan chain. It has length 2.

$$\{\mathbf{w}_3 = (0, -1, -1, 0, 1, 1, -4), \mathbf{w}_4 = (0, 1, 0, 0, 0, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(B - \lambda I_7)$	3	$\mathbf{w}_1$	$\mathbf{w}_3$	1
$\ker(B - \lambda I_7)^2$	5	$\mathbf{w}_2$	$\mathbf{w}_4$	0

with

$$\mathbf{w}_1 = (0, -2, -1, 0, 1, 1, -5)$$
$$\mathbf{w}_2 = (0, 0, 0, 1, 0, 0, 0)$$
$$\mathbf{w}_3 = (0, -1, -1, 0, 1, 1, -4)$$
$$\mathbf{w}_4 = (0, 1, 0, 0, 0, 0, 0)$$

Calculation of the third Jordan chain.
We look for a linearly independent vector $\{\mathbf{w}_5\}$ satisfying

$$(B - \lambda I_7)\,\mathbf{w_5} = \mathbf{0}.$$

This vector must be independent from the vectors of height 1 that were already chosen in $\ker(B - \lambda I_7)$.

We have at this point chosen in $\ker(B - \lambda I_7)$ already the vectors $\mathbf{w_1} = (0, -2, -1, 0, 1, 1, -5)$ and $\mathbf{w_3} = (0, -1, -1, 0, 1, 1, -4)$.

We know that a vector in $\ker(B - \lambda I_7)$ must be of the form

$$\begin{aligned} a\,(0,1,0,0,0,0,1) + b\,(0,0,1,0,0,-1,2) + c\,(0,0,0,0,1,0,-1) \\ = (0, a, b, 0, c, -b, a + 2\,b - c). \end{aligned}$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 0 & -2 & -1 & 0 & 1 & 1 & -5 \\ 0 & -1 & -1 & 0 & 1 & 1 & -4 \\ 0 & a & b & 0 & c & -b & a + 2\,b - c \end{pmatrix}.$$

We row reduce this matrix H and if we impose $b + c \neq 0$ we find

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $b + c \neq 0$. So we can choose $a = 0$, $b = 1$, $c = 0$.

So we have the generating vector

$$\mathbf{w_5} = (0, 0, 1, 0, 0, -1, 2).$$

This vector gives rise to a third Jordan. It has length 1.

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	chain 3	remaining dim
$\ker(B - \lambda I_7)$	3	$\mathbf{w_1}$	$\mathbf{w_3}$	$\mathbf{w_5}$	0
$\ker(B - \lambda I_7)^2$	5	$\mathbf{w_2}$	$\mathbf{w_4}$		0

with

$$\mathbf{w_1} = (0, -2, -1, 0, 1, 1, -5)$$
$$\mathbf{w_2} = (0, 0, 0, 1, 0, 0, 0)$$
$$\mathbf{w_3} = (0, -1, -1, 0, 1, 1, -4)$$
$$\mathbf{w_4} = (0, 1, 0, 0, 0, 0, 0)$$
$$\mathbf{w_5} = (0, 0, 1, 0, 0, -1, 2)$$

6. Investigation of the second eigenvalue.

We work now with the eigenvalue $\lambda = 1$.

7. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 8 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 7. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ 0 \quad 0 & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ 0 \quad 0 & 0 & 0 & (2) & 0 & 0 \\ 0 \quad 0 & 0 & 0 & 0 & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ 0 \quad 0 & 0 & 0 & 0 & \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_7.

$$A - \lambda I_7 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

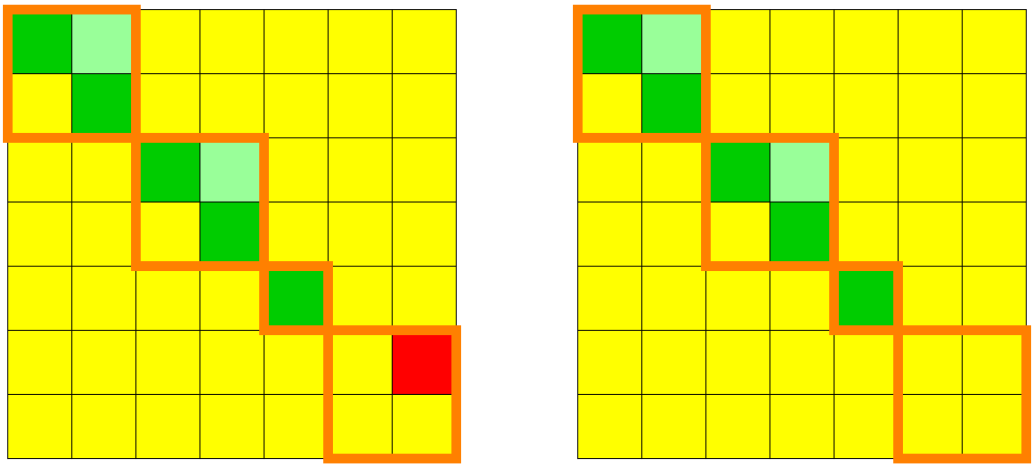
We see that the space span$\{\mathbf{e}_6, \mathbf{e}_7\}$ is invariant relative to $A - \lambda I_7$. If we restrict $A - \lambda I_7$ to this space, we have a nilpotent operator and we can apply the techniques learned in part 2.

We want to investigate the endomorphism $A - \lambda I_7$ restricted to this space. We compute now the powers of $A - \lambda I_7$.

$$A - \lambda I_7 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(A - \lambda I_7)^2 = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $(A - \lambda I_7)$. (b) Visualisation of $(A - \lambda I_7)^2$.

Figure 34

We give here some comments about the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero.

All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix. If we restrict the mapping $A - \lambda I_7$ to the subspace span$\{\mathbf{e}_6, \mathbf{e}_7\}$ which is an invariant subspace with respect to the operator $A - \lambda I_7$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 2. Let us take a look at the fourth elementary Jordan block in this matrix. We observe that the original superdiagonal of 1's in the fourth block of $A - \lambda I_7$ is going upwards in its elementary Jordan submatrix associated with the nilpotent part of the transformation $A - \lambda I_7$ when increasing the powers of the matrix $A - \lambda I_7$. They finally disappear when taking the second power of $A - \lambda I_7$.

Investigation of the first Jordan chain.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_7) = \operatorname{span}\{\mathbf{e}_6\}; \\ \ker(A - \lambda I_7)^2 = \operatorname{span}\{\mathbf{e}_6, \mathbf{e}_7\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_7)$	1	$1 = \dim(\ker(A - \lambda I_7))$
$\ker(A - \lambda I_7)^2$	2	$1 = \dim(\ker(A - \lambda I_7)^2) - \dim(\ker(A - \lambda I_7))$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A-\lambda I_7)^i)-\dim(\ker(A-\lambda I_7)^{i-1})$. In the first row, we have $\dim(\ker(A - \lambda I_7))$.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A - \lambda I_7) \subsetneq \ker(A - \lambda I_7)^2 = \ker(A - \lambda I_7)^3 = \cdots .$$

We remark by looking at the matrices $(A - \lambda I_7)^i$ that we have the following mappings

$$\begin{cases} (A - \lambda I_7)\,\mathbf{e}_7 = \mathbf{e}_6, \\ (A - \lambda I_7)\,\mathbf{e}_6 = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A - \lambda I_7)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases} (A - \lambda I_7)\,\mathbf{e}_7 = \mathbf{e}_6, \\ (A - \lambda I_7)^2\,\mathbf{e}_7 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_6 = (A - \lambda I_7)\,\mathbf{e}_7, \mathbf{e}_7\}.$$

After we have found the first Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A - \lambda I_7)$	1	$\mathbf{e_6}$	0
$\ker(A - \lambda I_7)^2$	1	$\mathbf{e_7}$	0

We have now in the last column only 0's and this means that we are done looking for Jordan chains.

8. Kernels of $(\mathbf{B} - \lambda\,\mathbf{I_7})^{\mathbf{i}}$.

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I_7})$.
We calculate the kernel of $B - \lambda\,I_7$.

The matrix $B - \lambda\,I_7$ is

$$B - \lambda\,I_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -2 & 1 & 2 & 1 \\ -3 & -1 & 1 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & -2 & -1 \\ 4 & 1 & -1 & 1 & -1 & -2 & -1 \\ -15 & -4 & 2 & -5 & 4 & 10 & 5 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -2 & 1 & 2 & 1 \\ -3 & -1 & 1 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & -2 & -1 \\ 4 & 1 & -1 & 1 & -1 & -2 & -1 \\ -15 & -4 & 2 & -5 & 4 & 10 & 5 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -2z_1 \quad -2z_4 + z_5 + 2z_6 + z_7 = 0, \\ -3z_1 - z_2 + z_3 - z_4 + z_5 + 2z_6 + z_7 = 0, \\ z_4 = 0, \\ 3z_1 + z_2 + z_4 - 2z_6 - z_7 = 0, \\ 4z_1 + z_2 - z_3 + z_4 - z_5 - 2z_6 - z_7 = 0, \\ -15z_1 - 4z_2 + 2z_3 - 5z_4 + 4z_5 + 10z_6 + 5z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_7) &= \{(0,0,0,0,0,r_6,-2r_6) \mid r_6 \in \mathbf{K}\} \\ &= \operatorname{span}\{(0,0,0,0,0,1,-2)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda \mathbf{I_7})^2$.
We calculate the kernel of $(B - \lambda I_7)^2$.

The matrix $(B - \lambda I_7)^2$ is

$$(B - \lambda I_7)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -1 & 0 & -4 & 2 & 4 & 2 \\ -5 & -2 & 1 & -2 & 2 & 4 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 5 & 2 & 0 & 2 & -1 & -4 & -2 \\ 5 & 2 & -1 & 2 & -2 & -4 & -2 \\ -21 & -8 & 2 & -10 & 8 & 18 & 9 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -1 & 0 & -4 & 2 & 4 & 2 \\ -5 & -2 & 1 & -2 & 2 & 4 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 5 & 2 & 0 & 2 & -1 & -4 & -2 \\ 5 & 2 & -1 & 2 & -2 & -4 & -2 \\ -21 & -8 & 2 & -10 & 8 & 18 & 9 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -4z_1 - z_2 - 4z_4 + 2z_5 + 4z_6 + 2z_7 = 0, \\ -5z_1 - 2z_2 + z_3 - 2z_4 + 2z_5 + 4z_6 + 2z_7 = 0, \\ z_4 = 0, \\ 5z_1 + 2z_2 + 2z_4 - z_5 - 4z_6 - 2z_7 = 0, \\ 5z_1 + 2z_2 - z_3 + 2z_4 - 2z_5 - 4z_6 - 2z_7 = 0, \\ -21z_1 - 8z_2 + 2z_3 - 10z_4 + 8z_5 + 18z_6 + 9z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_7)^2 &= \{(r_1, 0, r_1, 0, -r_1, r_6, 3r_1 - 2r_6) \mid r_1, r_6 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,1,0,-1,0,3), (0,0,0,0,0,1,-2)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I})^3$.
We calculate the kernel of $(B - \lambda I_7)^3$.

The matrix $(B - \lambda I_7)^3$ is

$$(B - \lambda I_7)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -2 & 0 & -6 & 3 & 6 & 3 \\ -7 & -3 & 1 & -3 & 3 & 6 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 7 & 3 & 0 & 3 & -2 & -6 & -3 \\ 7 & 3 & -1 & 3 & -3 & -6 & -3 \\ -29 & -12 & 2 & -15 & 12 & 26 & 13 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -2 & 0 & -6 & 3 & 6 & 3 \\ -7 & -3 & 1 & -3 & 3 & 6 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 7 & 3 & 0 & 3 & -2 & -6 & -3 \\ 7 & 3 & -1 & 3 & -3 & -6 & -3 \\ -29 & -12 & 2 & -15 & 12 & 26 & 13 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -6z_1 - 2z_2 - 6z_4 + 3z_5 + 6z_6 + 3z_7 = 0, \\ -7z_1 - 3z_2 + z_3 - 3z_4 + 3z_5 + 6z_6 + 3z_7 = 0, \\ z_4 = 0, \\ 7z_1 + 3z_2 + 3z_4 - 2z_5 - 6z_6 - 3z_7 = 0, \\ 7z_1 + 3z_2 - z_3 + 3z_4 - 3z_5 - 6z_6 - 3z_7 = 0, \\ -29z_1 - 12z_2 + 2z_3 - 15z_4 + 12z_5 + 26z_6 + 13z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_7)^3 &= \{(r_1, 0, r_1, 0, -r_1, r_6, 3r_1 - 2r_6) \mid r_1, r_6 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,1,0,-1,0,3), (0,0,0,0,0,1,-2)\}. \end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the second power onwards in the following chain of inclusion of sets.

$$\ker(B - \lambda I_7) \subsetneq \ker((B - \lambda I_7)^2) = \ker((B - \lambda I_7)^3) = \cdots.$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B - \lambda I_7)$	1	$1 = \dim(\ker(B - \lambda I_7))$
$\ker(B - \lambda I_7)^2$	2	$1 = \dim(\ker(B - \lambda I_7)^2) - \dim(\ker(B - \lambda I_7))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B - \lambda I_7)^i)) - \dim(\ker((B - \lambda I_7)^{i-1}))$. The first number of this last column is $\dim(\ker(B - \lambda I_7))$.

9. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w}_6, \mathbf{w}_7\}$ satisfying

$$\begin{cases}(B-\lambda\, I_7)\,\mathbf{w_6} = \mathbf{0},\\ (B-\lambda\, I_7)\,\mathbf{w_7} = \mathbf{w_6}.\end{cases}$$

where $\mathbf{w_7}$ is in the vector space $\ker((B-\lambda\, I_7)^2)$ but not in $\ker(B-\lambda\, I_7)$.

We look for a generating vector $\mathbf{w_7}$. This vector must be linearly independent of all vectors in $\ker(B-\lambda\, I_7)$ and must be independent from vectors that were already chosen in $\ker(B-\lambda\, I_7)^2$. We know that

$$\ker((B-\lambda\, I_7)^2) = \text{span}\{(1,0,1,0,-1,0,3),(0,0,0,0,0,1,-2)\}.$$

We know that a vector in $\ker(B-\lambda\, I_7)^2$ must be of the form

$$a\,((1,0,1,0,-1,0,3)) + b\,((0,0,0,0,0,1,-2)) = (a,0,a,0,-a,b,3\,a-2\,b).$$

We remember that

$$\ker(B-\lambda\, I_7) = \text{span}\{(0,0,0,0,0,1,-2)\}.$$

We have at this point not chosen any vector in $\ker(B-\lambda\, I_7)^2$.

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & -2\\ a & 0 & a & 0 & -a & b & 3\,a-2\,b\end{pmatrix}.$$

We row reduce this matrix H and if we impose $a \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 1 & 0 & -1 & 0 & 3\\ 0 & 0 & 0 & 0 & 0 & 1 & -2\end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a \neq 0$. So we can choose $a = 1$, $b = 0$.

We choose the generating vector

$$\mathbf{w_7} = 1\,(1,0,1,0,-1,0,3) + 0\,(0,0,0,0,0,1,-2) = (1,0,1,0,-1,0,3).$$

We start with $\mathbf{w_7} = (1,0,1,0,-1,0,3)$.

We calculate $\mathbf{w_6}$.

$$\mathbf{w_6} = (B - \lambda I_7)\,\mathbf{w_7}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -2 & 1 & 2 & 1 \\ -3 & -1 & 1 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & -2 & -1 \\ 4 & 1 & -1 & 1 & -1 & -2 & -1 \\ -15 & -4 & 2 & -5 & 4 & 10 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{pmatrix}.$$

So we have the Jordan chain

$$\{\mathbf{w_6} = (0,0,0,0,0,1,-2), \mathbf{w_7} = (1,0,1,0,-1,0,3)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_7)$	1	$\mathbf{w_6}$	0
$\ker(B - \lambda I_7)^2$	2	$\mathbf{w_7}$	0

with

$$\mathbf{w_6} = (0,0,0,0,0,1,-2)$$
$$\mathbf{w_7} = (1,0,1,0,-1,0,3)$$

10. Result and check of the result.

We construct the matrix of base change P with the coordinates of the vectors $\mathbf{w_i}$ found in the Jordan chains in the columns of P.

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 & 1 & 0 \\ -5 & 0 & -4 & 0 & 2 & -2 & 3 \end{pmatrix}.$$

$$
\begin{aligned}
A &= P^{-1}\,B\,P \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & -2 & -2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 & 1 \\ 2 & 1 & 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&\times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & -2 & 1 & 2 & 1 \\ -3 & -1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 1 & -2 & -1 \\ 4 & 1 & -1 & 1 & -1 & -1 & -1 \\ -15 & -4 & 2 & -5 & 4 & 10 & 6 \end{pmatrix} \\
&\times \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 & 1 & 0 \\ -5 & 0 & -4 & 0 & 2 & -2 & 3 \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{2} & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & \mathbf{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\mathbf{2}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} \end{pmatrix}.
\end{aligned}
$$

26 exercise. (7×7); $(\mathbf{J_4(1), J_1(1), J_2(-1)})$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} -2 & 1 & -2 & -1 & 0 & 0 & -2 \\ 1 & 1 & 2 & 0 & 0 & -1 & 1 \\ 2 & 0 & 2 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 2 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_7| = -(\lambda - 1)^5 (\lambda + 1)^2.$$

We see that the characteristic Cayley-Hamilton polynomial $p_{\text{C-H}}(\lambda)$ can be factorised in linear polynomials over the field $\mathbf{K}$. So we can apply the Jordan normalisation machinery.

The eigenvalue $\lambda = 1$ has algebraic multiplicity 5, The eigenvalue $\lambda = -1$ has algebraic multiplicity 2.

2. Investigation of the first eigenvalue.

We work now with the eigenvalue $\lambda = 1$.

3. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of

this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 4 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 3. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} (1) & 0 & 0 \\ 0 & \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_7.

$$A - \lambda I_7 = \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} (0) & 0 & 0 \\ 0 & \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

We compute also the powers of $A - \lambda I_7$.

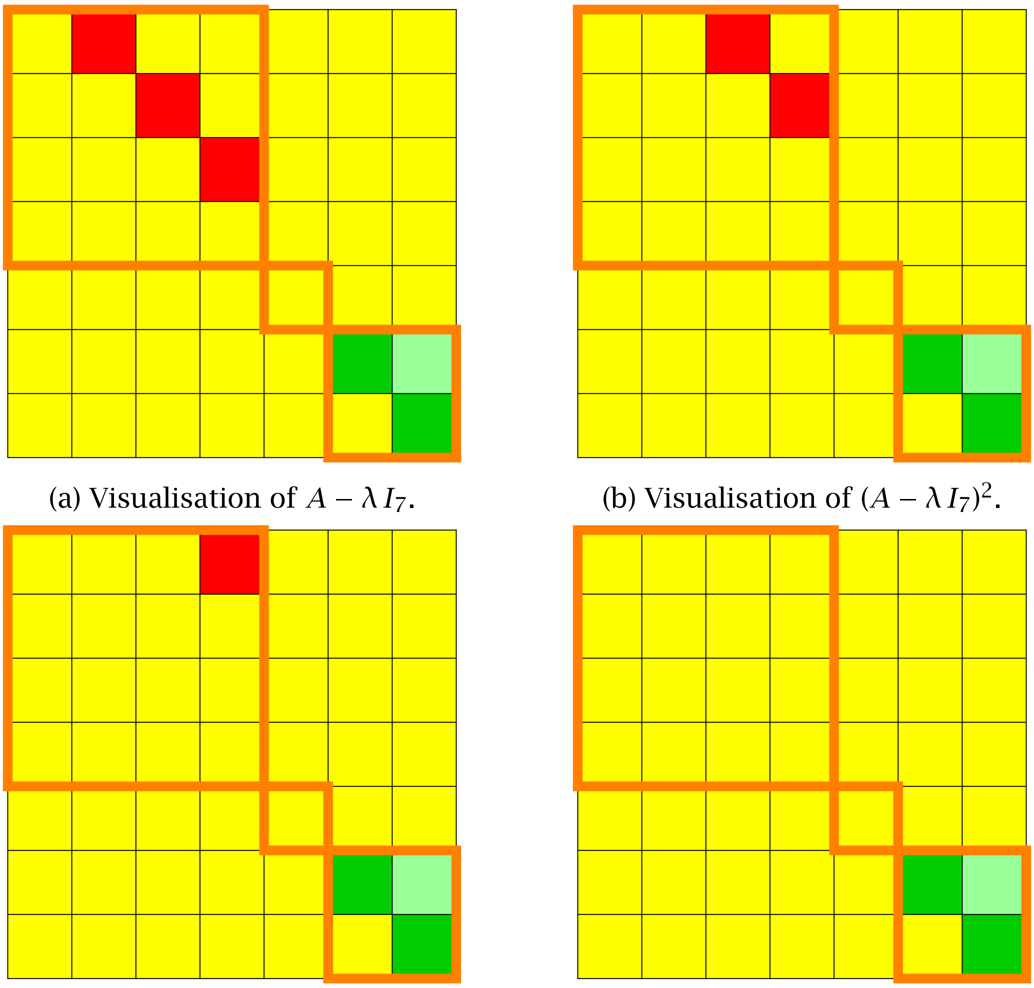
$$A - \lambda I_7 = \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} (0) & 0 & 0 \\ 0 & \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \end{matrix} \end{pmatrix},$$

$$(A-\lambda I_7)^2 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} (\mathbf{0}) & 0 & 0 \\ 0 & \begin{pmatrix} \mathbf{4} & \mathbf{-4} \\ \mathbf{0} & \mathbf{4} \end{pmatrix} \end{matrix} \end{pmatrix},$$

$$(A-\lambda I_7)^3 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} (\mathbf{0}) & 0 & 0 \\ 0 & \begin{pmatrix} \mathbf{-8} & \mathbf{12} \\ \mathbf{0} & \mathbf{-8} \end{pmatrix} \end{matrix} \end{pmatrix},$$

$$(A-\lambda I_7)^4 = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} (\mathbf{0}) & 0 & 0 \\ 0 & \begin{pmatrix} \mathbf{16} & \mathbf{-32} \\ \mathbf{0} & \mathbf{16} \end{pmatrix} \end{matrix} \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $A - \lambda I_7$.

(b) Visualisation of $(A - \lambda I_7)^2$.

(c) Visualisation of $(A - \lambda I_7)^3$.

(d) Visualisation of $(A - \lambda I_7)^4$.

Figure 35

We give some comments about the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. These cells keep their green colour however large we increase the exponents of $A - \lambda I_7$. The yellow cells are representing the number zero. The red cells are representing the number one. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix. If we restrict the mapping $A - \lambda I_7$ to the subspace span$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$ which is an invariant subspace with respect to the operator $A - \lambda I_7$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 4. Let us take a look at the two first elementary Jordan blocks in this matrix. We observe that the original superdiagonals of 1's in the two first blocks of $A - \lambda I_7$ are going upwards in their respective elementary Jordan blocks associated with the nilpotent part of the transformation $A - \lambda I_7$ when increasing the powers of the matrix $A - \lambda I_7$. They finally disappear when taking the fourth power of $A - \lambda I_7$.

Investigation of the first Jordan chain.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_7) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_5\}; \\ \ker(A - \lambda I_7)^2 = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5\}; \\ \ker(A - \lambda I_7)^3 = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_5\}; \\ \ker(A - \lambda I_7)^4 = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_7)$	2	$2 = \dim(\ker(A - \lambda I_7))$
$\ker(A - \lambda I_7)^2$	3	$1 = \dim(\ker(A - \lambda I_7)^2) - \dim(\ker(A - \lambda I_7))$
$\ker(A - \lambda I_7)^3$	4	$1 = \dim(\ker(A - \lambda I_7)^3) - \dim(\ker(A - \lambda I_7)^2)$
$\ker(A - \lambda I_7)^4$	5	$1 = \dim(\ker(A - \lambda I_7)^4) - \dim(\ker(A - \lambda I_7)^3)$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A - \lambda I_7)^i) - \dim(\ker(A - \lambda I_7)^{i-1})$. In the first row, we have $\dim(\ker(A - \lambda I_7))$.

We see also that there is equality in the inclusion of sets from the fourth power onwards.

$$\ker(A-\lambda I_7) \subsetneq \ker(A-\lambda I_7)^2 \subsetneq \ker(A-\lambda I_7)^3 \subsetneq \ker(A-\lambda I_7)^4 = \ker(A-\lambda I_7)^5 = \cdots .$$

We remark by looking at the matrices $(A-\lambda I_7)^i$ that we have the following mappings

$$\begin{cases} (A-\lambda I_7)\,\mathbf{e}_4 = \mathbf{e}_3, \\ (A-\lambda I_7)\,\mathbf{e}_3 = \mathbf{e}_2, \\ (A-\lambda I_7)\,\mathbf{e}_2 = \mathbf{e}_1, \\ (A-\lambda I_7)\,\mathbf{e}_1 = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A-\lambda I_7)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases} (A-\lambda I_7)\,\mathbf{e}_4 = \mathbf{e}_3, \\ (A-\lambda I_7)^2\,\mathbf{e}_4 = \mathbf{e}_2, \\ (A-\lambda I_7)^3\,\mathbf{e}_4 = \mathbf{e}_1, \\ (A-\lambda I_7)^4\,\mathbf{e}_4 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = (A-\lambda I_7)^3\,\mathbf{e}_4, \mathbf{e}_2 = (A-\lambda I_7)^2\,\mathbf{e}_4, \mathbf{e}_3 = (A-\lambda I_7)\,\mathbf{e}_4, \mathbf{e}_4\}.$$

After we have found the first Jordan chain of length 4, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A-\lambda I_7)$	2	$\mathbf{e}_1$	1
$\ker(A-\lambda I_7)^2$	3	$\mathbf{e}_2$	0
$\ker(A-\lambda I_7)^3$	4	$\mathbf{e}_3$	0
$\ker(A-\lambda I_7)^4$	5	$\mathbf{e}_4$	0

We have that the last number in the last column is not zero, and this means that there is still a chain left of length 1.

Investigating the second Jordan chain.
We remark by looking at the matrices $(A - \lambda I_7)^i$ that we have the following mapping

$$(A - \lambda I_7)\,\mathbf{e}_5 = \mathbf{0}.$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_5\}.$$

After we have found the second Jordan chain with length 1, we have then the following table.

Keeping track of chains and dimensions.

	dim	first chain	second chain	remaining dim
$\ker(A - \lambda I_7)$	2	$\mathbf{e}_1$	$\mathbf{e}_5$	0
$\ker(A - \lambda I_7)^2$	3	$\mathbf{e}_2$		0
$\ker(A - \lambda I_7)^3$	4	$\mathbf{e}_3$		0
$\ker(A - \lambda I_7)^4$	5	$\mathbf{e}_4$		0

The last column consists entirely of zero's and this means that we have finished investigating the first eigenvalue.

4. Kernels of $(\mathbf{B} - \lambda\,\mathbf{I}_7)^{\mathbf{i}}$.

We calculate the kernel of $B - \lambda I_7$.

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I}_7)$.
The matrix $B - \lambda I_7$ is

$$B-\lambda I_7 = \begin{pmatrix} -3 & 1 & -2 & -1 & 0 & 0 & -2 \\ 1 & 0 & 2 & 0 & 0 & -1 & 1 \\ 2 & 0 & 1 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -3 & 1 & -2 & -1 & 0 & 0 & -2 \\ 1 & 0 & 2 & 0 & 0 & -1 & 1 \\ 2 & 0 & 1 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -3z_1 + z_2 - 2z_3 - z_4 - 2z_7 = 0, \\ z_1 + 2z_3 - z_6 + z_7 = 0, \\ 2z_1 + z_3 + z_4 - z_6 + z_7 = 0, \\ z_1 - z_2 + z_4 + 2z_6 = 0, \\ z_1 + 2z_3 - z_6 + z_7 = 0, \\ z_1 + z_3 - 3z_6 + z_7 = 0, \\ - z_2 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B-\lambda I_7) &= \{(r_1, 0, 0, -r_1, r_5, 0, -r_1) \mid r_1, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,-1,0,0,-1), (0,0,0,0,1,0,0)\}. \end{aligned}$$

Kernel of $(\mathbf{B}-\lambda \mathbf{I_7})^2$.
We calculate the kernel of $(B-\lambda I_7)^2$.

The matrix $(B-\lambda I_7)^2$ is

$$(B-\lambda I_7)^2 = \begin{pmatrix} 5 & 0 & 6 & 0 & 0 & -1 & 5 \\ 0 & 0 & -1 & 1 & 0 & 1 & -1 \\ -4 & 0 & -4 & 0 & 0 & 4 & -4 \\ -1 & 0 & -2 & 0 & 0 & -3 & -1 \\ 0 & 0 & -1 & 1 & 0 & 1 & -1 \\ -4 & 0 & -4 & 0 & 0 & 8 & -4 \\ -1 & 0 & -2 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 5 & 0 & 6 & 0 & 0 & -1 & 5 \\ 0 & 0 & -1 & 1 & 0 & 1 & -1 \\ -4 & 0 & -4 & 0 & 0 & 4 & -4 \\ -1 & 0 & -2 & 0 & 0 & -3 & -1 \\ 0 & 0 & -1 & 1 & 0 & 1 & -1 \\ -4 & 0 & -4 & 0 & 0 & 8 & -4 \\ -1 & 0 & -2 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 5z_1 + 6z_3 - z_6 + 5z_7 = 0, \\ - z_3 + z_4 + z_6 - z_7 = 0, \\ -4z_1 - 4z_3 + 4z_6 - 4z_7 = 0, \\ - z_1 - 2z_3 - 3z_6 - z_7 = 0, \\ - z_3 + z_4 + z_6 - z_7 = 0, \\ -4z_1 - 4z_3 + 8z_6 - 4z_7 = 0, \\ - z_1 - 2z_3 + z_6 - z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B-\lambda I_7)^2 &= \{(r_1, r_2, 0, -r_1, r_5, 0, -r_1) \mid r_1, r_2, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,-1,0,0,-1), (0,1,0,0,0,0,0), \\ &\qquad (0,0,0,0,1,0,0)\}. \end{aligned}$$

Kernel of $(\mathbf{B}-\lambda\,\mathbf{I_7})^3$.
We calculate the kernel of $(B-\lambda I_7)^3$.

The matrix $(B-\lambda I_7)^3$ is

$$(B - \lambda I_7)^3 = \begin{pmatrix} -4 & 0 & -5 & 1 & 0 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 8 & 0 & 0 & -8 & 8 \\ -4 & 0 & -3 & -1 & 0 & 11 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 12 & 0 & 0 & -20 & 12 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -4 & 0 & -5 & 1 & 0 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 8 & 0 & 0 & -8 & 8 \\ -4 & 0 & -3 & -1 & 0 & 11 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 12 & 0 & 0 & -20 & 12 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -4z_1 - 5z_3 + z_4 - 3z_6 - 5z_7 = 0, \\ 8z_1 + 8z_3 - 8z_6 + 8z_7 = 0, \\ -4z_1 - 3z_3 - z_4 + 11z_6 - 3z_7 = 0, \\ 12z_1 + 12z_3 - 20z_6 + 12z_7 = 0, \\ z_3 - z_4 - z_6 + z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_7)^3 &= \{(r_1, r_2, r_3, -r_1, r_5, 0, -r_1 - r_3) \mid r_1, r_2, r_3, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,-1,0,0,-1), (0,1,0,0,0,0,0), \\ &\qquad (0,0,1,0,0,0,-1), (0,0,0,0,1,0,0)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda \mathbf{I}_7)^4$.
We calculate the kernel of $(B - \lambda I_7)^4$.

The matrix $(B - \lambda I_7)^4$ is

$$(B-\lambda I_7)^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -16 & 0 & -16 & 0 & 0 & 16 & -16 \\ 16 & 0 & 16 & 0 & 0 & -32 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -32 & 0 & -32 & 0 & 0 & 48 & -32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -16 & 0 & -16 & 0 & 0 & 16 & -16 \\ 16 & 0 & 16 & 0 & 0 & -32 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -32 & 0 & -32 & 0 & 0 & 48 & -32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 16z_6 = 0, \\ -16z_1 - 16z_3 + 16z_6 - 16z_7 = 0, \\ 16z_1 + 16z_3 - 32z_6 + 16z_7 = 0, \\ -32z_1 - 32z_3 + 48z_6 - 32z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B-\lambda I_7)^4 &= \{(r_1, r_2, r_3, r_4, r_5, 0, -r_1 - r_3) \mid r_1, r_2, r_3, r_4, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,0,0,-1), (0,1,0,0,0,0,0), \\ &\qquad (0,0,1,0,0,0,-1), (0,0,0,1,0,0,0), \\ &\qquad (0,0,0,0,1,0,0)\}. \end{aligned}$$

Kernel of $(\mathbf{B}-\lambda\,\mathbf{I}_7)^5$.
We calculate the kernel of $(B-\lambda I_7)^5$.

The matrix $(B-\lambda I_7)^5$ is

$$(B-\lambda I_7)^5 = \begin{pmatrix} 16 & 0 & 16 & 0 & 0 & -48 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 32 & 0 & 32 & 0 & 0 & -32 & 32 \\ -48 & 0 & -48 & 0 & 0 & 80 & -48 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 80 & 0 & 80 & 0 & 0 & -112 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 16 & 0 & 16 & 0 & 0 & -48 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 32 & 0 & 32 & 0 & 0 & -32 & 32 \\ -48 & 0 & -48 & 0 & 0 & 80 & -48 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 80 & 0 & 80 & 0 & 0 & -112 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 16z_1 + 16z_3 - 48z_6 + 16z_7 = 0, \\ 32z_1 + 32z_3 - 32z_6 + 32z_7 = 0, \\ -48z_1 - 48z_3 + 80z_6 - 48z_7 = 0, \\ 80z_1 + 80z_3 - 112z_6 + 80z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B-\lambda I_7)^5 &= \{(r_1, r_2, r_3, r_4, r_5, 0, -r_1 - r_3) \mid r_1, r_2, r_3, r_4, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,0,0,-1), (0,1,0,0,0,0,0), \\ &\qquad (0,0,1,0,0,0,-1), (0,0,0,1,0,0,0), \\ &\qquad (0,0,0,0,1,0,0)\}. \end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the fourth power onwards in the following chain of inclusion of sets.

$$\begin{aligned} \ker(B-\lambda I_7) &\subsetneq \ker((B-\lambda I_7)^2) \subsetneq \ker((B-\lambda I_7)^3) \\ &\subsetneq \ker((B-\lambda I_7)^4) = \ker((B-\lambda I_7)^5) \cdots . \end{aligned}$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B-\lambda I_7)$	2	$2 = \dim(\ker(B-\lambda I_7))$
$\ker(B-\lambda I_7)^2$	3	$1 = \dim(\ker(B-\lambda I_7)^2) - \dim(\ker(B-\lambda I_7))$
$\ker(B-\lambda I_7)^3$	4	$1 = \dim(\ker(B-\lambda I_7)^3) - \dim(\ker(B-\lambda I_7)^2)$
$\ker(B-\lambda I_7)^4$	5	$1 = \dim(\ker(B-\lambda I_7)^4) - \dim(\ker(B-\lambda I_7)^3)$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker(B-\lambda I_7)^i) - \dim(\ker(B-\lambda I_7)^{i-1})$. The first number of this last column is $\dim(\ker(B-\lambda I_7))$.

5. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ satisfying

$$\begin{cases} (B-\lambda I_7)\,\mathbf{w}_1 = \mathbf{0}, \\ (B-\lambda I_7)\,\mathbf{w}_2 = \mathbf{w}_1, \\ (B-\lambda I_7)\,\mathbf{w}_3 = \mathbf{w}_2, \\ (B-\lambda I_7)\,\mathbf{w}_4 = \mathbf{w}_3, \end{cases}$$

or

$$\begin{cases} (B-\lambda I_7)\,\mathbf{w}_4 = \mathbf{w}_3, \\ (B-\lambda I_7)^2\,\mathbf{w}_4 = \mathbf{w}_2, \\ (B-\lambda I_7)^3\,\mathbf{w}_4 = \mathbf{w}_1, \\ (B-\lambda I_7)^4\,\mathbf{w}_4 = \mathbf{0} \end{cases}$$

where $\mathbf{w}_4$ is in the vector space $\ker((B-\lambda I_7)^4)$ but not in $\ker((B-\lambda I_7)^3)$.

We look for a generating vector $\mathbf{w}_4$. This vector must be linearly independent of all vectors in $\ker(B-\lambda I_7)^3$ and must be independent from vectors of height 4 that were already chosen in $\ker(B-\lambda I_7)^4$. We know that

$$\ker((B - \lambda I_7)^3) = \text{span}\{(1,0,0,-1,0,0,-1),(0,1,0,0,0,0,0),\\ (0,0,1,0,0,0,-1),(0,0,0,0,1,0,0)\}.$$

We have at this point not chosen any vector in a previous Jordan chain of $\ker((B - \lambda I_7)^4)$.

We know that a vector in $\ker((B - \lambda I_7)^4)$ must be of the form

$$\begin{aligned} a\,(1,0,0,0,0,0,-1) + b\,(0,1,0,0,0,0,0) + c\,(0,0,1,0,0,0,-1)\\ + d\,(0,0,0,1,0,0,0) + e\,(0,0,0,0,1,0,0)\\ = (a,b,c,d,e,0,-a-c). \end{aligned}$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ a & b & c & d & e & 0 & -a-c \end{pmatrix}.$$

We row reduce this matrix H and if we impose $a + d \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + d \neq 0$. So we can chose $a = 1$, $b = 0$, $c = 0$, $d = 0$ and $e = 0$.

So we have the generating vector

$$\mathbf{w_4} = (1,0,0,0,0,0,-1).$$

We start with $\mathbf{w_4} = (1,0,0,0,0,0,-1)$.

We calculate

$$\mathbf{w_3} = (B - \lambda I_7)\,\mathbf{w_4}$$
$$= \begin{pmatrix} -3 & 1 & -2 & -1 & 0 & 0 & -2 \\ 1 & 0 & 2 & 0 & 0 & -1 & 1 \\ 2 & 0 & 1 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\mathbf{w_2} = (B - \lambda I_7)^2\,\mathbf{w_4}$$
$$= \begin{pmatrix} 5 & 0 & 6 & 0 & 0 & -1 & 5 \\ 0 & 0 & -1 & 1 & 0 & 1 & -1 \\ -4 & 0 & -4 & 0 & 0 & 4 & -4 \\ -1 & 0 & -2 & 0 & 0 & -3 & -1 \\ 0 & 0 & -1 & 1 & 0 & 1 & -1 \\ -4 & 0 & -4 & 0 & 0 & 8 & -4 \\ -1 & 0 & -2 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\mathbf{w_1} = (B - \lambda I_7)^3\,\mathbf{w_4}$$
$$= \begin{pmatrix} -4 & 0 & -5 & 1 & 0 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 8 & 0 & 0 & -8 & 8 \\ -4 & 0 & -3 & -1 & 0 & 11 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 12 & 0 & 0 & -20 & 12 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

So we have the Jordan chain

$$\{\mathbf{w}_1 = (1,0,0,-1,0,0,-1), \mathbf{w}_2 = (0,1,0,0,1,0,0),$$
$$\mathbf{w}_3 = (-1,0,1,1,0,0,0), \mathbf{w}_4 = (1,0,0,0,0,0,-1)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B-\lambda I_7)$	2	$\mathbf{w}_1$	1
$\ker(B-\lambda I_7)^2$	3	$\mathbf{w}_2$	0
$\ker(B-\lambda I_7)^3$	4	$\mathbf{w}_3$	0
$\ker(B-\lambda I_7)^4$	5	$\mathbf{w}_4$	0

with

$$\mathbf{w}_1 = (1,0,0,-1,0,0,-1)$$
$$\mathbf{w}_2 = (0,1,0,0,1,0,0)$$
$$\mathbf{w}_3 = (-1,0,1,1,0,0,0)$$
$$\mathbf{w}_4 = (1,0,0,0,0,0,-1)$$

Calculation of the second Jordan chain.
We look for a vector $\{\mathbf{w}_5\}$ where $\mathbf{w}_5$ is in the vector space $\ker(B-\lambda I_7)$ and must be linearly independent from vectors of height 1 that were already chosen in $\ker(B-\lambda I_7)$, and that is $\mathbf{w}_1 = (1,0,0,-1,0,0,-1)$.

We remember that

$$\ker(B-\lambda I_7) = \text{span}\{(1,0,0,-1,0,0,-1), (0,0,0,0,1,0,0)\}.$$

We know that a vector in $\ker(B-\lambda I_7)$ must be of the form

$$a\,(1,0,0,-1,0,0,-1)+b\,(0,0,0,0,1,0,0)=(a,0,0,-a,b,0,-a).$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H=\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ a & 0 & 0 & -a & b & 0 & -a \end{pmatrix}.$$

We row reduce this matrix H and if we impose $b\neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We see that these vectors are linearly independent if we impose the condition $b\neq 0$. We can choose $a=0$ and $b=1$. We have the generating vector

$$\mathbf{w}_5=(0,0,0,0,1,0,0).$$

So we have a second Jordan chain and the length of it is 1.

$$\{\mathbf{w}_5=(0,0,0,0,1,0,0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(B-\lambda I_7)$	2	$\mathbf{w}_1$	$\mathbf{w}_5$	0
$\ker(B-\lambda I_7)^2$	3	$\mathbf{w}_2$		0
$\ker(B-\lambda I_7)^3$	4	$\mathbf{w}_3$		0
$\ker(B-\lambda I_7)^4$	5	$\mathbf{w}_4$		0

with

$$\mathbf{w_1} = (1, 0, 0, -1, 0, 0, -1)$$
$$\mathbf{w_2} = (0, 1, 0, 0, 1, 0, 0)$$
$$\mathbf{w_3} = (-1, 0, 1, 1, 0, 0, 0)$$
$$\mathbf{w_4} = (1, 0, 0, 0, 0, 0, -1)$$
$$\mathbf{w_5} = (0, 0, 0, 0, 1, 0, 0)$$

6. Investigation of the second eigenvalue.

We work now with the eigenvalue $\lambda = -1$.

7. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 8 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 7. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_7.

$$A - \lambda I_7 = \begin{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} (2) & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

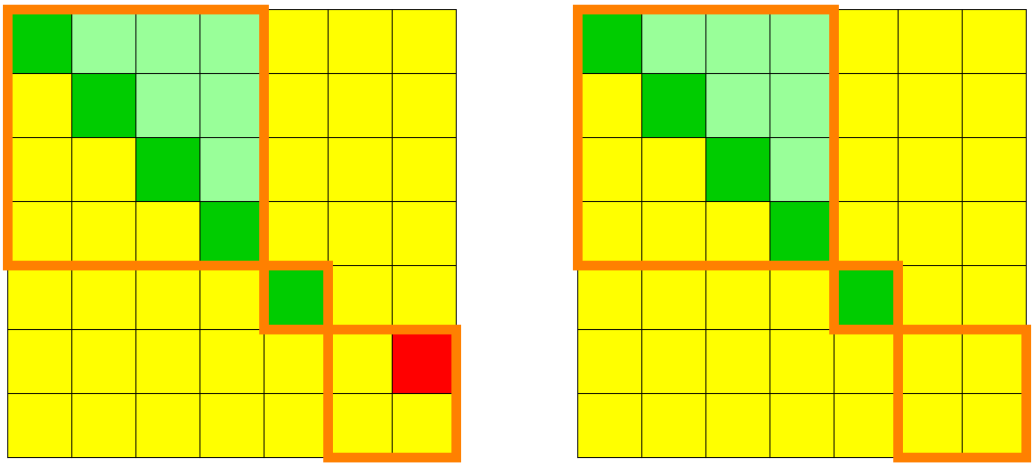
We see that the space $\operatorname{span}\{\mathbf{e}_6, \mathbf{e}_7\}$ is invariant relative to $A - \lambda I_7$. If we restrict $A - \lambda I_7$ to this space, we have a nilpotent operator and we can apply the techniques learned in part 2.

We want to investigate the endomorphism $A - \lambda I_7$ restricted to this space. We compute now the powers of $A - \lambda I_7$. We remember that $\lambda = -1$.

$$A - \lambda I_7 = \begin{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} (2) & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{matrix} \end{pmatrix},$$

$$(A - \lambda I_7)^2 = \begin{pmatrix} \begin{pmatrix} 4 & 4 & 1 & 0 \\ 0 & 4 & 4 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} (4) & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $(A - \lambda I_7)$. (b) Visualisation of $(A - \lambda I_7)^2$.

Figure 36

We give here some comments about the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells are representing the number one. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix.

If we restrict the mapping $A - \lambda I_7$ to the subspace $\text{span}\{\mathbf{e}_6, \mathbf{e}_7\}$ which is an invariant subspace with respect to the operator $A - \lambda I_7$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 2.

Let us take a look at the third elementary Jordan block in this matrix. We observe that the original superdiagonal of 1's in the last Jordan block of $A - \lambda I_7$ is going upwards in its elementary Jordan block associated with the nilpotent part of the transformation $A - \lambda I_7$ when increasing the powers of the matrix $A - \lambda I_7$. They finally disappear when taking the second power of $A - \lambda I_7$.

Investigation of the first Jordan chain.
Let us take a look at the third elementary Jordan block in this matrix.

We see that the space $\text{span}\{\mathbf{e}_6, \mathbf{e}_7\}$ is invariant with respect to the operator $A - \lambda I_7$.

We observe that the original superdiagonal of 1's in the third elementary Jordan block of the matrix $A - \lambda I_7$ is going upwards in its block in the powers of the matrix $A - \lambda I_7$ until it finally disappears when taking the second power of A.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_7) = \operatorname{span}\{\mathbf{e}_6\}; \\ \ker(A - \lambda I_7)^2 = \operatorname{span}\{\mathbf{e}_6, \mathbf{e}_7\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_7)$	1	$1 = \dim(\ker(A - \lambda I_7))$
$\ker(A - \lambda I_7)^2$	2	$1 = \dim(\ker(A - \lambda I_7)^2) - \dim(\ker(A - \lambda I_7))$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A-\lambda I_7)^i) - \dim(\ker(A-\lambda I_7)^{i-1})$. In the first row, we have $\dim(\ker(A - \lambda I_7))$.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A - \lambda I_7) \subsetneq \ker(A - \lambda I_7)^2 = \ker(A - \lambda I_7)^3 = \cdots .$$

We remark by looking at the matrices $(A - \lambda I_7)^i$ that we have the following mappings

$$\begin{cases} (A - \lambda I_7)\,\mathbf{e}_7 = \mathbf{e}_6, \\ (A - \lambda I_7)\,\mathbf{e}_6 = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A - \lambda I_7)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases}(A-\lambda I_7)\,\mathbf{e}_7=\mathbf{e}_6,\\(A-\lambda I_7)^2\,\mathbf{e}_7=\mathbf{0}.\end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_6=(A-\lambda I_7)\,\mathbf{e}_7,\mathbf{e}_7\}.$$

After we have found the first Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A-\lambda I_7)$	1	$\mathbf{e}_6$	0
$\ker(A-\lambda I_7)^2$	2	$\mathbf{e}_7$	0

We have now in the last column only 0's and this means that we are done looking for Jordan chains.

8. Kernels of $(B-\lambda I_7)^i$.

We remember that we are working here with $\lambda=-1$.

Kernel of $(\mathbf{B}-\lambda\,\mathbf{I}_7)$.

We calculate the kernel of $B-\lambda I_7$.

The matrix $B-\lambda I_7$ is

$$B-\lambda I_7=\begin{pmatrix}-1&1&-2&-1&0&0&-2\\1&2&2&0&0&-1&1\\2&0&3&1&0&-1&1\\1&-1&0&3&0&2&0\\1&0&2&0&2&-1&1\\1&0&1&0&0&-1&1\\0&-1&0&0&0&0&2\end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -1 & 1 & -2 & -1 & 0 & 0 & -2 \\ 1 & 2 & 2 & 0 & 0 & -1 & 1 \\ 2 & 0 & 3 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 3 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 2 & -1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -z_1 + z_2 - 2z_3 - z_4 - 2z_7 = 0, \\ z_1 + 2z_2 + 2z_3 - z_6 + z_7 = 0, \\ 2z_1 + 3z_3 + z_4 - z_6 + z_7 = 0, \\ z_1 - z_2 + 3z_4 + 2z_6 = 0, \\ z_1 + 2z_3 + 2z_5 - z_6 + z_7 = 0, \\ z_1 + z_3 - z_6 + z_7 = 0, \\ -z_2 + 2z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_7) &= \{(r_1, 0, 0, -r_1, 0, r_1, 0) \mid r_1 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, 0, 0, -1, 0, 1, 0)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda \mathbf{I_7})^2$.
We calculate the kernel of $(B - \lambda I_7)^2$.

The matrix $(B - \lambda I_7)^2$ is

$$(B - \lambda I_7)^2 = \begin{pmatrix} -3 & 4 & -2 & -4 & 0 & -1 & -3 \\ 4 & 4 & 7 & 1 & 0 & -3 & 3 \\ 4 & 0 & 4 & 4 & 0 & 0 & 0 \\ 3 & -4 & -2 & 8 & 0 & 5 & -1 \\ 4 & 0 & 7 & 1 & 4 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -4 & -2 & 0 & 0 & 1 & 3 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -3 & 4 & -2 & -4 & 0 & -1 & -3 \\ 4 & 4 & 7 & 1 & 0 & -3 & 3 \\ 4 & 0 & 4 & 4 & 0 & 0 & 0 \\ 3 & -4 & -2 & 8 & 0 & 5 & -1 \\ 4 & 0 & 7 & 1 & 4 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -4 & -2 & 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -3z_1 + 4z_2 - 2z_3 - 4z_4 - z_6 - 3z_7 = 0, \\ 4z_1 + 4z_2 + 7z_3 + z_4 - 3z_6 + 3z_7 = 0, \\ 4z_1 + 4z_3 + 4z_4 = 0, \\ 3z_1 - 4z_2 - 2z_3 + 8z_4 + 5z_6 - z_7 = 0, \\ 4z_1 + 7z_3 + z_4 + 4z_5 - 3z_6 + 3z_7 = 0, \\ - z_1 - 4z_2 - 2z_3 + z_6 + 3z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_7)^2 &= \{(r_1, 0, r_3, -r_1 - r_3, 0, r_1 + 2r_3, 0) \mid r_1, r_3 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, 0, 0, -1, 0, 1, 0), (0, 0, 1, -1, 0, 2, 0)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda \mathbf{I}_7)^3$.
We calculate the kernel of $(B - \lambda I_7)^3$.

The matrix $(B - \lambda I_7)^3$ is

$$(B - \lambda I_7)^3 = \begin{pmatrix} -3 & 4 & -2 & -4 & 0 & -1 & -3 \\ 4 & 4 & 7 & 1 & 0 & -3 & 3 \\ 4 & 0 & 4 & 4 & 0 & 0 & 0 \\ 3 & -4 & -2 & 8 & 0 & 5 & -1 \\ 4 & 0 & 7 & 1 & 4 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -4 & -2 & 0 & 0 & 1 & 3 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -3 & 4 & -2 & -4 & 0 & -1 & -3 \\ 4 & 4 & 7 & 1 & 0 & -3 & 3 \\ 4 & 0 & 4 & 4 & 0 & 0 & 0 \\ 3 & -4 & -2 & 8 & 0 & 5 & -1 \\ 4 & 0 & 7 & 1 & 4 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -4 & -2 & 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -3z_1 + 4z_2 - 2z_3 - 4z_4 - z_6 - 3z_7 = 0, \\ 4z_1 + 4z_2 + 7z_3 + z_4 - 3z_6 + 3z_7 = 0, \\ 4z_1 + 4z_3 + 4z_4 = 0, \\ 3z_1 - 4z_2 - 2z_3 + 8z_4 + 5z_6 - z_7 = 0, \\ 4z_1 + 7z_3 + z_4 + 4z_5 - 3z_6 + 3z_7 = 0, \\ -z_1 - 4z_2 - 2z_3 + z_6 + 3z_7 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_7)^3 &= \{(r_1, 0, r_3, -r_1 - r_3, 0, r_1 + 2r_3, 0) \mid r_1, r_3 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,-1,0,1,0), (0,0,1,-1,0,2,0)\}. \end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the second power onwards in the following chain of inclusion of sets.

$$\ker(B - \lambda I_7) \subsetneq \ker((B - \lambda I_7)^2) = \ker((B - \lambda I_7)^3) = \cdots.$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B - \lambda I_7)$	1	$1 = \dim(\ker(B - \lambda I_7))$
$\ker(B - \lambda I_7)^2$	2	$1 = \dim(\ker(B - \lambda I_7)^2) - \dim(\ker(B - \lambda I_7))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B-\lambda I_7)^i)) - \dim(\ker((B-\lambda I_7)^{i-1}))$. The first number of this last column is $\dim(\ker(B-\lambda I_7))$.

9. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w_6}, \mathbf{w_7}\}$ satisfying

$$\begin{cases} (B-\lambda I_7)\,\mathbf{w_6} = \mathbf{0}, \\ (B-\lambda I_7)\,\mathbf{w_7} = \mathbf{w_6}, \end{cases}$$

or

$$\begin{cases} (B-\lambda I_7)^2\,\mathbf{w_7} = \mathbf{0}, \\ (B-\lambda I_7)\,\mathbf{w_7} = \mathbf{w_6} \end{cases}$$

where $\mathbf{w_7}$ is in the vector space $\ker((B-\lambda I_7)^2)$ but not in $\ker(B-\lambda I_7)$.

We look for a generating vector $\mathbf{w_7}$. This vector must be linearly independent of all vectors in $\ker(B-\lambda I_7)$ and must be independent from vectors of height 2 that were already chosen in $\ker(B-\lambda I_7)^2$. We know that

$$\ker((B-\lambda I_7)^2) = \text{span}\{(1,0,0,-1,0,1,0),(0,0,1,-1,0,2,0)\}.$$

We have at this point not chosen any vector in a previous Jordan chain of of height 2 in $\ker((B-\lambda I_7)^2)$.

We know that a vector in $\ker((B-\lambda I_7)^2)$ must be of the form

$$a\,(1,0,0,-1,0,1,0) + b\,(0,0,1,-1,0,2,0) = (a,0,b,-a-b,0,a+2\,b,0).$$

We remember that

$$\ker(B-\lambda I_7) = \text{span}\{(1,0,0,-1,0,1,0)\}.$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ a & 0 & b & -a-b & 0 & a+2\,b & 0 \end{pmatrix}.$$

We row reduce this matrix H and if we impose $b \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 & 0 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $b \neq 0$. So we have $a = 0$ and $b = 1$.

We choose the generating vector

$$\mathbf{w_7} = (0, 0, 1, -1, 0, 2, 0).$$

We start with $\mathbf{w_7} = (0, 0, 1, -1, 0, 2, 0)$.

We calculate $\mathbf{w_6}$.

$$\begin{aligned} \mathbf{w_6} &= (B - \lambda I_7)\,\mathbf{w_7} \\ &= \begin{pmatrix} -1 & 1 & -2 & -1 & 0 & 0 & -2 \\ 1 & 2 & 2 & 0 & 0 & -1 & 1 \\ 2 & 0 & 3 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 3 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 2 & -1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \end{aligned}$$

We have now the Jordan chain

$$\{\mathbf{w_6} = (-1, 0, 0, 1, 0, -1, 0), \mathbf{w_7} = (0, 0, 1, -1, 0, 2, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_7)$	1	$\mathbf{w_6}$	0
$\ker(B - \lambda I_7)^2$	2	$\mathbf{w_7}$	0

with

$$\mathbf{w_6} = (-1, 0, 0, 1, 0, -1, 0)$$

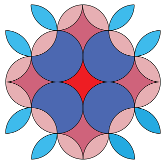
$$\mathbf{w_7} = (0, 0, 1, -1, 0, 2, 0)$$

10. Result and check of the result.

We construct the matrix of base change P with the coordinates of the vectors $\mathbf{w_i}$ found in the Jordan chains in the columns of P.

$$P = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

$$
\begin{aligned}
A &= P^{-1} B P \\
&= \begin{pmatrix}
0 & 0 & 1 & -1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & -1 & 1 \\
0 & 0 & -1 & 1 & 0 & 1 & -1 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 \\
-2 & 0 & -2 & 0 & 0 & 1 & -2 \\
-1 & 0 & -1 & 0 & 0 & 1 & -1
\end{pmatrix} \\
&\quad \times \begin{pmatrix}
-2 & 1 & -2 & -1 & 0 & 0 & -2 \\
1 & 1 & 2 & 0 & 0 & -1 & 1 \\
2 & 0 & 2 & 1 & 0 & -1 & 1 \\
1 & -1 & 0 & 2 & 0 & 2 & 0 \\
1 & 0 & 2 & 0 & 1 & -1 & 1 \\
1 & 0 & 1 & 0 & 0 & -2 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \\
&\quad \times \begin{pmatrix}
1 & 0 & -1 & 1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
-1 & 0 & 1 & 0 & 0 & 1 & -1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix} \\
&= \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} \\
&= \begin{pmatrix}
\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\
\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} (\mathbf{1}) & 0 & 0 \\ 0 & \begin{pmatrix} \mathbf{-1} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} \\ 0 & & \end{matrix}
\end{pmatrix}.
\end{aligned}
$$



27 exercise. (6×6); $(\mathbf{J}_3(-1), \mathbf{J}_1(-1), \mathbf{J}_2(1))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} -1 & -2 & 3 & 2 & -4 & 2 \\ 0 & -1 & -4 & -3 & -2 & -3 \\ 0 & 0 & 2 & 2 & 1 & 2 \\ 0 & 4 & -2 & -4 & 4 & -3 \\ 0 & 0 & 3 & 2 & 0 & 2 \\ 0 & -8 & 0 & 3 & -10 & 2 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_6| = (\lambda - 1)^2 (\lambda + 1)^4 = 0.$$

We see that the characteristic Cayley-Hamilton polynomial $p_{\text{C-H}}(\lambda)$ can be factorised in linear polynomials over the field $\mathbf{K}$. So we can apply the Jordan normalisation machinery.

The eigenvalue $\lambda = -1$ has algebraic multiplicity 4. The eigenvalue $\lambda = 1$ has algebraic multiplicity 2.

2. Investigation of the first eigenvalue.

We work now with the eigenvalue $\lambda = -1$.

3. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or

calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 4 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 3. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (-1) & 0 & 0 \\ 0 & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_6.

$$A - \lambda I_6 = \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (0) & 0 & 0 \\ 0 & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

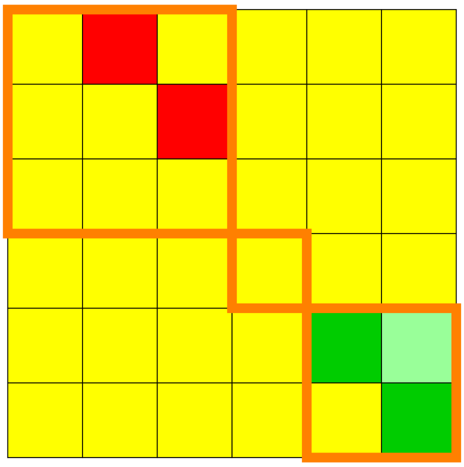
We compute also the powers of $A - \lambda I_6$.

$$A - \lambda I_6 = \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (0) & 0 & 0 \\ 0 & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{matrix} \end{pmatrix},$$

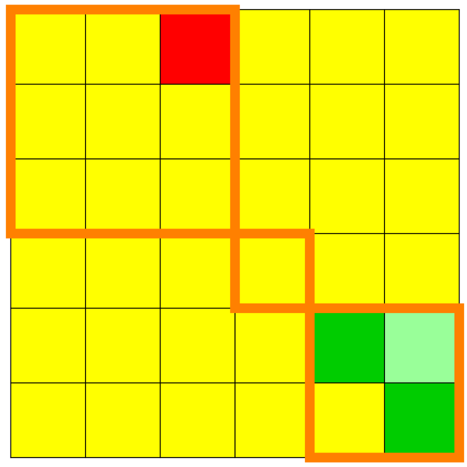
$$(A - \lambda I_6)^2 = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (0) & 0 & 0 \\ 0 & \begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix} \end{matrix} \end{pmatrix},$$

$$(A - \lambda I_6)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (0) & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 12 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{pmatrix}.$$

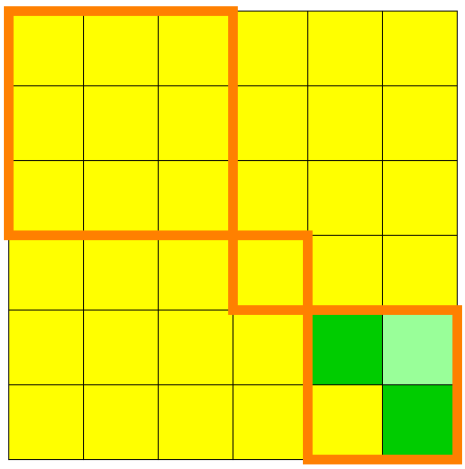
Let us visualise this situation.



(a) Visualisation of $A - \lambda I_6$.



(b) Visualisation of $(A - \lambda I_6)^2$.



(c) Visualisation of $(A - \lambda I_6)^3$.

Figure 37

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells are representing the number 1. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix.

If we restrict the mapping $A - \lambda I_6$ to the subspace $\operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ which is an invariant subspace with respect to the operator $A - \lambda I_6$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 3. Let us take a look at the two first elementary Jordan blocks in this matrix. We observe that the original superdiagonals of 1's in the two first blocks of $A - \lambda I_6$ are going upwards in their respective elementary Jordan blocks associated with the nilpotent part of the transformation $A - \lambda I_6$ when increasing the powers of the matrix $A - \lambda I_6$. They finally disappear when taking the third power of $A - \lambda I_6$.

Investigation of the first Jordan chain.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_6) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_4\}; \\ \ker(A - \lambda I_6)^2 = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4\}; \\ \ker(A - \lambda I_6)^3 = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_6)$	2	$2 = \dim(\ker(A - \lambda I_6))$
$\ker(A - \lambda I_6)^2$	3	$1 = \dim(\ker(A - \lambda I_6)^2) - \dim(\ker(A - \lambda I_6))$
$\ker(A - \lambda I_6)^3$	4	$1 = \dim(\ker(A - \lambda I_6)^3) - \dim(\ker(A - \lambda I_6)^2)$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A-\lambda I_6)^i)-\dim(\ker(A-\lambda I_6)^{i-1})$. In the first row, we have $\dim(\ker(A-\lambda I_6))$.

We see also that there is equality in the inclusion of sets from the third power onwards

$$\ker(A-\lambda I_6) \subsetneq \ker(A-\lambda I_6)^2 \subsetneq \ker(A-\lambda I_6)^3 = \ker(A-\lambda I_6)^4 = \cdots.$$

We remark by looking at the matrices $(A-\lambda I_6)^i$ that we have the following mappings

$$\begin{cases}(A-\lambda I_6)\,\mathbf{e}_3 = \mathbf{e}_2,\\ (A-\lambda I_6)\,\mathbf{e}_2 = \mathbf{e}_1,\\ (A-\lambda I_6)\,\mathbf{e}_1 = \mathbf{0}.\end{cases}$$

We remark by looking at the matrices $(A-\lambda I_6)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases}(A-\lambda I_6)\,\mathbf{e}_3 = \mathbf{e}_2,\\ (A-\lambda I_6)^2\,\mathbf{e}_3 = \mathbf{e}_1.\\ (A-\lambda I_6)^3\,\mathbf{e}_3 = \mathbf{0}.\end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = (A-\lambda I_6)^2\,\mathbf{e}_3, \mathbf{e}_2 = (A-\lambda I_6)\,\mathbf{e}_3, \mathbf{e}_3\}.$$

After we have found the first Jordan chain of length 4, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A-\lambda I_6)$	2	$\mathbf{e}_1$	1
$\ker(A-\lambda I_6)^2$	3	$\mathbf{e}_2$	0
$\ker(A-\lambda I_6)^3$	4	$\mathbf{e}_3$	0

We have that the last number in the last column is not zero, and this means that there is still a chain left of length 1.

Investigating the second Jordan chain.
We remark by looking at the matrices $(A - \lambda I_6)^i$ that we have the following mapping

$$(A - \lambda I_6)\,\mathbf{e_4} = \mathbf{0}.$$

We see that we have a second Jordan chain and its length is one.

$$\{\mathbf{e_4}\}.$$

After we have found this second Jordan chain of length 1, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(A - \lambda I_6)$	2	$\mathbf{e_1}$	$\mathbf{e_4}$	0
$\ker(A - \lambda I_6)^2$	3	$\mathbf{e_2}$		0
$\ker(A - \lambda I_6)^3$	4	$\mathbf{e_3}$		0

4. Kernels of $(\mathbf{B} - \lambda\,\mathbf{I_6})^{\mathbf{i}}$.

We calculate the kernel of $B - \lambda I_6$.

Kernel of $\mathbf{B} - \lambda\,\mathbf{I_6}$.
The matrix $B - \lambda I_6$ is

$$B - \lambda I_6 = \begin{pmatrix} 0 & -2 & 3 & 2 & -4 & 2 \\ 0 & 0 & -4 & -3 & -2 & -3 \\ 0 & 0 & 3 & 2 & 1 & 2 \\ 0 & 4 & -2 & -3 & 4 & -3 \\ 0 & 0 & 3 & 2 & 1 & 2 \\ 0 & -8 & 0 & 3 & -10 & 3 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & -2 & 3 & 2 & -4 & 2 \\ 0 & 0 & -4 & -3 & -2 & -3 \\ 0 & 0 & 3 & 2 & 1 & 2 \\ 0 & 4 & -2 & -3 & 4 & -3 \\ 0 & 0 & 3 & 2 & 1 & 2 \\ 0 & -8 & 0 & 3 & -10 & 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -2z_2 + 3z_3 + 2z_4 - 4z_5 + 2z_6 = 0, \\ -4z_3 - 3z_4 - 2z_5 - 3z_6 = 0, \\ 3z_3 + 2z_4 + z_5 + 2z_6 = 0, \\ 4z_2 - 2z_3 - 3z_4 + 4z_5 - 3z_6 = 0, \\ 3z_3 + 2z_4 + z_5 + 2z_6 = 0, \\ -8z_2 + 3z_4 - 10z_5 + 3z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_6) &= \{(r_1, 0, 0, r_4, 0, -r_4) \mid r_1, r_4 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,0,0), (0,0,0,1,0,-1)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda \mathbf{I_6})^2$.
We calculate the kernel of $(B - \lambda I_6)^2$.

The matrix $(B - \lambda I_6)^2$ is

$$(B - \lambda I_6)^2 = \begin{pmatrix} 0 & -8 & 1 & 4 & -9 & 4 \\ 0 & 12 & -12 & -12 & 12 & -12 \\ 0 & -8 & 8 & 8 & -8 & 8 \\ 0 & 12 & -4 & -8 & 12 & -8 \\ 0 & -8 & 8 & 8 & -8 & 8 \\ 0 & -12 & -4 & 4 & -12 & 4 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & -8 & 1 & 4 & -9 & 4 \\ 0 & 12 & -12 & -12 & 12 & -12 \\ 0 & -8 & 8 & 8 & -8 & 8 \\ 0 & 12 & -4 & -8 & 12 & -8 \\ 0 & -8 & 8 & 8 & -8 & 8 \\ 0 & -12 & -4 & 4 & -12 & 4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -8z_2 + z_3 + 4z_4 - 9z_5 + 4z_6 = 0, \\ 12z_2 - 12z_3 - 12z_4 + 12z_5 - 12z_6 = 0, \\ -8z_2 + 8z_3 + 8z_4 - 8z_5 + 8z_6 = 0, \\ 12z_2 - 4z_3 - 8z_4 + 12z_5 - 8z_6 = 0, \\ -8z_2 + 8z_3 + 8z_4 - 8z_5 + 8z_6 = 0, \\ -12z_2 - 4z_3 + 4z_4 - 12z_5 + 4z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_6)^2 &= \{(r_1, r_2, -r_2/2, r_4, -r_2/2, r_2 - r_4) \mid r_1, r_2, r_4 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,0,0), (0,1,-1/2,0,-1/2,1), \\ &\qquad (0,0,0,1,0,-1)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda \mathbf{I_6})^3$.
We calculate the kernel of $(B - \lambda I_6)^3$.

The matrix $(B - \lambda I_6)^3$ is

$$(B - \lambda I_6)^3 = \begin{pmatrix} 0 & -16 & 0 & 8 & -16 & 8 \\ 0 & 48 & -24 & -36 & 48 & -36 \\ 0 & -32 & 16 & 24 & -32 & 24 \\ 0 & 32 & -8 & -20 & 32 & -20 \\ 0 & -32 & 16 & 24 & -32 & 24 \\ 0 & -16 & -8 & 4 & -16 & 4 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & -16 & 0 & 8 & -16 & 8 \\ 0 & 48 & -24 & -36 & 48 & -36 \\ 0 & -32 & 16 & 24 & -32 & 24 \\ 0 & 32 & -8 & -20 & 32 & -20 \\ 0 & -32 & 16 & 24 & -32 & 24 \\ 0 & -16 & -8 & 4 & -16 & 4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -16z_2 \qquad\quad + \ 8\,z_4 - 16z_5 + \ 8\,z_6 = 0, \\ \ \ 48z_2 - 24z_3 - 36z_4 + 48z_5 - 36z_6 = 0, \\ -32z_2 + 16z_3 + 24z_4 - 32z_5 + 24z_6 = 0, \\ \ \ 32z_2 - \ 8\,z_3 - 20z_4 + 32z_5 - 20z_6 = 0, \\ -32z_2 + 16z_3 + 24z_4 - 32z_5 + 24z_6 = 0, \\ -16z_2 - \ 8\,z_3 + \ 4\,z_4 - 16z_5 + \ 4\,z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda\, I_6)^3 &= \{(r_1, r_2, r_3, r_4, -r_2 - r_3, -2\,r_3 - r_4) \mid r_1, r_2, r_3, r_4 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,0,0), (0,1,0,0,-1,0), \\ &\qquad (0,0,1,0,-1,-2), (0,0,0,1,0,-1)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\, \mathbf{I_6})^4$.
We calculate the kernel of $(B - \lambda\, I_6)^4$.

The matrix $(B - \lambda\, I_6)^4$ is

$$(B - \lambda\, I_6)^4 = \begin{pmatrix} 0 & -32 & 0 & 16 & -32 & 16 \\ 0 & 144 & -48 & -96 & 144 & -96 \\ 0 & -96 & 32 & 64 & -96 & 64 \\ 0 & 80 & -16 & -48 & 80 & -48 \\ 0 & -96 & 32 & 64 & -96 & 64 \\ 0 & -16 & -16 & 0 & -16 & 0 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & -32 & 0 & 16 & -32 & 16 \\ 0 & 144 & -48 & -96 & 144 & -96 \\ 0 & -96 & 32 & 64 & -96 & 64 \\ 0 & 80 & -16 & -48 & 80 & -48 \\ 0 & -96 & 32 & 64 & -96 & 64 \\ 0 & -16 & -16 & 0 & -16 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -32z_2 + 16z_4 - 32z_5 + 16z_6 = 0, \\ 144z_2 - 48z_3 - 96z_4 + 144z_5 - 96z_6 = 0, \\ -96z_2 + 32z_3 + 64z_4 - 96z_5 + 64z_6 = 0, \\ 80z_2 - 16z_3 - 48z_4 + 80z_5 - 48z_6 = 0, \\ -96z_2 + 32z_3 + 64z_4 - 96z_5 + 64z_6 = 0, \\ -16z_2 - 16z_3 - 16z_5 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_6)^4 &= \{(r_1, r_2, r_3, r_4, -r_2 - r_3, -2r_3 - r_4) \mid r_1, r_2, r_3, r_4 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,0,0,0), (0,1,0,0,-1,0), \\ &\qquad (0,0,1,0,-1,-2), (0,0,0,1,0,-1)\}. \end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the third power onwards in the following chain of inclusion of sets.

$$\ker(B - \lambda I_6) \subsetneq \ker((B - \lambda I_6)^2) \subsetneq \ker((B - \lambda I_6)^3) = \ker((B - \lambda I_6)^4) = \cdots.$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B - \lambda I_6)$	2	$2 = \dim(\ker(B - \lambda I_6))$
$\ker(B - \lambda I_6)^2$	3	$1 = \dim(\ker((B - \lambda I_6)^2)) - \dim(\ker(B - \lambda I_6))$
$\ker(B - \lambda I_6)^3$	4	$1 = \dim(\ker((B - \lambda I_6)^3)) - \dim(\ker((B - \lambda I_6)^2))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B - \lambda I_6)^i)) - \dim(\ker((B - \lambda I_6)^{i-1}))$. The first number of this last column is $\dim(\ker(B - \lambda I_6))$.

5. Calculation of Jordan chains.

Calculation of the first Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ satisfying

$$\begin{cases} (B - \lambda I_6)\,\mathbf{w_1} = \mathbf{0}, \\ (B - \lambda I_6)\,\mathbf{w_2} = \mathbf{w_1}, \\ (B - \lambda I_6)\,\mathbf{w_3} = \mathbf{w_2} \end{cases}$$

or

$$\begin{cases} (B - \lambda I_6)^3\,\mathbf{w_3} = \mathbf{0}, \\ (B - \lambda I_6)^2\,\mathbf{w_3} = \mathbf{w_1}, \\ (B - \lambda I_6)\,\mathbf{w_3} = \mathbf{w_2} \end{cases}$$

where $\mathbf{w_3}$ is in the vector space $\ker((B - \lambda I_6)^3)$ but not in $\ker((B - \lambda I_6)^2)$.

We look for a generating vector $\mathbf{w_3}$. This vector must be linearly independent of all vectors in $\ker(B - \lambda I_6)^2$ and must be independent from vectors of height 3 that were already chosen in $\ker(B - \lambda I_6)^3$. We know that

$$\begin{aligned} \ker((B - \lambda I_6)^3) = \operatorname{span}\{&(1,0,0,0,0,0),(0,1,0,0,-1,0), \\ &(0,0,1,0,-1,-2),(0,0,0,1,0,-1)\}. \end{aligned}$$

We have at this point not chosen in $\ker((B - \lambda I_6)^3)$ not chosen any vector.

We know that a vector in $\ker((B - \lambda I_6)^3)$ must be of the form

$$\begin{aligned} a\,(1,0,0,0,0,0) + b\,(0,1,0,0,-1,0) + c\,(0,0,1,0,-1,-2) \\ + d\,(0,0,0,1,0,-1) \\ = (a, b, c, d, -b-c, -2c-d). \end{aligned}$$

We remember that

$$\begin{aligned} \ker(B - \lambda I_6)^2 = \operatorname{span}\{&(1,0,0,0,0,0),(0,1,-1/2,0,-1/2,1), \\ &(0,0,0,1,0,-1)\}. \end{aligned}$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & -1/2 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ a & b & c & d & -b-c & -2c-d \end{pmatrix}.$$

We row reduce this matrix H and if we impose $b + 2c \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $b + 2c \neq 0$. So we can choose $a = 0$, $b = 1$, $c = 0$ and $d = 0$.

So we have the generating vector

$$\mathbf{w_3} = (0, 1, 0, 0, -1, 0).$$

We start with $\mathbf{w_3} = (0, 1, 0, 0, -1, 0)$.

We calculate $\mathbf{w_2}$.

$$\begin{aligned} \mathbf{w_2} &= (B - \lambda I_6)\mathbf{w_3} \\ &= \begin{pmatrix} 0 & -2 & 3 & 2 & -4 & 2 \\ 0 & 0 & -4 & -3 & -2 & -3 \\ 0 & 0 & 3 & 2 & 1 & 2 \\ 0 & 4 & -2 & -3 & 4 & -3 \\ 0 & 0 & 3 & 2 & 1 & 2 \\ 0 & -8 & 0 & 3 & -10 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \\ -1 \\ 2 \end{pmatrix} \end{aligned}$$

and

$$\mathbf{w_1} = (B - \lambda I_6)^2 \mathbf{w_3}$$
$$= \begin{pmatrix} 0 & -8 & 1 & 4 & -9 & 4 \\ 0 & 12 & -12 & -12 & 12 & -12 \\ 0 & -8 & 8 & 8 & -8 & 8 \\ 0 & 12 & -4 & -8 & 12 & -8 \\ 0 & -8 & 8 & 8 & -8 & 8 \\ 0 & -12 & -4 & 4 & -12 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

So we have the Jordan chain

$$\{\mathbf{w_1} = (1,0,0,0,0,0), \mathbf{w_2} = (2,2,-1,0,-1,2), \mathbf{w_3} = (0,1,0,0,-1,0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_6)$	2	$\mathbf{w_1}$	1
$\ker(B - \lambda I_6)^2$	3	$\mathbf{w_2}$	0
$\ker(B - \lambda I_6)^3$	4	$\mathbf{w_3}$	0

with

$$\mathbf{w_1} = (1,0,0,0,0,0)$$
$$\mathbf{w_2} = (2,2,-1,0,-1,2)$$
$$\mathbf{w_3} = (0,1,0,0,-1,0)$$

Calculation of the second Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w_4}\}$ satisfying

$$(B - \lambda I_6)\mathbf{w_4} = \mathbf{0},$$

where $\mathbf{w_4}$ is in the vector space $\ker((B - \lambda I_6))$.

We look for a generating vector $\mathbf{w_4}$. This vector must be linearly independent of vectors that were already chosen in $\ker(B - \lambda I_6)$.

We have at this point chosen in $\ker((B - \lambda I_6))$ already the vector $\mathbf{w_1} = (1, 0, 0, 0, 0, 0)$.

We know that a vector in $\ker(B - \lambda I_6)$ must be of the form

$$a\,(1, 0, 0, 0, 0, 0) + b\,(0, 0, 0, 1, 0, -1) = (a, 0, 0, b, 0, -b).$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & b & 0 & -b \end{pmatrix}.$$

We row reduce this matrix H and if we impose $b \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $b \neq 0$. We can choose $a = 0$ and $b = 1$.

We have the generating vector

$$\mathbf{w_4} = (0, 0, 0, 1, 0, -1).$$

We have found the second Jordan chain for this eigenvalue. It has length 1.

$$\{(0, 0, 0, 1, 0, -1)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(B - \lambda I_6)$	2	$\mathbf{w_1}$	$\mathbf{w_4}$	0
$\ker(B - \lambda I_6)^2$	3	$\mathbf{w_2}$		0
$\ker(B - \lambda I_6)^3$	4	$\mathbf{w_3}$		0

with

$$\begin{array}{l}\hline\hline \mathbf{w}_1 = (1,0,0,0,0,0) \\ \mathbf{w}_2 = (2,2,-1,0,-1,2) \\ \mathbf{w}_3 = (0,1,0,0,-1,0) \\ \hline \mathbf{w}_4 = (0,0,0,1,0,-1) \\ \hline\hline\end{array}$$

6. Investigation of the second eigenvalue.

We work now with the eigenvalue $\lambda = 1$.

7. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 8 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 7. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (-1) & 0 & 0 \\ 0 & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ 0 & & \end{matrix} \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_6.

$$A - \lambda I_6 = \begin{pmatrix} \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (-2) & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

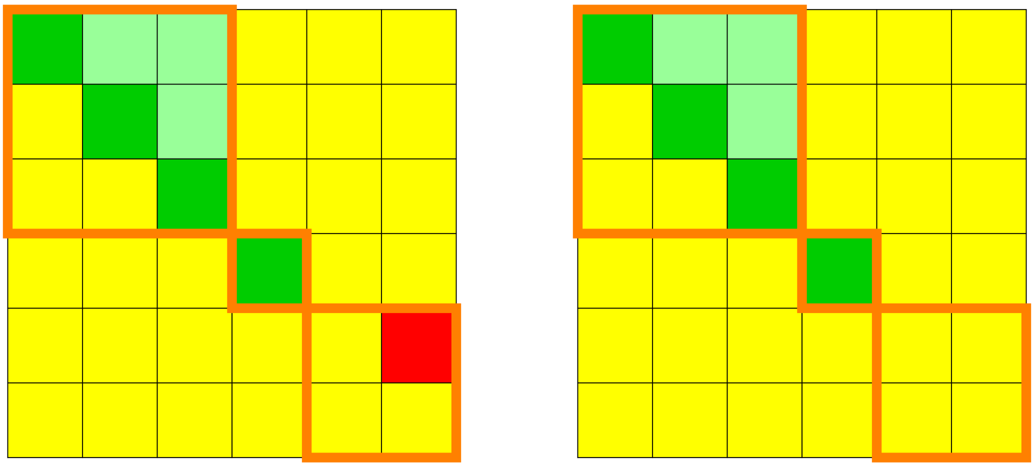
We see that the space $\mathrm{span}\{\mathbf{e}_5, \mathbf{e}_6\}$ is invariant relative to $A - \lambda I_6$. If we restrict $A - \lambda I_6$ to this space, we have a nilpotent operator and we can apply the techniques learned in part 2.

We want to investigate the endomorphism $A - \lambda I_6$ restricted to this space. We compute now the powers of $A - \lambda I_6$.

$$A - \lambda I_6 = \begin{pmatrix} \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (-2) & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{matrix} \end{pmatrix},$$

$$(A - \lambda I_6)^2 = \begin{pmatrix} \begin{pmatrix} 4 & -4 & 1 \\ 0 & 4 & -4 \\ 0 & 0 & 4 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (4) & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $(A - \lambda I_6)$. (b) Visualisation of $(A - \lambda I_6)^2$.

Figure 38

We give some comment on the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow colour represents the number 0 and the red colour represents the number 1. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix.

If we restrict the mapping $A - \lambda I_6$ to the subspace $\text{span}\{\mathbf{e}_5, \mathbf{e}_6\}$ which is an invariant subspace with respect to the operator $A - \lambda I_6$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 2. Let us take a look at the third elementary Jordan block in this matrix. We observe that the original superdiagonal of 1's in the third block of $A - \lambda I_6$ is going upwards in its elementary Jordan block associated with the nilpotent part of the transformation $A - \lambda I_6$ when increasing the powers of the matrix $A - \lambda I_6$. It finally disappears when taking the second power of $A - \lambda I_6$.

Investigation of the first Jordan chain.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_6) = \operatorname{span}\{\mathbf{e}_5\}; \\ \ker(A - \lambda I_6)^2 = \operatorname{span}\{\mathbf{e}_5, \mathbf{e}_6\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_6)$	1	$1 = \dim(\ker(A - \lambda I_6))$
$\ker(A - \lambda I_6)^2$	2	$1 = \dim(\ker(A - \lambda I_6)^2) - \dim(\ker(A - \lambda I_6))$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A - \lambda I_6)^i) - \dim(\ker(A - \lambda I_6)^{i-1})$. In the first row, we have $\dim(\ker(A - \lambda I_6))$.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A - \lambda I_6) \subsetneq \ker(A - \lambda I_6)^2 = \ker(A - \lambda I_6)^3 = \cdots .$$

We remark by looking at the matrices $(A - \lambda I_6)^i$ that we have the following mappings

$$\begin{cases} (A - \lambda I_6)\,\mathbf{e}_6 = \mathbf{e}_5, \\ (A - \lambda I_6)\,\mathbf{e}_5 = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A - \lambda I_6)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases} (A - \lambda I_6)\,\mathbf{e}_6 = \mathbf{e}_5, \\ (A - \lambda I_6)^2\,\mathbf{e}_6 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_5 = (A - \lambda I_6)\,\mathbf{e}_6, \mathbf{e}_6\}.$$

After we have found the first Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A - \lambda I_6)$	1	$\mathbf{e_5}$	0
$\ker(A - \lambda I_6)^2$	2	$\mathbf{e_6}$	0

We have now in the last column only 0's and this means that we are done looking for Jordan chains.

8. Kernels of $(\mathbf{B} - \lambda \mathbf{I_6})^{\mathbf{i}}$.

Kernel of $\mathbf{B} - \lambda \mathbf{I_6}$.
We remember that $\lambda = 1$. We calculate the kernel of $B - \lambda I_6$.

The matrix $B - \lambda I_6$ is

$$B - \lambda I_6 = \begin{pmatrix} -2 & -2 & 3 & 2 & -4 & 2 \\ 0 & -2 & -4 & -3 & -2 & -3 \\ 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 4 & -2 & -5 & 4 & -3 \\ 0 & 0 & 3 & 2 & -1 & 2 \\ 0 & -8 & 0 & 3 & -10 & 1 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -2 & -2 & 3 & 2 & -4 & 2 \\ 0 & -2 & -4 & -3 & -2 & -3 \\ 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 4 & -2 & -5 & 4 & -3 \\ 0 & 0 & 3 & 2 & -1 & 2 \\ 0 & -8 & 0 & 3 & -10 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -2z_1 - 2z_2 + 3z_3 + 2z_4 - 4z_5 + 2z_6 = 0, \\ \quad - 2z_2 - 4z_3 - 3z_4 - 2z_5 - 3z_6 = 0, \\ \quad z_3 + 2z_4 + z_5 + 2z_6 = 0, \\ \quad 4z_2 - 2z_3 - 5z_4 + 4z_5 - 3z_6 = 0, \\ \quad 3z_3 + 2z_4 - z_5 + 2z_6 = 0, \\ \quad - 8z_2 + 3z_4 - 10z_5 + z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_6) &= \{(0, r_2, -2r_2/3, r_2/3, -2r_2/3, r_2/3) \mid r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(0, 1, -2/3, 1/3, -2/3, 1/3)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I_6})^2$.
We calculate the kernel of $(B - \lambda I_6)^2$.

The matrix $(B - \lambda I_6)^2$ is

$$(B - \lambda I_6)^2 = \begin{pmatrix} 4 & 0 & -11 & -4 & 7 & -4 \\ 0 & 16 & 4 & 0 & 20 & 0 \\ 0 & -8 & 0 & 0 & -12 & 0 \\ 0 & -4 & 4 & 8 & -4 & 4 \\ 0 & -8 & -4 & 0 & -8 & 0 \\ 0 & 20 & -4 & -8 & 28 & -4 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 4 & 0 & -11 & -4 & 7 & -4 \\ 0 & 16 & 4 & 0 & 20 & 0 \\ 0 & -8 & 0 & 0 & -12 & 0 \\ 0 & -4 & 4 & 8 & -4 & 4 \\ 0 & -8 & -4 & 0 & -8 & 0 \\ 0 & 20 & -4 & -8 & 28 & -4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 4z_1 - 11z_3 - 4z_4 + 7z_5 - 4z_6 = 0, \\ \quad 16z_2 + 4z_3 + 20z_5 = 0, \\ \quad - 8z_2 - 12z_5 = 0, \\ \quad - 4z_2 + 4z_3 + 8z_4 - 4z_5 + 4z_6 = 0, \\ \quad - 8z_2 - 4z_3 - 8z_5 = 0, \\ \quad 20z_2 - 4z_3 - 8z_4 + 28z_5 - 4z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B-\lambda I_6)^2 &= \{(r_1, r_2, -2\,r_2/3, -r_1 + r_2/3, -2\,r_2/3, 2\,r_1 + r_2/3) \\ &\qquad\qquad\qquad\qquad \mid r_1, r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,-1,0,2), (0,1,-2/3,1/3,-2/3,1/3)\}.\end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I_6})^3$.
We calculate the kernel of $(B - \lambda\,I_6)^3$.

The matrix $(B - \lambda\,I_6)^3$ is

$$(B-\lambda\,I_6)^3 = \begin{pmatrix} -8 & 8 & 30 & 8 & -10 & 8 \\ 0 & -32 & 0 & 0 & -48 & 0 \\ 0 & 16 & -4 & 0 & 28 & 0 \\ 0 & 8 & -8 & -16 & 8 & -8 \\ 0 & 16 & 4 & 0 & 20 & 0 \\ 0 & -40 & 16 & 16 & -64 & 8 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -8 & 8 & 30 & 8 & -10 & 8 \\ 0 & -32 & 0 & 0 & -48 & 0 \\ 0 & 16 & -4 & 0 & 28 & 0 \\ 0 & 8 & -8 & -16 & 8 & -8 \\ 0 & 16 & 4 & 0 & 20 & 0 \\ 0 & -40 & 16 & 16 & -64 & 8 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -8z_1 + 8\,z_2 + 30z_3 + 8\,z_4 - 10z_5 + 8z_6 = 0, \\ \quad - 32z_2 - 48z_5 = 0, \\ \quad 16z_2 - 4\,z_3 + 28z_5 = 0, \\ \quad 8\,z_2 - 8\,z_3 - 16z_4 + 8\,z_5 - 8z_6 = 0, \\ \quad 16z_2 + 4\,z_3 + 20z_5 = 0, \\ \quad - 40z_2 + 16z_3 + 16z_4 - 64z_5 + 8z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B - \lambda I_6)^3 &= \{(r_1, r_2, -2r_2/3, -r_1 + r_2/3, -2r_2/3, 2r_1 + r_2/3) \\ &\qquad \mid r_1, r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,-1,0,2),(0,1,-2/3,1/3,-2/3,1/3)\}.\end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the second power onwards in the following chain of inclusion of sets.

$$\ker(B - \lambda I_6) \subsetneq \ker((B - \lambda I_6)^2) = \ker((B - \lambda I_6)^3) = \cdots .$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B - \lambda I_6)$	1	$1 = \dim(\ker(B - \lambda I_6))$
$\ker(B - \lambda I_6)^2$	2	$1 = \dim(\ker((B - \lambda I_6)^2)) - \dim(\ker(B - \lambda I_6))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B - \lambda I_6)^i)) - \dim(\ker((B - \lambda I_6)^{i-1}))$. The first number of this last column is $\dim(\ker(B - \lambda I_6))$.

9. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w_5}, \mathbf{w_6}\}$ satisfying

$$\begin{cases}(B - \lambda I_6)\mathbf{w_5} = \mathbf{0}, \\ (B - \lambda I_6)\mathbf{w_6} = \mathbf{w_5}.\end{cases}$$

or

$$\begin{cases}(B - \lambda I_6)^2\mathbf{w_6} = \mathbf{0}, \\ (B - \lambda I_6)\mathbf{w_6} = \mathbf{w_5}.\end{cases}$$

where $\mathbf{w_6}$ is in the vector space $\ker((B - \lambda I_6)^2)$ but not in $\ker(B - \lambda I_6)$.

We look for a generating vector $\mathbf{w_6}$. This vector must be linearly independent of all vectors in $\ker(B - \lambda I_6)$ and must be independent from vectors of height 2 that were already chosen in $\ker(B - \lambda I_6)^2$. We know that

$$\ker(B - \lambda I_6)^2 = \operatorname{span}\{(1,0,0,-1,0,2),(0,1,-2/3,1/3,-2/3,1/3)\}.$$

We have at this point not chosen any vector in $\ker(B - \lambda I_6)^2$.

We know that a vector in $\ker((B - \lambda I_6)^2)$ must be of the form

$$\begin{aligned} &a\,(1,0,0,-1,0,2) + b\,(0,1,-2/3,1/3,-2/3,1/3) \\ &\qquad = (a,b,-2\,b/3,-a+b/3,-2\,b/3,2\,a+b/3). \end{aligned}$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 0 & 1 & -2/3 & 1/3 & -2/3 & 1/3 \\ a & b & -2\,b/3 & b/3-a & -2\,b/3 & 2a+b/3 \end{pmatrix}.$$

We row reduce this matrix H and if we impose $a \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & -2/3 & 1/3 & -2/3 & 1/3 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a \neq 0$. So we can choose $a = 1$ and $b = 0$.

We have the generating vector

$$\mathbf{w_6} = (1,0,0,-1,0,2).$$

We start with $\mathbf{w_6} = (1,0,0,-1,0,2)$.

We calculate $\mathbf{w_5}$.

$$\begin{aligned} \mathbf{w_5} &= (B - \lambda I_6)\,\mathbf{w_6} \\ &= \begin{pmatrix} -2 & -2 & 3 & 2 & -4 & 2 \\ 0 & -2 & -4 & -3 & -2 & -3 \\ 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 4 & -2 & -5 & 4 & -3 \\ 0 & 0 & 3 & 2 & -1 & 2 \\ 0 & -8 & 0 & 3 & -10 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 2 \\ -1 \\ 2 \\ -1 \end{pmatrix}. \end{aligned}$$

We have now the first Jordan chain for this eigenvalue. In total we have found 3 chains. The length of this chain is 2.

$$\{\mathbf{w_5} = (0, -3, 2, -1, 2, -1), \mathbf{w_6} = (1, 0, 0, -1, 0, 2)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_6)$	1	$\mathbf{w_5}$	0
$\ker((B - \lambda I_6)^2)$	2	$\mathbf{w_6}$	0

with

$\mathbf{w_5}$	$= (0, -3, 2, -1, 2, -1)$
$\mathbf{w_6}$	$= (1, 0, 0, -1, 0, 2)$

10. Result and check of the result.

We construct the matrix of base change P with the coordinates of the vectors $\mathbf{w_i}$ found in the Jordan chains in the columns of P.

$$P = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & -3 & 0 \\ 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & -1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 & -1 & 2 \end{pmatrix}.$$

$$
\begin{aligned}
A &= P^{-1} B P \\
&= \begin{pmatrix} 1 & -2 & -2 & -1 & -2 & -1 \\ 0 & 2 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 & -2 & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} -1 & -2 & 3 & 2 & -4 & 2 \\ 0 & -1 & -4 & -3 & -2 & -3 \\ 0 & 0 & 2 & 2 & 1 & 2 \\ 0 & 4 & -2 & -4 & 4 & -3 \\ 0 & 0 & 3 & 2 & 0 & 2 \\ 0 & -8 & 0 & 3 & -10 & 2 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & -3 & 0 \\ 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & -1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 & -1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

28 exercise. (6×6); $(\mathbf{J_2(-1), J_2(-1), J_2(1)})$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} -4 & 5 & 0 & 3 & 8 & 6 \\ -8 & 5 & 2 & -1 & 15 & 8 \\ -6 & 8 & -1 & 3 & 14 & 9 \\ 7 & -3 & -2 & 1 & -11 & -6 \\ 9 & -5 & -2 & 2 & -16 & -8 \\ -13 & 9 & 2 & 0 & 23 & 13 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_6| = (\lambda - 1)^2 (\lambda + 1)^4.$$

We see that the characteristic Cayley-Hamilton polynomial $p_{\text{C-H}}(\lambda)$ can be factorised in linear polynomials over the field $\mathbf{K}$. So we can apply the Jordan normalisation machinery.

The eigenvalue $\lambda = -1$ has algebraic multiplicity 4. The eigenvalue $\lambda = 1$ has algebraic multiplicity 2.

2. Investigation of the first eigenvalue.

We work now with the eigenvalue $\lambda = -1$.

3. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or

calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 4 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 3. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_6.

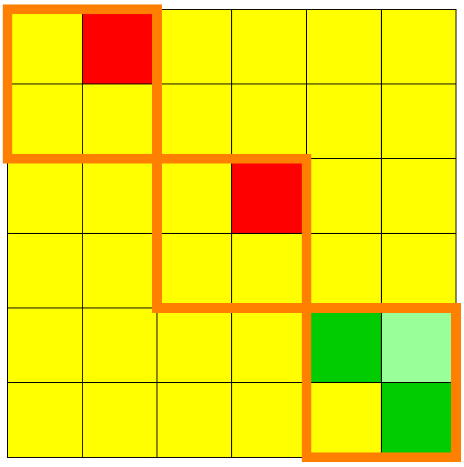
$$A - \lambda I_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

We compute also the powers of $A - \lambda I_6$.

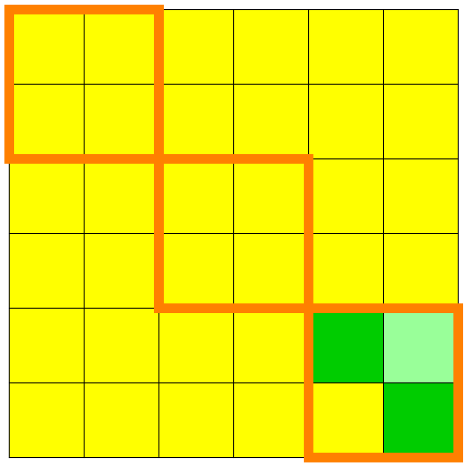
$$A - \lambda I_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$(A - \lambda I_6)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $A - \lambda I_6$.



(b) Visualisation of $(A - \lambda I_6)^2$.

Figure 39

We give here some comments about this figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number 0. The red cells are representing the number 1. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix.

If we restrict the mapping $A - \lambda I_6$ to the subspace $\text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ which is an invariant subspace with respect to the operator $A - \lambda I_6$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 2. Let us take a look at the two first elementary Jordan blocks in this matrix.
We observe that the original superdiagonals of 1's in the two first blocks of $A - \lambda I_6$ are going upwards in their respective elementary Jordan blocks associated with the nilpotent part of the transformation $A - \lambda I_6$ when increasing the powers of the matrix $A - \lambda I_6$. They finally disappear when taking the second power of $A - \lambda I_6$.

Investigation of the first Jordan chain.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A-\lambda I_6) = \operatorname{span}\{\mathbf{e}_1,\mathbf{e}_3\}; \\ \ker(A-\lambda I_6)^2 = \operatorname{span}\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A-\lambda I_6)$	2	$2 = \dim(\ker(A-\lambda I_6))$
$\ker(A-\lambda I_6)^2$	4	$2 = \dim(\ker(A-\lambda I_6)^2) - \dim(\ker(A-\lambda I_6))$

We observe that the superdiagonals consisting of 1's in the two first blocks of $A-\lambda I_6$ are going upwards in their respective elementary Jordan blocks when increasing the powers of the matrix $A-\lambda I_6$ until they ultimately disappear when taking the second power of $A-\lambda I_7$.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A-\lambda I_6) \subsetneq \ker(A-\lambda I_6)^2 = \ker(A-\lambda I_6)^3 = \cdots .$$

We remark by looking at the matrices $(A-\lambda I_6)^i$ that we have the following mappings

$$\begin{cases} (A-\lambda I_6)\,\mathbf{e}_2 = \mathbf{e}_1, \\ (A-\lambda I_6)\,\mathbf{e}_1 = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A-\lambda I_6)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases} (A-\lambda I_6)\,\mathbf{e}_2 = \mathbf{e}_1, \\ (A-\lambda I_6)\,\mathbf{e}_1 = \mathbf{0} \end{cases}$$

or

$$\begin{cases} (A-\lambda I_6)\,\mathbf{e}_2 = \mathbf{e}_1, \\ (A-\lambda I_6)^2\,\mathbf{e}_2 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = (A - \lambda I_6)\,\mathbf{e}_2, \mathbf{e}_2\}.$$

After we have found the first Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A - \lambda I_6)$	2	$\mathbf{e}_1$	1
$\ker(A - \lambda I_6)^2$	4	$\mathbf{e}_2$	1

We have that the last number in the last column is not zero, and this means that there is still a chain left of length 2.

Investigating the second Jordan chain.
We remark by looking at the matrices $(A - \lambda I_6)^i$ that we have the following mappings

$$\begin{cases}(A - \lambda I_6)\,\mathbf{e}_4 = \mathbf{e}_3, \\ (A - \lambda I_6)\,\mathbf{e}_3 = \mathbf{0}\end{cases}$$

or

$$\begin{cases}(A - \lambda I_6)\,\mathbf{e}_4 = \mathbf{e}_3, \\ (A - \lambda I_6)^2\,\mathbf{e}_4 = \mathbf{0}.\end{cases}$$

We remark by looking at the matrices $(A - \lambda I_6)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases}(A - \lambda I_6)\,\mathbf{e}_4 = \mathbf{e}_3, \\ (A - \lambda I_6)^2\,\mathbf{e}_4 = \mathbf{0}.\end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_3 = (A - \lambda I_6)\,\mathbf{e}_4, \mathbf{e}_4\}.$$

After we have found the second Jordan chain of length 4, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(A - \lambda I_6)$	1	$\mathbf{e}_1$	$\mathbf{e}_3$	0
$\ker(A - \lambda I_6)^2$	1	$\mathbf{e}_2$	$\mathbf{e}_4$	0

4. Kernels of $(\mathbf{B} - \lambda\, \mathbf{I}_6)^{\mathbf{i}}$.

Kernel of $(\mathbf{B} - \lambda\, \mathbf{I}_6)$.
The matrix $B - \lambda\, I_6$ is

$$B - \lambda\, I_6 = \begin{pmatrix} -3 & 5 & 0 & 3 & 8 & 6 \\ -8 & 6 & 2 & -1 & 15 & 8 \\ -6 & 8 & 0 & 3 & 14 & 9 \\ 7 & -3 & -2 & 2 & -11 & -6 \\ 9 & -5 & -2 & 2 & -15 & -8 \\ -13 & 9 & 2 & 0 & 23 & 14 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -3 & 5 & 0 & 3 & 8 & 6 \\ -8 & 6 & 2 & -1 & 15 & 8 \\ -6 & 8 & 0 & 3 & 14 & 9 \\ 7 & -3 & -2 & 2 & -11 & -6 \\ 9 & -5 & -2 & 2 & -15 & -8 \\ -13 & 9 & 2 & 0 & 23 & 14 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -3z_1 + 5z_2 + 3z_4 + 8z_5 + 6z_6 = 0, \\ -8z_1 + 6z_2 + 2z_3 - z_4 + 15z_5 + 8z_6 = 0, \\ -6z_1 + 8z_2 + 3z_4 + 14z_5 + 9z_6 = 0, \\ 7z_1 - 3z_2 - 2z_3 + 2z_4 - 11z_5 - 6z_6 = 0, \\ 9z_1 - 5z_2 - 2z_3 + 2z_4 - 15z_5 - 8z_6 = 0, \\ -13z_1 + 9z_2 + 2z_3 + 23z_5 + 14z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B - \lambda I_6) &= \{(r_1, r_2, -r_1/2, -r_1 - r_2, -r_2, r_1 + r_2) \mid r_1, r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, 0, -1/2, -1, 0, 1), (0, 1, 0, -1, -1, 1)\}.\end{aligned}$$

Kernel of $(\mathbf{B} - \lambda \mathbf{I_6})^2$.
We calculate the kernel of $(B - \lambda I_6)^2$.

The matrix $(B - \lambda I_6)^2$ is

$$(B - \lambda I_6)^2 = \begin{pmatrix} -16 & 20 & 0 & 8 & 36 & 24 \\ -12 & 12 & 0 & 4 & 24 & 16 \\ -16 & 20 & 0 & 8 & 36 & 24 \\ 8 & -4 & 0 & 0 & -12 & -8 \\ 8 & -4 & 0 & 0 & -12 & -8 \\ -20 & 16 & 0 & 4 & 36 & 24 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -16 & 20 & 0 & 8 & 36 & 24 \\ -12 & 12 & 0 & 4 & 24 & 16 \\ -16 & 20 & 0 & 8 & 36 & 24 \\ 8 & -4 & 0 & 0 & -12 & -8 \\ 8 & -4 & 0 & 0 & -12 & -8 \\ -20 & 16 & 0 & 4 & 36 & 24 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -16z_1 + 20z_2 + 8z_4 + 36z_5 + 24z_6 = 0, \\ -12z_1 + 12z_2 + 4z_4 + 24z_5 + 16z_6 = 0, \\ -16z_1 + 20z_2 + 8z_4 + 36z_5 + 24z_6 = 0, \\ 8\,z_1 - 4\,z_2 - 12z_5 - 8\,z_6 = 0, \\ 8\,z_1 - 4\,z_2 - 12z_5 - 8\,z_6 = 0, \\ -20z_1 + 16z_2 + 4z_4 + 36z_5 + 24z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B - \lambda I_6)^2 &= \{(r_1, r_2, r_3, -r_1 - r_2, r_5, r_1 - r_2/2 - 3\,r_5/2) \\ &\qquad \mid r_1, r_2, r_3, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, 0, 0, -1, 0, 1), (0, 1, 0, -1, 0, -1/2), \\ &\qquad (0, 0, 1, 0, 0, 0), (0, 0, 0, 0, 1, -3/2)\}.\end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\, \mathbf{I_6})^3$.
We calculate the kernel of $(B - \lambda\, I_6)^3$.

The matrix $(B - \lambda\, I_6)^3$ is

$$(B - \lambda\, I_6)^3 = \begin{pmatrix} -44 & 52 & 0 & 20 & 96 & 64 \\ -24 & 24 & 0 & 8 & 48 & 32 \\ -44 & 52 & 0 & 20 & 96 & 64 \\ 4 & 4 & 0 & 4 & 0 & 0 \\ 4 & 4 & 0 & 4 & 0 & 0 \\ -28 & 20 & 0 & 4 & 48 & 32 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -44 & 52 & 0 & 20 & 96 & 64 \\ -24 & 24 & 0 & 8 & 48 & 32 \\ -44 & 52 & 0 & 20 & 96 & 64 \\ 4 & 4 & 0 & 4 & 0 & 0 \\ 4 & 4 & 0 & 4 & 0 & 0 \\ -28 & 20 & 0 & 4 & 48 & 32 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -44z_1 + 52z_2 + 20z_4 + 96z_5 + 64z_6 = 0, \\ -24z_1 + 24z_2 + 8\, z_4 + 48z_5 + 32z_6 = 0, \\ -44z_1 + 52z_2 + 20z_4 + 96z_5 + 64z_6 = 0, \\ 4\, z_1 + 4\, z_2 + 4\, z_4 = 0, \\ 4\, z_1 + 4\, z_2 + 4\, z_4 = 0, \\ -28z_1 + 20z_2 + 4\, z_4 + 48z_5 + 32z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda\, I_6)^3 &= \{(r_1, r_2, r_3, -r_1 - r_2, r_5, r_1 - r_2/2 - 3\, r_5/2) \\ &\qquad \mid r_1, r_2, r_3, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, 0, 0, -1, 0, 1), (0, 1, 0, -1, 0, -1/2), \\ &\qquad (0, 0, 1, 0, 0, 0), (0, 0, 0, 0, 1, -3/2)\}. \end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality starting from the second power onwards in the following chain of inclusion of sets.

$$\ker(B - \lambda I_6) \subsetneq \ker((B - \lambda I_6)^2) = \ker((B - \lambda I_6)^3) = \cdots .$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B - \lambda I_6)$	2	$2 = \dim(\ker(B - \lambda I_6))$
$\ker(B - \lambda I_6)^2$	4	$2 = \dim(\ker(B - \lambda I_6)^2) - \dim(\ker(B - \lambda I_6))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B - \lambda I_6)^i)) - \dim(\ker((B - \lambda I_6)^{i-1}))$. The first number of this last column is $\dim(\ker(B - \lambda I_6))$.

5. Calculation of Jordan chains.

Calculation of the first Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w_1}, \mathbf{w_2}\}$ satisfying

$$\begin{cases} (B - \lambda I_6)\, \mathbf{w_1} = \mathbf{0}, \\ (B - \lambda I_6)\, \mathbf{w_2} = \mathbf{w_1}, \end{cases}$$

or

$$\begin{cases} (B - \lambda I_6)^2\, \mathbf{w_2} = \mathbf{0}, \\ (B - \lambda I_6)\, \mathbf{w_2} = \mathbf{w_1}, \end{cases}$$

where $\mathbf{w_2}$ is in the vector space $\ker((B - \lambda I_6)^2)$ but not in $\ker(B - \lambda I_6)$.

We look for a generating vector $\mathbf{w_2}$. This vector must be linearly independent of all vectors in $\ker(B - \lambda I_6)$ and must be independent from vectors that were already chosen in $\ker(B - \lambda I_6)^2$. We know that

$$\ker(B - \lambda I_6) = \mathrm{span}\{(1, 0, -1/2, -1, 0, 1), (0, 1, 0, -1, -1, 1)\}.$$

We have at this point not chosen in $\ker(B - \lambda I_6)$ any vectors.

We know that a vector in $\ker((B - \lambda I_6)^2)$ must be of the form

$$\begin{aligned} a\,(1,0,0,-1,0,1) + b\,(0,1,0,-1,0,-(1/2)) + c\,(0,0,1,0,0,0) \\ + d\,(0,0,0,0,1,-3/2) \\ = (a,b,c,-a-b,d,a-b/2-3\,d/2). \end{aligned}$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 0 & -1/2 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ a & b & c & -a-b & d & a-b/2-3\,d/2 \end{pmatrix}.$$

We row reduce this matrix H and if we impose $a + 2\,c \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 & -1 & (b+d)/(a+2\,c) & (2\,a-3\,b+4\,c-3\,d)/(2\,(a+2\,c)) \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & (2\,(b+d))/(a+2\,c) & -3\,(b+d)/(a+2\,c) \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + 2\,c \neq 0$. So we can choose $a = 0$, $b = 0$, $c = 1$, $d = 0$.
So we have the generating vector

$$\mathbf{w_2} = (0,0,1,0,0,0).$$

We start with $\mathbf{w_2} = (0,0,1,0,0,0)$.

We calculate $\mathbf{w_1}$.

$$\begin{aligned} \mathbf{w_1} &= (B - \lambda\, I_6)\,\mathbf{w_2} \\ &= \begin{pmatrix} -3 & 5 & 0 & 3 & 8 & 6 \\ -8 & 6 & 2 & -1 & 15 & 8 \\ -6 & 8 & 0 & 3 & 14 & 9 \\ 7 & -3 & -2 & 2 & -11 & -6 \\ 9 & -5 & -2 & 2 & -15 & -8 \\ -13 & 9 & 2 & 0 & 23 & 14 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \\ -2 \\ 2 \end{pmatrix}. \end{aligned}$$

We have now found the first Jordan chain. It has length 2.

$$\{\mathbf{w_1} = (0,2,0,-2,-2,2), \mathbf{w_2} = (0,0,1,0,0,0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_6)$	2	$\mathbf{w_1}$	1
$\ker(B - \lambda I_6)^2$	4	$\mathbf{w_2}$	1

with

$$\mathbf{w_1} = (0, 2, 0, -2, -2, 2)$$
$$\mathbf{w_2} = (0, 0, 1, 0, 0, 0)$$

Calculation of the second Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w_3}, \mathbf{w_4}\}$ satisfying

$$\begin{cases} (B - \lambda I_6)\,\mathbf{w_3} = \mathbf{0}, \\ (B - \lambda I_6)\,\mathbf{w_4} = \mathbf{w_3}, \end{cases}$$

or

$$\begin{cases} (B - \lambda I_6)^2\,\mathbf{w_4} = \mathbf{0}, \\ (B - \lambda I_6)\,\mathbf{w_4} = \mathbf{w_3} \end{cases}$$

where $\mathbf{w_4}$ is in the vector space $\ker((B - \lambda I_6)^2)$ but not in $\ker(B - \lambda I_6)$.

We look for a generating vector $\mathbf{w_4}$. This vector must be linearly independent of all vectors in $\ker(B - \lambda I_6)^2$ and must be independent from vectors that were already chosen in $\ker(B - \lambda I_6)^2$. We know that

$$\ker(B - \lambda I_6) = \text{span}\{(1, 0, -1/2, -1, 0, 1), (0, 1, 0, -1, -1, 1)\}.$$

We have at this point chosen in $\ker((B - \lambda I_6)^2)$ already the vector $\mathbf{w_2} = (0, 0, 1, 0, 0, 0)$ of height 2.

We know that a vector in $\ker((B - \lambda I_6)^2)$ must be of the form

$$\begin{aligned} a\,(1, 0, 0, -1, 0, 1) + b\,(0, 1, 0, -1, 0, -1/2) + c\,(0, 0, 1, 0, 0, 0) \\ + d\,(0, 0, 0, 0, 1, -3/2) \\ = (a, b, c, -a - b, d, a - b/2 - 3\,d/2). \end{aligned}$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 0 & -1/2 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ a & b & c & -a-b & d & a-b/2-3d/2 \end{pmatrix}.$$

We row reduce this matrix H and if we impose $b+d \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3/2 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $b+d \neq 0$. We can choose $a = 0$, $b = 2$, $c = 0$, $d = 0$.

We have the generating vector

$$\mathbf{w}_4 = (0, 2, 0, -2, 0, -1).$$

We start with $\mathbf{w}_4 = (0, 2, 0, -2, 0, -1)$.

We calculate $\mathbf{w}_3$.

$$\begin{aligned} \mathbf{w}_3 &= (B - \lambda I_6)\mathbf{w}_4 \\ &= \begin{pmatrix} -3 & 5 & 0 & 3 & 8 & 6 \\ -8 & 6 & 2 & -1 & 15 & 8 \\ -6 & 8 & 0 & 3 & 14 & 9 \\ 7 & -3 & -2 & 2 & -11 & -6 \\ 9 & -5 & -2 & 2 & -15 & -8 \\ -13 & 9 & 2 & 0 & 23 & 14 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 1 \\ -4 \\ -6 \\ 4 \end{pmatrix}. \end{aligned}$$

We have now found a second Jordan chain. It has length 2.

$$\{\mathbf{w}_3 = (-2, 6, 1, -4, -6, 4), \mathbf{w}_4 = (0, 2, 0, -2, 0, -1)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(B - \lambda I_6)$	2	$\mathbf{w_1}$	$\mathbf{w_3}$	0
$\ker(B - \lambda I_6)^2$	4	$\mathbf{w_2}$	$\mathbf{w_4}$	0

with

$$\mathbf{w_1} = (0, 2, 0, -2, -2, 2)$$
$$\mathbf{w_2} = (0, 0, 1, 0, 0, 0)$$
$$\mathbf{w_3} = (-2, 6, 1, -4, -6, 4)$$
$$\mathbf{w_4} = (0, 2, 0, -2, 0, -1)$$

6. Investigation of the second eigenvalue.

We work now with the eigenvalue $\lambda = 1$.

7. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 8 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 7. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_6.

$$A - \lambda I_6 = \begin{pmatrix} \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

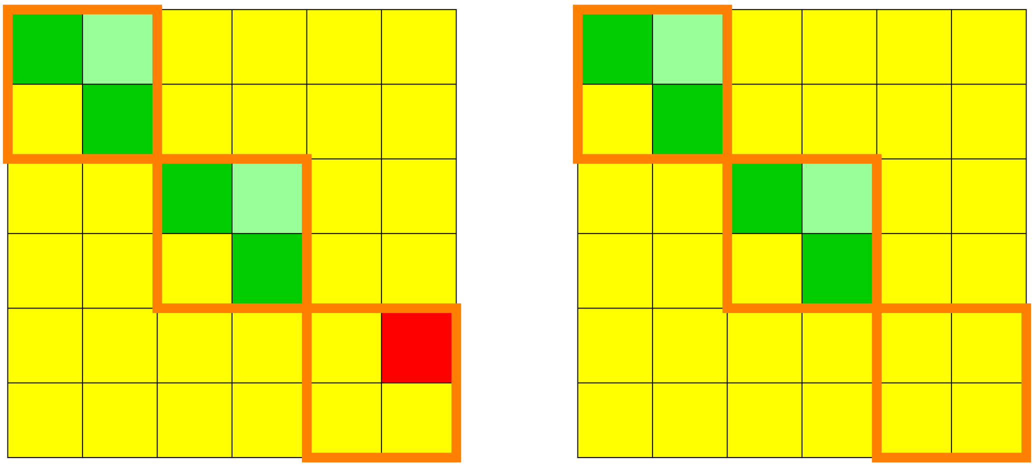
We see that the space $\text{span}\{\mathbf{e}_5, \mathbf{e}_6\}$ is invariant relative to $A - \lambda I_6$. If we restrict $A - \lambda I_6$ to this space, we have a nilpotent operator and we can apply the techniques learned in part 2.

We want to investigate the endomorphism $A - \lambda I_6$ restricted to this space. We compute now the powers of $A - \lambda I_6$.

$$A - \lambda I_6 = \begin{pmatrix} \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$(A - \lambda I_6)^2 = \begin{pmatrix} \begin{pmatrix} 4 & -4 \\ 0 & 4 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 4 & -4 \\ 0 & 4 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $(A - \lambda I_6)$. (b) Visualisation of $(A - \lambda I_6)^2$.

Figure 40

We give here some comment about the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells represent the number 1. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix. If we restrict the mapping $A - \lambda I_6$ to the subspace $\text{span}\{\mathbf{e}_5, \mathbf{e}_6\}$ which is an invariant subspace with respect to the operator $A - \lambda I_6$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 2.

Let us take a look at the third elementary Jordan block in this matrix. We observe that the original superdiagonal of 1's in the third block of $A - \lambda I_6$ is going upwards in its elementary Jordan block associated with the nilpotent part of the transformation $A - \lambda I_6$ when increasing the powers of the matrix $A - \lambda I_6$. They finally disappear when taking the second power of $A - \lambda I_6$.

Investigation of the first Jordan chain.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_6) = \text{span}\{\mathbf{e}_5\}; \\ \ker(A - \lambda I_6)^2 = \text{span}\{\mathbf{e}_5, \mathbf{e}_6\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_6)$	1	$1 = \dim(\ker(A - \lambda I_6))$
$\ker(A - \lambda I_6)^2$	2	$1 = \dim(\ker(A - \lambda I_6)^2) - \dim(\ker(A - \lambda I_6))$

We observe that the superdiagonals consisting of 1's in the third block of $A - \lambda I_6$ is going upwards in its block when increasing the powers of the matrix $A - \lambda I_6$ until it ultimately disappears when taking the second power of $A - \lambda I_6$.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A - \lambda I_6) \subsetneq \ker(A - \lambda I_6)^2 = \ker(A - \lambda I_6)^3 = \cdots .$$

We remark by looking at the matrices $(A - \lambda I_6)^i$ that we have the following mappings

$$\begin{cases} (A - \lambda I_6)\,\mathbf{e_6} = \mathbf{e}_5, \\ (A - \lambda I_6)\,\mathbf{e}_5 = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A - \lambda I_6)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases} (A - \lambda I_6)^2\,\mathbf{e_6} = \mathbf{0}, \\ (A - \lambda I_6)\,\mathbf{e_6} = \mathbf{e}_5. \end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_5 = (A - \lambda I_6)\,\mathbf{e_6}, \mathbf{e_6}\}.$$

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A - \lambda I_6)$	1	$\mathbf{e}_5$	0
$\ker(A - \lambda I_6)^2$	2	$\mathbf{e}_6$	0

After we have found the first Jordan chain of length 2, we have then the following table.

We have now in the last column only 0's and this means that we are done looking for Jordan chains.

8. Kernels of $(\mathbf{B} - \lambda\, \mathbf{I}_6)^{\mathbf{i}}$.

Kernel of $(\mathbf{B} - \lambda\, \mathbf{I}_6)$.
The matrix $B - \lambda\, I_6$ is

$$B - \lambda\, I_6 = \begin{pmatrix} -5 & 5 & 0 & 3 & 8 & 6 \\ -8 & 4 & 2 & -1 & 15 & 8 \\ -6 & 8 & -2 & 3 & 14 & 9 \\ 7 & -3 & -2 & 0 & -11 & -6 \\ 9 & -5 & -2 & 2 & -17 & -8 \\ -13 & 9 & 2 & 0 & 23 & 12 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -5 & 5 & 0 & 3 & 8 & 6 \\ -8 & 4 & 2 & -1 & 15 & 8 \\ -6 & 8 & -2 & 3 & 14 & 9 \\ 7 & -3 & -2 & 0 & -11 & -6 \\ 9 & -5 & -2 & 2 & -17 & -8 \\ -13 & 9 & 2 & 0 & 23 & 12 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -5\,z_1 + 5z_2 + 3z_4 + 8\,z_5 + 6\,z_6 = 0, \\ -8\,z_1 + 4z_2 + 2z_3 - z_4 + 15z_5 + 8\,z_6 = 0, \\ -6\,z_1 + 8z_2 - 2z_3 + 3z_4 + 14z_5 + 9\,z_6 = 0, \\ 7\,z_1 - 3z_2 - 2z_3 - 11z_5 - 6\,z_6 = 0, \\ 9\,z_1 - 5z_2 - 2z_3 + 2z_4 - 17z_5 - 8\,z_6 = 0, \\ -13z_1 + 9z_2 + 2z_3 + 23z_5 + 12z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B-\lambda I_6) &= \{(r_1,0,r_1,r_1,r_1,-r_1) \mid r_1 \in \mathbf{K}\}\\ &= \operatorname{span}\{(1,0,1,1,1,-1)\}.\end{aligned}$$

Kernel of $(\mathbf{B}-\lambda\,\mathbf{I_6})^2$.
We calculate the kernel of $(B-\lambda I_6)^2$.

The matrix $(B-\lambda I_6)^2$ is

$$(B-\lambda I_6)^2 = \begin{pmatrix} 0 & 0 & 0 & -4 & 4 & 0\\ 20 & -8 & -8 & 8 & -36 & -16\\ 8 & -12 & 4 & -4 & -20 & -12\\ -20 & 8 & 8 & -4 & 32 & 16\\ -28 & 16 & 8 & -8 & 52 & 24\\ 32 & -20 & -8 & 4 & -56 & -28 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & -4 & 4 & 0\\ 20 & -8 & -8 & 8 & -36 & -16\\ 8 & -12 & 4 & -4 & -20 & -12\\ -20 & 8 & 8 & -4 & 32 & 16\\ -28 & 16 & 8 & -8 & 52 & 24\\ 32 & -20 & -8 & 4 & -56 & -28 \end{pmatrix}\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\\ z_5\\ z_6\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -4z_4+4z_5=0,\\ 20z_1-8z_2-8z_3+8z_4-36z_5-16z_6=0,\\ 8z_1-12z_2+4z_3-4z_4-20z_5-12z_6=0,\\ -20z_1+8z_2+8z_3-4z_4+32z_5+16z_6=0,\\ -28z_1+16z_2+8z_3-8z_4+52z_5+24z_6=0,\\ 32z_1-20z_2-8z_3+4z_4-56z_5-28z_6=0.\end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B-\lambda I_6)^2 &= \{(r_1,r_2,r_1,r_1-2r_2,r_1-2r_2,-r_1+3r_2) \mid r_1,r_2 \in \mathbf{K}\}\\ &= \operatorname{span}\{(1,0,1,1,1,-1),(0,1,0,-2,-2,3)\}.\end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\, \mathbf{I}_6)^3$.
We calculate the kernel of $(B - \lambda\, I_6)^3$.

The matrix $(B - \lambda\, I_6)^3$ is

$$(B - \lambda\, I_6)^3 = \begin{pmatrix} 8 & -8 & 0 & 8 & -24 & -8 \\ -48 & 16 & 24 & -28 & 84 & 32 \\ -20 & 28 & -8 & 8 & 48 & 28 \\ 40 & -8 & -24 & 20 & -60 & -24 \\ 64 & -32 & -24 & 28 & -116 & -48 \\ -64 & 32 & 24 & -20 & 108 & 48 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 8 & -8 & 0 & 8 & -24 & -8 \\ -48 & 16 & 24 & -28 & 84 & 32 \\ -20 & 28 & -8 & 8 & 48 & 28 \\ 40 & -8 & -24 & 20 & -60 & -24 \\ 64 & -32 & -24 & 28 & -116 & -48 \\ -64 & 32 & 24 & -20 & 108 & 48 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 8\,z_1 - 8\,z_2 + 8\,z_4 - 24\,z_5 - 8\,z_6 = 0, \\ -48z_1 + 16z_2 + 24z_3 - 28z_4 + 84\,z_5 + 32z_6 = 0, \\ -20z_1 + 28z_2 - 8\,z_3 + 8\,z_4 + 48\,z_5 + 28z_6 = 0, \\ 40z_1 - 8\,z_2 - 24z_3 + 20z_4 - 60\,z_5 - 24z_6 = 0, \\ 64z_1 - 32z_2 - 24z_3 + 28z_4 - 116z_5 - 48z_6 = 0, \\ -64z_1 + 32z_2 + 24z_3 - 20z_4 + 108z_5 + 48z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda\, I_6)^3 &= \{(r_1, r_2, r_1, r_1 - 2\,r_2, r_1 - 2\,r_2, -r_1 + 3\,r_2) \mid r_1, r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,1,1,1,-1), (0,1,0,-2,-2,3)\}. \end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the second power onwards in the following chain of inclusion of sets.

$$\ker(B - \lambda\, I_6) \subsetneq \ker((B - \lambda\, I_6)^2) = \ker((B - \lambda\, I_6)^3) = \cdots.$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B-\lambda I_6)$	1	$1 = \dim(\ker(B-\lambda I_6))$
$\ker(B-\lambda I_6)^2$	2	$1 = \dim(\ker(B-\lambda I_6)^2) - \dim(\ker(B-\lambda I_6))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B-\lambda I_6)^i)) - \dim(\ker((B-\lambda I_6)^{i-1}))$. The first number of this last column is $\dim(\ker(B-\lambda I_6))$.

9. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w_5}, \mathbf{w_6}\}$ satisfying

$$\begin{cases}(B-\lambda I_6)\,\mathbf{w_5} = \mathbf{0},\\ (B-\lambda I_6)\,\mathbf{w_6} = \mathbf{w_5}\end{cases}$$

or

$$\begin{cases}(B-\lambda I_6)^2\,\mathbf{w_6} = \mathbf{0},\\ (B-\lambda I_6)\,\mathbf{w_6} = \mathbf{w_5}\end{cases}$$

where $\mathbf{w_6}$ is in the vector space $\ker((B-\lambda I_6)^2)$ but not in $\ker(B-\lambda I_6)$.

We look for a generating vector $\mathbf{w_6}$. This vector must be linearly independent of all vectors in $\ker(B-\lambda I_6)$ and must be independent from vectors of height 2 that were already chosen in $\ker(B-\lambda I_6)^2$. We know that

$$\ker(B-\lambda I_6)^2 = \text{span}\{(1,0,1,1,1,-1),(0,1,0,-2,-2,3)\}.$$

We have at this point not chosen any vector in a previous Jordan chain of $\ker(B-\lambda I_6)^2$.

We know that a vector in $\ker(B-\lambda I_6)^2$ must be of the form

$$a\,(1,0,1,1,1,-1)+b\,(0,1,0,-2,-2,3) = (a,b,a,a-2\,b,a-2\,b,-a+3\,b).$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & -1 \\ a & b & a & a-2b & a-2b & 3b-a \end{pmatrix}.$$

We row reduce this matrix H and if we impose $b \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -2 & -2 & 3 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $b \neq 0$. We can choose $a = 0$ and $b = 1$.

We have the generating vector

$$\mathbf{w_6} = (0, 1, 0, -2, -2, 3).$$

We start with $\mathbf{w_6} = (0, 1, 0, -2, -2, 3)$.

We calculate $\mathbf{w_5}$.

$$\begin{aligned} \mathbf{w_5} &= (B - \lambda I_6)\,\mathbf{w_6} \\ &= \begin{pmatrix} -5 & 5 & 0 & 3 & 8 & 6 \\ -8 & 4 & 2 & -1 & 15 & 8 \\ -6 & 8 & -2 & 3 & 14 & 9 \\ 7 & -3 & -2 & 0 & -11 & -6 \\ 9 & -5 & -2 & 2 & -17 & -8 \\ -13 & 9 & 2 & 0 & 23 & 12 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}. \end{aligned}$$

We have now found the first Jordan chain for this eigenvalue. It is the third Jordan chain found in total. It has length 2.

$$\{\mathbf{w_5} = (1, 0, 1, 1, 1, -1), \mathbf{w_6} = (0, 1, 0, -2, -2, 3)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_6)$	1	$\mathbf{w_5}$	0
$\ker(B - \lambda I_6)^2$	2	$\mathbf{w_6}$	0

with

$$\mathbf{w_5} = (1, 0, 1, 1, 1, -1)$$

$$\mathbf{w_6} = (0, 1, 0, -2, -2, 3)$$

10. Result and check of the result.

We construct the matrix of base change P with the coordinates of the vectors $\mathbf{w_i}$ found in the Jordan chains in the columns of P.

$$P = \begin{pmatrix} 0 & 0 & -2 & 0 & 1 & 0 \\ 2 & 0 & 6 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -2 & 0 & -4 & -2 & 1 & -2 \\ -2 & 0 & -6 & 0 & 1 & -2 \\ 2 & 0 & 4 & -1 & -1 & 3 \end{pmatrix}.$$

$$P^{-1}BP = \begin{pmatrix} 11/2 & -5 & 0 & -2 & -19/2 & -6 \\ 2 & -3 & 1 & -3/2 & -9/2 & -3 \\ -1 & 1 & 0 & 1/2 & 3/2 & 1 \\ -1 & 1 & 0 & 0 & 2 & 1 \\ -1 & 2 & 0 & 1 & 3 & 2 \\ -3 & 3 & 0 & 1 & 6 & 4 \end{pmatrix}$$
$$\times \begin{pmatrix} -4 & 5 & 0 & 3 & 8 & 6 \\ -8 & 5 & 2 & -1 & 15 & 8 \\ -6 & 8 & -1 & 3 & 14 & 9 \\ 7 & -3 & -2 & 1 & -11 & -6 \\ 9 & -5 & -2 & 2 & -16 & -8 \\ -13 & 9 & 2 & 0 & 23 & 13 \end{pmatrix}$$
$$\times \begin{pmatrix} 0 & 0 & -2 & 0 & 1 & 0 \\ 2 & 0 & 6 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -2 & 0 & -4 & -2 & 1 & -2 \\ -2 & 0 & -6 & 0 & 1 & -2 \\ 2 & 0 & 4 & -1 & -1 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} \begin{pmatrix} \mathbf{-1} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{-1} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \end{pmatrix}.$$

29 exercise. (6×6); $(\mathbf{J}_3(-1), \mathbf{J}_1(-1), \mathbf{J}_2(1))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} -7 & 7 & 1 & 2 & 13 & 8 \\ -8 & 5 & 2 & -1 & 15 & 8 \\ -5 & 7 & -1 & 3 & 12 & 8 \\ 10 & -5 & -3 & 2 & -16 & -8 \\ 9 & -5 & -2 & 2 & -16 & -8 \\ -16 & 11 & 3 & -1 & 28 & 15 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_6| = (\lambda - 1)^2 (\lambda + 1)^4.$$

We see that the characteristic Cayley-Hamilton polynomial $p_{\text{C-H}}(\lambda)$ can be factorised in linear polynomials over the field $\mathbf{K}$. So we can apply the Jordan normalisation machinery.

The eigenvalue $\lambda = 1$ has algebraic multiplicity 2. The eigenvalue $\lambda = -1$ has algebraic multiplicity 4.

2. Investigation of the first eigenvalue.

We work now with the eigenvalue $\lambda = -1$.

3. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or

calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 4 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 3. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (-1) & 0 & 0 \\ 0 & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_6.

$$A - \lambda I_6 = \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (0) & 0 & 0 \\ 0 & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

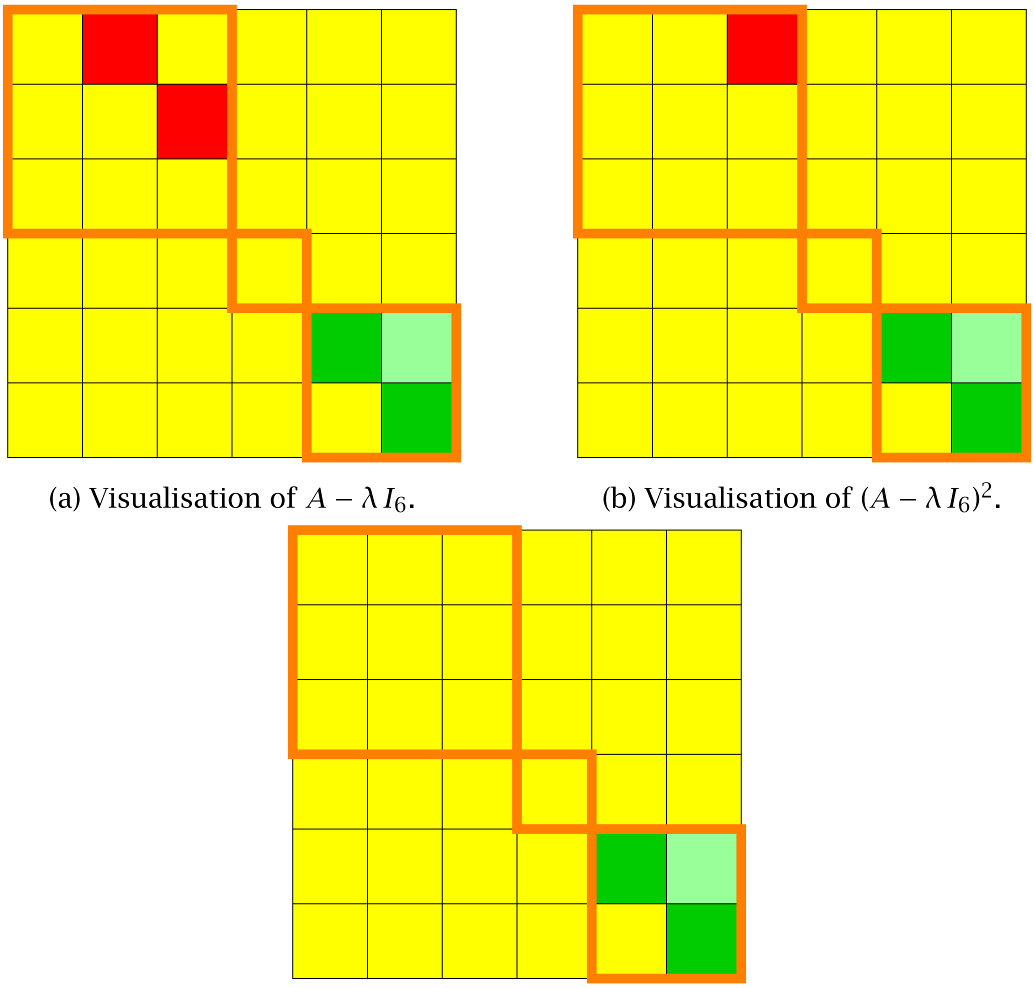
We compute also the powers of $A - \lambda I_6$.

$$A - \lambda I_6 = \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (0) & 0 & 0 \\ 0 & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{matrix} \end{pmatrix},$$

$$(A - \lambda I_6)^2 = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (0) & 0 & 0 \\ 0 & \begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix} \end{matrix} \end{pmatrix},$$

$$(A - \lambda I_6)^3 = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (0) & 0 & 0 \\ 0 & \begin{pmatrix} 8 & 12 \\ 0 & 8 \end{pmatrix} \\ 0 & & \end{matrix} \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $A - \lambda I_6$.

(b) Visualisation of $(A - \lambda I_6)^2$.

(c) Visualisation of $(A - \lambda I_6)^3$.

Figure 41

We give some comment on the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells represent the number 1. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix. If we restrict the mapping $A - \lambda I_6$ to the subspace $\text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ which is an invariant subspace with respect to the operator $A - \lambda I_6$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 3. Let us take a look at the two first elementary Jordan blocks in this matrix.

We observe that the original superdiagonals of 1's in the two first blocks of $A - \lambda I_6$ are going upwards in their respective elementary Jordan blocks associated with the nilpotent part of the transformation $A - \lambda I_6$ when increasing the powers of the matrix $A - \lambda I_6$. They finally disappear when taking the third power of $A - \lambda I_6$.

Investigation of the first Jordan chain.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_6) = \text{span}\{\mathbf{e}_1, \mathbf{e}_4\}; \\ \ker(A - \lambda I_6)^2 = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4\}; \\ \ker(A - \lambda I_6)^3 = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_6)$	2	$2 = \dim(\ker(A - \lambda I_6))$
$\ker(A - \lambda I_6)^2$	3	$1 = \dim(\ker(A - \lambda I_6)^2) - \dim(\ker(A - \lambda I_6))$
$\ker(A - \lambda I_6)^3$	4	$1 = \dim(\ker(A - \lambda I_6)^3) - \dim(\ker(A - \lambda I_6)^2)$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A-\lambda I_6)^i)-\dim(\ker(A-\lambda I_6)^{i-1})$. In the first row, we have $\dim(\ker(A-\lambda I_6))$.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A-\lambda I_6) \subsetneq \ker(A-\lambda I_6)^2 \subsetneq \ker(A-\lambda I_6)^3 = \ker(A-\lambda I_6)^4 = \cdots .$$

We remark by looking at the matrices $(A-\lambda I_6)^i$ that we have the following mappings

$$\begin{cases} (A-\lambda I_6)\,\mathbf{e}_3 = \mathbf{e}_2, \\ (A-\lambda I_6)\,\mathbf{e}_2 = \mathbf{e}_1, \\ (A-\lambda I_6)\,\mathbf{e}_1 = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A-\lambda I_6)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases} (A-\lambda I_6)\,\mathbf{e}_3 = \mathbf{e}_2, \\ (A-\lambda I_6)^2\,\mathbf{e}_3 = \mathbf{e}_1, \\ (A-\lambda I_6)^3\,\mathbf{e}_3 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = (A-\lambda I_6)^2\,\mathbf{e}_3, \mathbf{e}_2 = (A-\lambda I_6)\,\mathbf{e}_3, \mathbf{e}_3\}.$$

After we have found the first Jordan chain of length 3, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A-\lambda I_6)$	2	$\mathbf{e}_1$	1
$\ker(A-\lambda I_6)^2$	3	$\mathbf{e}_2$	0
$\ker(A-\lambda I_6)^3$	4	$\mathbf{e}_3$	0

We have that the last number in the last column is not zero, and this means that there is still a chain left of length 1.

Investigating the second Jordan chain.
We remark by looking at the matrices $(A - \lambda I_6)^i$ that we have the following mapping

$$(A - \lambda I_6)\,\mathbf{e_4} = \mathbf{0}.$$

We see that we have a second Jordan chain. It has length one.

$$\{\mathbf{e_4}\}.$$

We have now the following table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(A - \lambda I_6)$	2	$\mathbf{e_1}$	$\mathbf{e_4}$	0
$\ker(A - \lambda I_6)^2$	3	$\mathbf{e_2}$		0
$\ker(A - \lambda I_6)^3$	4	$\mathbf{e_3}$		0

4. Kernels of $((\mathbf{B} - \lambda\,\mathbf{I_6})^{\mathbf{i}})$.

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I_6})$.
The matrix $B - \lambda I_6$ is

$$B - \lambda I_6 = \begin{pmatrix} -6 & 7 & 1 & 2 & 13 & 8 \\ -8 & 6 & 2 & -1 & 15 & 8 \\ -5 & 7 & 0 & 3 & 12 & 8 \\ 10 & -5 & -3 & 3 & -16 & -8 \\ 9 & -5 & -2 & 2 & -15 & -8 \\ -16 & 11 & 3 & -1 & 28 & 16 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -6 & 7 & 1 & 2 & 13 & 8 \\ -8 & 6 & 2 & -1 & 15 & 8 \\ -5 & 7 & 0 & 3 & 12 & 8 \\ 10 & -5 & -3 & 3 & -16 & -8 \\ 9 & -5 & -2 & 2 & -15 & -8 \\ -16 & 11 & 3 & -1 & 28 & 16 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -6z_1 + 7z_2 + z_3 + 2z_4 + 13z_5 + 8z_6 = 0, \\ -8z_1 + 6z_2 + 2z_3 - z_4 + 15z_5 + 8z_6 = 0, \\ -5z_1 + 7z_2 + 3z_4 + 12z_5 + 8z_6 = 0, \\ 10z_1 - 5z_2 - 3z_3 + 3z_4 - 16z_5 - 8z_6 = 0, \\ 9z_1 - 5z_2 - 2z_3 + 2z_4 - 15z_5 - 8z_6 = 0, \\ -16z_1 + 11z_2 + 3z_3 - z_4 + 28z_5 + 16z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_6) &= \{(r_1, r_2, r_1, -r_1 - r_2, -r_1 - r_2, 5r_1/2 + r_2) \mid r_1, r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,1,-1,-1,5/2), (0,1,0,-1,-1,1)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\, \mathbf{I_6})^2$.

We calculate the kernel of $(B - \lambda I_6)^2$.

The matrix $(B - \lambda I_6)^2$ is

$$(B - \lambda I_6)^2 = \begin{pmatrix} -16 & 20 & 0 & 8 & 36 & 24 \\ -13 & 12 & 1 & 3 & 25 & 16 \\ -16 & 20 & 0 & 8 & 36 & 24 \\ 9 & -4 & -1 & 1 & -13 & -8 \\ 9 & -4 & -1 & 1 & -13 & -8 \\ -21 & 16 & 1 & 3 & 37 & 24 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -16 & 20 & 0 & 8 & 36 & 24 \\ -13 & 12 & 1 & 3 & 25 & 16 \\ -16 & 20 & 0 & 8 & 36 & 24 \\ 9 & -4 & -1 & 1 & -13 & -8 \\ 9 & -4 & -1 & 1 & -13 & -8 \\ -21 & 16 & 1 & 3 & 37 & 24 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -16z_1 + 20z_2 \phantom{{}+z_3} + 8z_4 + 36z_5 + 24z_6 = 0, \\ -13z_1 + 12z_2 + z_3 + 3z_4 + 25z_5 + 16z_6 = 0, \\ -16z_1 + 20z_2 \phantom{{}+z_3} + 8z_4 + 36z_5 + 24z_6 = 0, \\ 9\,z_1 - 4\,z_2 - z_3 + z_4 - 13z_5 - 8\,z_6 = 0, \\ 9\,z_1 - 4\,z_2 - z_3 + z_4 - 13z_5 - 8\,z_6 = 0, \\ -21z_1 + 16z_2 + z_3 + 3z_4 + 37z_5 + 24z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda\, I_6)^2 &= \{(r_1, r_2, r_3, -r_1 - r_2, -r_2 - r_3, r_1 + r_2 + 3\,r_3/2) \\ &\qquad \mid r_1, r_2, r_3 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,-1,0,1), (0,1,0,-1,-1,1), \\ &\qquad (0,0,1,0,-1,3/2)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I_6})^3$.
We calculate the kernel of $(B - \lambda\, I_6)^3$.

The matrix $(B - \lambda\, I_6)^3$ is

$$(B - \lambda\, I_6)^3 = \begin{pmatrix} -44 & 52 & 0 & 20 & 96 & 64 \\ -24 & 24 & 0 & 8 & 48 & 32 \\ -44 & 52 & 0 & 20 & 96 & 64 \\ 4 & 4 & 0 & 4 & 0 & 0 \\ 4 & 4 & 0 & 4 & 0 & 0 \\ -28 & 20 & 0 & 4 & 48 & 32 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -44 & 52 & 0 & 20 & 96 & 64 \\ -24 & 24 & 0 & 8 & 48 & 32 \\ -44 & 52 & 0 & 20 & 96 & 64 \\ 4 & 4 & 0 & 4 & 0 & 0 \\ 4 & 4 & 0 & 4 & 0 & 0 \\ -28 & 20 & 0 & 4 & 48 & 32 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -44z_1 + 52z_2 + 20z_4 + 96z_5 + 64z_6 = 0, \\ -24z_1 + 24z_2 + 8z_4 + 48z_5 + 32z_6 = 0, \\ -44z_1 + 52z_2 + 20z_4 + 96z_5 + 64z_6 = 0, \\ 4z_1 + 4z_2 + 4z_4 = 0, \\ 4z_1 + 4z_2 + 4z_4 = 0, \\ -28z_1 + 20z_2 + 4z_4 + 48z_5 + 32z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_6)^3 &= \{(r_1, r_2, r_3, -r_1 - r_2, r_5, r_1 - r_2/2 - 3r_5/2) \\ &\qquad \mid r_1, r_2, r_3, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,-1,0,1), (0,1,0,-1,0,-1/2), \\ &\qquad (0,0,1,0,0,0), (0,0,0,0,1,-3/2)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda_6)^4$.
We calculate the kernel of $(B - \lambda I_6)^4$.

The matrix $(B - \lambda I_6)^4$ is

$$(B - \lambda I_6)^4 = \begin{pmatrix} -112 & 128 & 0 & 48 & 240 & 160 \\ -48 & 48 & 0 & 16 & 96 & 64 \\ -112 & 128 & 0 & 48 & 240 & 160 \\ -16 & 32 & 0 & 16 & 48 & 32 \\ -16 & 32 & 0 & 16 & 48 & 32 \\ -32 & 16 & 0 & 0 & 48 & 32 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -112 & 128 & 0 & 48 & 240 & 160 \\ -48 & 48 & 0 & 16 & 96 & 64 \\ -112 & 128 & 0 & 48 & 240 & 160 \\ -16 & 32 & 0 & 16 & 48 & 32 \\ -16 & 32 & 0 & 16 & 48 & 32 \\ -32 & 16 & 0 & 0 & 48 & 32 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -112z_1 + 128z_2 + 48z_4 + 240z_5 + 160z_6 = 0, \\ -\ 48\ z_1 +\ 48\ z_2 + 16z_4 +\ 96\ z_5 +\ 64\ z_6 = 0, \\ -112z_1 + 128z_2 + 48z_4 + 240z_5 + 160z_6 = 0, \\ -\ 16\ z_1 +\ 32\ z_2 + 16z_4 +\ 48\ z_5 +\ 32\ z_6 = 0, \\ -\ 16\ z_1 +\ 32\ z_2 + 16z_4 +\ 48\ z_5 +\ 32\ z_6 = 0, \\ -\ 32\ z_1 +\ 16\ z_2 \qquad\quad +\ 48\ z_5 +\ 32\ z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_6)^4 &= \{(r_1, r_2, r_3, -r_1 - r_2, r_5, r_1 - r_2/2 - 3\, r_5/2) \\ &\qquad \mid r_1, r_2, r_3, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,-1,0,1), (0,1,0,-1,0,-1/2), \\ &\qquad (0,0,1,0,0,0), (0,0,0,0,1,-3/2)\}. \end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the third power onwards in the following chain of inclusion of sets.

$$\ker(B - \lambda I_6) \subsetneq \ker((B - \lambda I_6)^2) \subsetneq \ker((B - \lambda I_6)^3) = \ker((B - \lambda I_6)^4) = \cdots .$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B - \lambda I_6)$	2	$2 = \dim(\ker(B - \lambda I_6))$
$\ker(B - \lambda I_6)^2$	3	$1 = \dim(\ker((B - \lambda I_6)^2)) - \dim(\ker(B - \lambda I_6))$
$\ker(B - \lambda I_6)^3$	4	$1 = \dim(\ker((B - \lambda I_6)^3)) - \dim(\ker((B - \lambda I_6)^2))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B - \lambda I_6)^i)) - \dim(\ker((B - \lambda I_6)^{i-1}))$. The first number of this last column is $\dim(\ker(B - \lambda I_6))$.

5. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ satisfying

$$\begin{cases} (B - \lambda I_6)\,\mathbf{w}_1 = \mathbf{0}, \\ (B - \lambda I_6)\,\mathbf{w}_2 = \mathbf{w}_1, \\ (B - \lambda I_6)\,\mathbf{w}_3 = \mathbf{w}_2, \end{cases}$$

or

$$\begin{cases} (B - \lambda I_6)^3\,\mathbf{w}_3 = \mathbf{0}, \\ (B - \lambda I_6)^2\,\mathbf{w}_3 = \mathbf{w}_1, \\ (B - \lambda I_6)\,\mathbf{w}_3 = \mathbf{w}_2, \end{cases}$$

where $\mathbf{w}_3$ is in the vector space $\ker((B - \lambda I_6)^3)$ but not in $\ker((B - \lambda I_6)^2)$.

We look for a generating vector $\mathbf{w}_3$. This vector must be linearly independent of all vectors in $\ker(B - \lambda I_6)^2$ and must be independent from vectors of height 3 that were already chosen in $\ker(B - \lambda I_6)^3$.

We have at this point not chosen any vector of height 3 in a previous Jordan chain of $\ker((B - \lambda I_6)^3)$.

We remember that

$$\begin{aligned} \ker((B - \lambda I_6)^3) = \operatorname{span}\{&(1, 0, 0, -1, 0, 1), (0, 1, 0, -1, 0, -1/2), \\ &(0, 0, 1, 0, 0, 0)(0, 0, 0, 0, 1, -3/2)\}. \end{aligned}$$

We know that a vector in $\ker((B - \lambda I_6)^3)$ must be of the form

$$\begin{aligned} a\,(1, 0, 0, -1, 0, 1) + b\,(0, 1, 0, -1, 0, -1/2) + c\,(0, 0, 1, 0, 0, 0) \\ + d\,(0, 0, 0, 0, 1, -3/2) \\ = (a, b, c, -a - b, d, a - b/2 - 3\,d/2). \end{aligned}$$

We remember also that

$$\begin{aligned} \ker(B - \lambda I_6)^2 = \operatorname{span}\{&(1, 0, 0, -1, 0, 1), (0, 1, 0, -1, -1, 1), \\ &(0, 0, 1, 0, -1, 3/2)\}. \end{aligned}$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 3/2 \\ a & b & c & -a-b & d & a-b/2-3d/2 \end{pmatrix}.$$

We row reduce this matrix H and if we impose $b+c+d \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3/2 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $b+c+d \neq 0$. We can take $a = 0$, $b = 1$, $c = 0$ and $d = 0$.
We have the generating vector

$$\mathbf{w_3} = (0, 1, 0, -1, 0, -1/2).$$

We start with $\mathbf{w_3} = (0, 1, 0, -1, 0, -1/2)$.

We calculate $\mathbf{w_2}$.

$$\begin{aligned} \mathbf{w_2} &= (B - \lambda I_6)\mathbf{w_4} \\ &= \begin{pmatrix} -6 & 7 & 1 & 2 & 13 & 8 \\ -8 & 6 & 2 & -1 & 15 & 8 \\ -5 & 7 & 0 & 3 & 12 & 8 \\ 10 & -5 & -3 & 3 & -16 & -8 \\ 9 & -5 & -2 & 2 & -15 & -8 \\ -16 & 11 & 3 & -1 & 28 & 16 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \\ -4 \\ -3 \\ 4 \end{pmatrix} \end{aligned}$$

and

$$\mathbf{w_1} = (B - \lambda I_6)^2 \mathbf{w_3}$$
$$= \begin{pmatrix} -16 & 20 & 0 & 8 & 36 & 24 \\ -13 & 12 & 1 & 3 & 25 & 16 \\ -16 & 20 & 0 & 8 & 36 & 24 \\ 9 & -4 & -1 & 1 & -13 & -8 \\ 9 & -4 & -1 & 1 & -13 & -8 \\ -21 & 16 & 1 & 3 & 37 & 24 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

We have now found a first Jordan chain. It has length 3.

$$\{\mathbf{w_1} = (0, 1, 0, -1, -1, 1), \mathbf{w_2} = (1, 3, 0, -4, -3, 4),$$
$$\mathbf{w_3} = (0, 1, 0, -1, 0, -1/2)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_6)$	2	$\mathbf{w_1}$	1
$\ker((B - \lambda I_6)^2)$	3	$\mathbf{w_2}$	0
$\ker((B - \lambda I_6)^3)$	4	$\mathbf{w_3}$	0

with

$$\mathbf{w_1} = (0, 1, 0, -1, -1, 1)$$
$$\mathbf{w_2} = (1, 3, 0, -4, -3, 4)$$
$$\mathbf{w_3} = (0, 1, 0, -1, 0, -1/2)$$

Calculation of the second Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w_4}\}$ satisfying

$$(B - \lambda I_6)\,\mathbf{w_4} = \mathbf{0},$$

where $\mathbf{w_4}$ is in the vector space $\ker(B - \lambda I_6)$.

We look for a generating vector $\mathbf{w_4}$. This vector must be linearly independent of vectors that were already chosen in $\ker(B - \lambda I_6)$. We know that

$$\ker(B - \lambda I_6) = \text{span}\{(1, 0, 1, -1, -1, 5/2), (0, 1, 0, -1, -1, 1)\}.$$

We have at this point chosen in $\ker(B - \lambda I_6)$ already the vector $\mathbf{w_1} = (0, 1, 0, -1, -1, 1)$.

We know that a vector in $\ker(B - \lambda I_6)$ must be of the form

$$\begin{aligned} a\,(1, 0, 1, -1, -1, 5/2) &+ b\,(0, 1, 0, -1, -1, 1) \\ &= (a, b, a, -a - b, -a - b, 5\,a/2 + b). \end{aligned}$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 0 & 1 & 0 & -1 & -1 & 1 \\ a & b & a & -a - b & -a - b & 5\,a/2 + b \end{pmatrix}.$$

We row reduce this matrix H and if we impose $a \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 1 & -1 & -1 & 5/2 \\ 0 & 1 & 0 & -1 & -1 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a \neq 0$. So we have $a = 1$ and $b = 0$.

So we have the generating vector

$$\mathbf{w_4} = (1, 0, 1, -1, -1, 5/2).$$

We have now found a second Jordan chain. It has length one.

$$\{(1, 0, 1, -1, -1, 5/2)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(B - \lambda I_6)$	2	$\mathbf{w_1}$	$\mathbf{w_4}$	0
$\ker(B - \lambda I_6)^2$	3	$\mathbf{w_2}$		0
$\ker(B - \lambda I_6)^3$	4	$\mathbf{w_3}$		0

with

$\mathbf{w_1} = (0, 1, 0, -1, -1, 1)$
$\mathbf{w_2} = (1, 3, 0, -4, -3, 4)$
$\mathbf{w_3} = (0, 1, 0, -1, 0, -1/2)$
$\mathbf{w_4} = (1, 0, 1, -1, -1, 5/2)$

6. Investigation of the second eigenvalue.

We work now with the eigenvalue $\lambda = 1$.

7. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 8 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 7. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (-1) & 0 & 0 \\ 0 & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_6.

$$A - \lambda I_6 = \begin{pmatrix} \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (-2) & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

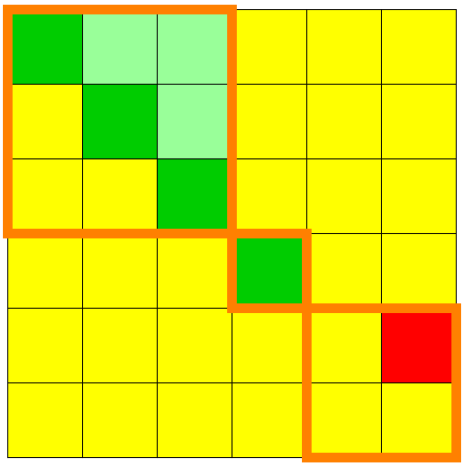
We see that the space $\operatorname{span}\{\mathbf{e}_5, \mathbf{e}_6\}$ is invariant relative to $A - \lambda I_6$. If we restrict $A - \lambda I_6$ to this space, we have a nilpotent operator and we can apply the techniques learned in part 2.

We want to investigate the endomorphism $A - \lambda I_6$ restricted to this space. We compute now the powers of $A - \lambda I_6$.

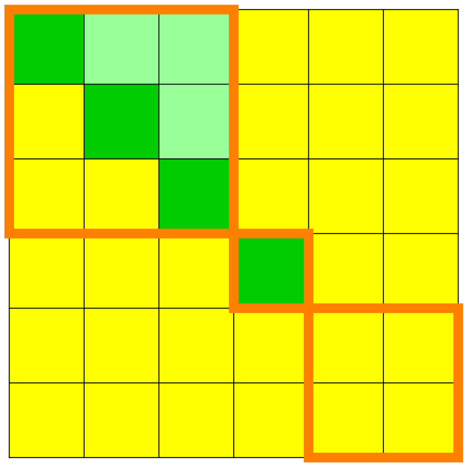
$$A - \lambda I_6 = \begin{pmatrix} \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (-2) & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{matrix} \end{pmatrix},$$

$$(A - \lambda I_6)^2 = \begin{pmatrix} \begin{pmatrix} 4 & -4 & 1 \\ 0 & 4 & -4 \\ 0 & 0 & 4 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (4) & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{matrix} \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $(A - \lambda I_6)$.



(b) Visualisation of $(A - \lambda I_6)^2$.

Figure 42

We give some comment on the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells represent the number 1. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix.

If we restrict the mapping $A - \lambda I_6$ to the subspace span$\{\mathbf{e}_5, \mathbf{e}_6\}$ which is an invariant subspace with respect to the operator $A - \lambda I_6$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 2. Let us take a look at the third elementary Jordan block in this matrix.

We observe that the original superdiagonal of 1's in the third block of $A - \lambda I_6$ is going upwards in its elementary Jordan block associated with the nilpotent part of the transformation $A - \lambda I_6$ when increasing the powers of the matrix $A - \lambda I_6$. It finally disappears when taking the second power of $A - \lambda I_6$.

Investigation of the first Jordan chain.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_6) = \operatorname{span}\{\mathbf{e}_5\}; \\ \ker(A - \lambda I_6)^2 = \operatorname{span}\{\mathbf{e}_5, \mathbf{e}_6\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_6)$	1	$1 = \dim(\ker(A - \lambda I_6))$
$\ker(A - \lambda I_6)^2$	2	$1 = \dim(\ker(A - \lambda I_6)^2) - \dim(\ker(A - \lambda I_6))$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A-\lambda I_6)^i)-\dim(\ker(A-\lambda I_6)^{i-1})$. In the first row, we have $\dim(\ker(A - \lambda I_6))$.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A - \lambda I_6) \subsetneq \ker(A - \lambda I_6)^2 = \ker(A - \lambda I_6)^3 = \cdots .$$

We remark by looking at the matrices $(A - \lambda I_6)^i$ that we have the following mappings

$$\begin{cases} (A - \lambda I_6)\,\mathbf{e}_6 = \mathbf{e}_5, \\ (A - \lambda I_6)\,\mathbf{e}_5 = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A - \lambda I_6)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases} (A - \lambda I_6)^2\,\mathbf{e}_6 = \mathbf{0}, \\ (A - \lambda I_6)\,\mathbf{e}_6 = \mathbf{e}_5. \end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_5 = (A - \lambda I_6)\,\mathbf{e}_6, \mathbf{e}_6\}.$$

After we have found the first Jordan chain of length 4, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A - \lambda I_6)$	1	$\mathbf{e_5}$	0
$\ker(A - \lambda I_6)^2$	2	$\mathbf{e_6}$	0

We have now in the last column only 0's and this means that we are done looking for Jordan chains.

8. Kernels of $(\mathbf{B} - \lambda\, \mathbf{I_6})^{\mathbf{i}}$.

Kernel of $(\mathbf{B} - \lambda\, \mathbf{I_6})$.
We calculate the kernel of $B - \lambda I_6$.

The matrix $B - \lambda I_6$ is

$$B - \lambda I_6 = \begin{pmatrix} -8 & 7 & 1 & 2 & 13 & 8 \\ -8 & 4 & 2 & -1 & 15 & 8 \\ -5 & 7 & -2 & 3 & 12 & 8 \\ 10 & -5 & -3 & 1 & -16 & -8 \\ 9 & -5 & -2 & 2 & -17 & -8 \\ -16 & 11 & 3 & -1 & 28 & 14 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -8 & 7 & 1 & 2 & 13 & 8 \\ -8 & 4 & 2 & -1 & 15 & 8 \\ -5 & 7 & -2 & 3 & 12 & 8 \\ 10 & -5 & -3 & 1 & -16 & -8 \\ 9 & -5 & -2 & 2 & -17 & -8 \\ -16 & 11 & 3 & -1 & 28 & 14 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -8z_1 + 7z_2 + z_3 + 2z_4 + 13z_5 + 8z_6 = 0, \\ -8z_1 + 4z_2 + 2z_3 - z_4 + 15z_5 + 8z_6 = 0, \\ -5z_1 + 7z_2 - 2z_3 + 3z_4 + 12z_5 + 8z_6 = 0, \\ 10z_1 - 5z_2 - 3z_3 + z_4 - 16z_5 - 8z_6 = 0, \\ 9z_1 - 5z_2 - 2z_3 + 2z_4 - 17z_5 - 8z_6 = 0, \\ -16z_1 + 11z_2 + 3z_3 - z_4 + 28z_5 + 14z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B-\lambda I_6) &= \{(r_1,0,r_1,r_1,r_1,-r_1) \mid r_1 \in \mathbf{K}\}\\ &= \operatorname{span}\{(1,0,1,1,1,-1)\}.\end{aligned}$$

Kernel of $(\mathbf{B}-\lambda\,\mathbf{I_6})^2$.
We calculate the kernel of $(B-\lambda I_6)^2$.

The matrix $(B-\lambda I_6)^2$ is

$$(B-\lambda I_6)^2 = \begin{pmatrix} 12 & -8 & -4 & 0 & -16 & -8\\ 19 & -8 & -7 & 7 & -35 & -16\\ 4 & -8 & 4 & -4 & -12 & -8\\ -31 & 16 & 11 & -7 & 51 & 24\\ -27 & 16 & 7 & -7 & 51 & 24\\ 43 & -28 & -11 & 7 & -75 & -36 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 12 & -8 & -4 & 0 & -16 & -8\\ 19 & -8 & -7 & 7 & -35 & -16\\ 4 & -8 & 4 & -4 & -12 & -8\\ -31 & 16 & 11 & -7 & 51 & 24\\ -27 & 16 & 7 & -7 & 51 & 24\\ 43 & -28 & -11 & 7 & -75 & -36 \end{pmatrix}\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\\ z_5\\ z_6\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 12z_1 - 8z_2 - 4z_3 - 16z_5 - 8z_6 = 0,\\ 19z_1 - 8z_2 - 7z_3 + 7z_4 - 35z_5 - 16z_6 = 0,\\ 4z_1 - 8z_2 + 4z_3 - 4z_4 - 12z_5 - 8z_6 = 0,\\ -31z_1 + 16z_2 + 11z_3 - 7z_4 + 51z_5 + 24z_6 = 0,\\ -27z_1 + 16z_2 + 7z_3 - 7z_4 + 51z_5 + 24z_6 = 0,\\ 43z_1 - 28z_2 - 11z_3 + 7z_4 - 75z_5 - 36z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B-\lambda I_6)^2 &= \{(r_1,r_2,r_1,r_1-2r_2,r_1-2r_2,-r_1+3r_2) \mid r_1,r_2 \in \mathbf{K}\}\\ &= \operatorname{span}\{(1,0,1,1,1,-1),(0,1,0,-2,-2,3)\}.\end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\, \mathbf{I_6})^3$.
We calculate the kernel of $(B - \lambda\, I_6)^3$.

The matrix $(B - \lambda\, I_6)^3$ is

$$(B - \lambda\, I_6)^3 = \begin{pmatrix} -28 & 16 & 12 & -4 & 36 & 16 \\ -42 & 16 & 18 & -22 & 78 & 32 \\ -8 & 16 & -8 & 8 & 24 & 16 \\ 70 & -32 & -30 & 26 & -114 & -48 \\ 58 & -32 & -18 & 22 & -110 & -48 \\ -94 & 56 & 30 & -26 & 162 & 72 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -28 & 16 & 12 & -4 & 36 & 16 \\ -42 & 16 & 18 & -22 & 78 & 32 \\ -8 & 16 & -8 & 8 & 24 & 16 \\ 70 & -32 & -30 & 26 & -114 & -48 \\ 58 & -32 & -18 & 22 & -110 & -48 \\ -94 & 56 & 30 & -26 & 162 & 72 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -28z_1 + 16z_2 + 12z_3 - 4\,z_4 + 36\,z_5 + 16z_6 = 0, \\ -42z_1 + 16z_2 + 18z_3 - 22z_4 + 78\,z_5 + 32z_6 = 0, \\ -8\,z_1 + 16z_2 - 8\,z_3 + 8\,z_4 + 24\,z_5 + 16z_6 = 0, \\ 70z_1 - 32z_2 - 30z_3 + 26z_4 - 114z_5 - 48z_6 = 0, \\ 58z_1 - 32z_2 - 18z_3 + 22z_4 - 110z_5 - 48z_6 = 0, \\ -94z_1 + 56z_2 + 30z_3 - 26z_4 + 162z_5 + 72z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda\, I_6)^3 &= \{(r_1, r_2, r_1, r_1 - 2\,r_2, r_1 - 2\,r_2, -r_1 + 3\,r_2) \mid r_1, r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,1,1,1,-1), (0,1,0,-2,-2,3)\}. \end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the second power onwards in the following chain of inclusion of sets.

$$\ker(B - \lambda\, I_6) \subsetneq \ker((B - \lambda\, I_6)^2) = \ker((B - \lambda\, I_6)^3) = \cdots .$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B-\lambda I_6)$	1	$1 = \dim(\ker(B-\lambda I_6))$
$\ker(B-\lambda I_6)^2$	2	$1 = \dim(\ker(B-\lambda I_6)^2) - \dim(\ker(B-\lambda I_6))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B-\lambda I_6)^i)) - \dim(\ker((B-\lambda I_6)^{i-1}))$. The first number of this last column is $\dim(\ker(B-\lambda I_6))$.

9. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w_5}, \mathbf{w_6}\}$ satisfying

$$\begin{cases} (B-\lambda I_6)\,\mathbf{w_5} = \mathbf{0}, \\ (B-\lambda I_6)\,\mathbf{w_6} = \mathbf{w_5} \end{cases}$$

or

$$\begin{cases} (B-\lambda I_6)^2\,\mathbf{w_6} = \mathbf{0}, \\ (B-\lambda I_6)\,\mathbf{w_6} = \mathbf{w_5} \end{cases}$$

where $\mathbf{w_6}$ is in the vector space $\ker((B-\lambda I_6)^2)$ but not in $\ker(B-\lambda I_6)$.

We look for a generating vector $\mathbf{w_6}$. This vector must be linearly independent of all vectors in $\ker(B-\lambda I_6)$ and must be independent from vectors of height 2 that were already chosen in $\ker(B-\lambda I_6)^2$. We know that

$$\ker((B-\lambda I_6)^2) = \operatorname{span}\{(1,0,1,1,1,-1),(0,1,0,-2,-2,3)\}.$$

We have at this point not chosen any vector in a previous Jordan chain of $\ker((B-\lambda I_6)^2)$.

We know that a vector in $\ker((B-\lambda I_6)^2)$ must be of the form

$$a\,(1,0,1,1,1,-1)+b\,(0,1,0,-2,-2,3) = (a,b,a,a-2\,b,a-2\,b,-a+3\,b).$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & -1 \\ a & b & a & a-2b & a-2b & 3b-a \end{pmatrix}.$$

We row reduce this matrix H and if we impose $b \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -2 & -2 & 3 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $b \neq 0$. We can choose $a = 0$ and $b = 1$.

We have the generating vector

$$\mathbf{w_6} = (0, 1, 0, -2, -2, 3).$$

We start with $\mathbf{w_6} = (0, 1, 0, -2, -2, 3)$.

We calculate $\mathbf{w_5}$.

$$\begin{aligned} \mathbf{w_5} &= (B - \lambda I_6)\mathbf{w_6} \\ &= \begin{pmatrix} -8 & 7 & 1 & 2 & 13 & 8 \\ -8 & 4 & 2 & -1 & 15 & 8 \\ -5 & 7 & -2 & 3 & 12 & 8 \\ 10 & -5 & -3 & 1 & -16 & -8 \\ 9 & -5 & -2 & 2 & -17 & -8 \\ -16 & 11 & 3 & -1 & 28 & 14 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}. \end{aligned}$$

We have found a first Jordan chain for this eigenvalue. We have found three Jordan chains in total. The length of this chain is 2.

$$\{\mathbf{w_5} = (1, 0, 1, 1, 1, -1), \mathbf{w_6} = (0, 1, 0, -2, -2, 3)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_6)$	1	$\mathbf{w_5}$	0
$\ker(B - \lambda I_6)^2$	2	$\mathbf{w_6}$	0

with

$$\mathbf{w_5} = (1, 0, 1, 1, 1, -1)$$

$$\mathbf{w_6} = (0, 1, 0, -2, -2, 3)$$

10. Result and check of the result.

We construct the matrix of base change P with the coordinates of the vectors $\mathbf{w_i}$ found in the Jordan chains in the columns of P.

$$P = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & -4 & -1 & -1 & 1 & -2 \\ -1 & -3 & 0 & -1 & 1 & -2 \\ 1 & 4 & -1/2 & 5/2 & -1 & 3 \end{pmatrix}.$$

$$P^{-1}\,B\,P = \begin{pmatrix} 1 & -2 & 2 & 0 & -7 & -4 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \\ 1 & -2 & 1 & -1 & -3 & -2 \\ -1 & 2 & 0 & 1 & 3 & 2 \\ -3 & 3 & 0 & 1 & 6 & 4 \end{pmatrix}$$

$$\times \begin{pmatrix} -7 & 7 & 1 & 2 & 13 & 8 \\ -8 & 5 & 2 & -1 & 15 & 8 \\ -5 & 7 & -1 & 3 & 12 & 8 \\ 10 & -5 & -3 & 2 & -16 & -8 \\ 9 & -5 & -2 & 2 & -16 & -8 \\ -16 & 11 & 3 & -1 & 28 & 15 \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & -4 & -1 & -1 & 1 & -2 \\ -1 & -3 & 0 & -1 & 1 & -2 \\ 1 & 4 & -1/2 & 5/2 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} \mathbf{-1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} (\mathbf{-1}) & 0 & 0 \\ 0 & \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \end{matrix} \end{pmatrix}.$$

30 exercise. (6×6); $(\mathbf{J}_2(-1), \mathbf{J}_2(-1), \mathbf{J}_2(1))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} -4 & 5 & 0 & 3 & 8 & 6 \\ -8 & 5 & 2 & -1 & 15 & 8 \\ -6 & 8 & -1 & 3 & 14 & 9 \\ 7 & -3 & -2 & 1 & -11 & -6 \\ 9 & -5 & -2 & 2 & -16 & -8 \\ -13 & 9 & 2 & 0 & 23 & 13 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_6| = (\lambda - 1)^2 (\lambda + 1)^4.$$

We see that the characteristic Cayley-Hamilton polynomial $p_{\text{C-H}}(\lambda)$ can be factorised in linear polynomials over the field $\mathbf{K}$. So we can apply the Jordan normalisation machinery.

The eigenvalue $\lambda = -1$ has algebraic multiplicity 4. The eigenvalue $\lambda = 1$ has algebraic multiplicity 2.

2. Investigation of the first eigenvalue.

We work now with the eigenvalue $\lambda = -1$.

3. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or

calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 4 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 3. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_6.

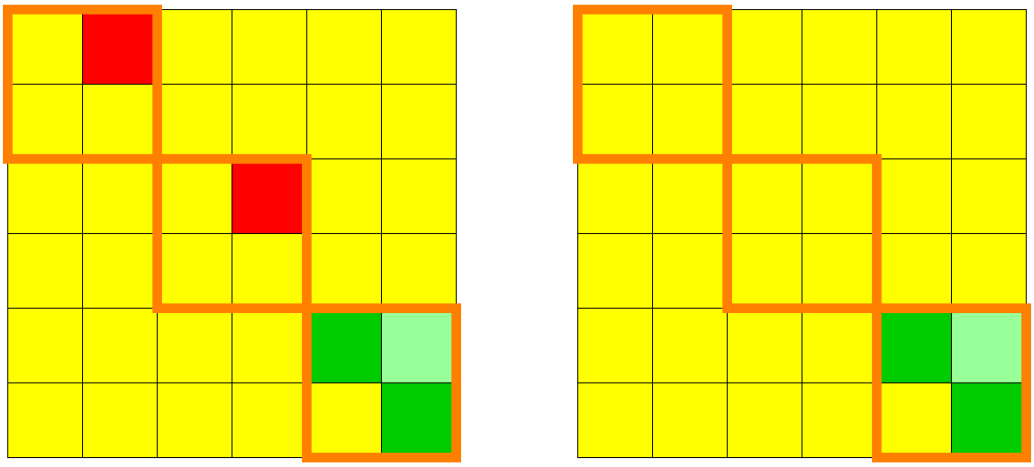
$$A - \lambda I_6 = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix}.$$

We compute also the powers of $A - \lambda I_6$.

$$A - \lambda I_6 = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix},$$

$$(A - \lambda I_6)^2 = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix} \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $A - \lambda I_6$. (b) Visualisation of $(A - \lambda I_6)^2$.

Figure 43

We give here some comment on the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells represent the number 1. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix.

If we restrict the mapping $A - \lambda I_6$ to the subspace $\text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ which is an invariant subspace with respect to the operator $A - \lambda I_6$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 2. Let us take a look at the two first elementary Jordan blocks in this matrix. We observe that the original superdiagonals of 1's in the two first blocks of $A - \lambda I_6$ are going upwards in their respective elementary Jordan blocks associated with the nilpotent part of the transformation $A - \lambda I_6$ when increasing the powers of the matrix $A - \lambda I_6$. They finally disappear when taking the second power of $A - \lambda I_6$.

Investigation of the first Jordan chain.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_6) = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_3\}; \\ \ker(A - \lambda I_6)^2 = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_6)$	2	$2 = \dim(\ker(A - \lambda I_6))$
$\ker(A - \lambda I_6)^2$	4	$2 = \dim(\ker(A - \lambda I_6)^2) - \dim(\ker(A - \lambda I_6))$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A-\lambda I_6)^i)-\dim(\ker(A-\lambda I_6)^{i-1})$. In the first row, we have $\dim(\ker(A - \lambda I_6))$.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A - \lambda I_6) \subsetneq \ker(A - \lambda I_6)^2 = \ker(A - \lambda I_6)^3 = \cdots .$$

We remark by looking at the matrices $(A - \lambda I_6)^i$ that we have the following mappings

$$\begin{cases} (A - \lambda I_6)\,\mathbf{e}_2 = \mathbf{e}_1, \\ (A - \lambda I_6)\,\mathbf{e}_1 = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A - \lambda I_6)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases} (A - \lambda I_6)\,\mathbf{e}_2 = \mathbf{e}_1, \\ (A - \lambda I_6)^2\,\mathbf{e}_2 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = (A - \lambda I_6)\,\mathbf{e}_2, \mathbf{e}_2\}.$$

After we have found the first Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A - \lambda I_6)$	2	$\mathbf{e}_1$	1
$\ker(A - \lambda I_6)^2$	4	$\mathbf{e}_2$	1

We have that the last number in the last column is not zero, and this means that there is still a chain left of length 2.

Investigating the second Jordan chain.
We remark by looking at the matrices $(A - \lambda I_6)^i$ that we have the following mappings

$$\begin{cases}(A - \lambda I_6)\,\mathbf{e}_4 = \mathbf{e}_3,\\(A - \lambda I_6)\,\mathbf{e}_3 = \mathbf{0}.\end{cases}$$

We remark by looking at the matrices $(A - \lambda I_6)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases}(A - \lambda I_6)\,\mathbf{e}_4 = \mathbf{e}_3,\\(A - \lambda I_6)^2\,\mathbf{e}_4 = \mathbf{0}.\end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_3 = (A - \lambda I_6)\,\mathbf{e}_4, \mathbf{e}_4\}.$$

We have found the second Jordan chain. It has length 2. We have now the following table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(A - \lambda I_6)$	2	$\mathbf{e}_1$	$\mathbf{e}_3$	0
$\ker(A - \lambda I_6)^2$	4	$\mathbf{e}_2$	$\mathbf{e}_4$	0

4. Kernels of $(\mathbf{B} - \lambda\, \mathbf{I_6})^{\mathbf{i}}$.

Kernel of $(\mathbf{B} - \lambda\, \mathbf{I_6})$.
The matrix $B - \lambda\, I_6$ is

$$B - \lambda\, I_6 = \begin{pmatrix} -3 & 5 & 0 & 3 & 8 & 6 \\ -8 & 6 & 2 & -1 & 15 & 8 \\ -6 & 8 & 0 & 3 & 14 & 9 \\ 7 & -3 & -2 & 2 & -11 & -6 \\ 9 & -5 & -2 & 2 & -15 & -8 \\ -13 & 9 & 2 & 0 & 23 & 14 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -3 & 5 & 0 & 3 & 8 & 6 \\ -8 & 6 & 2 & -1 & 15 & 8 \\ -6 & 8 & 0 & 3 & 14 & 9 \\ 7 & -3 & -2 & 2 & -11 & -6 \\ 9 & -5 & -2 & 2 & -15 & -8 \\ -13 & 9 & 2 & 0 & 23 & 14 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -3z_1 + 5z_2 + 3z_4 + 8z_5 + 6z_6 = 0, \\ -8z_1 + 6z_2 + 2z_3 - z_4 + 15z_5 + 8z_6 = 0, \\ -6z_1 + 8z_2 + 3z_4 + 14z_5 + 9z_6 = 0, \\ 7z_1 - 3z_2 - 2z_3 + 2z_4 - 11z_5 - 6z_6 = 0, \\ 9z_1 - 5z_2 - 2z_3 + 2z_4 - 15z_5 - 8z_6 = 0, \\ -13z_1 + 9z_2 + 2z_3 + 23z_5 + 14z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda\, I_6) &= \{(r_1, r_2, -r_1/2, -r_1 - r_2, -r_2, r_1 + r_2) \mid r_1, r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, 0, -1/2, -1, 0, 1), (0, 1, 0, -1, -1, 1)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\, \mathbf{I_6})^2$.
We calculate the kernel of $(B - \lambda\, I_6)^2$.

The matrix $(B - \lambda\, I_6)^2$ is

$$(B-\lambda I_6)^2 = \begin{pmatrix} -16 & 20 & 0 & 8 & 36 & 24 \\ -12 & 12 & 0 & 4 & 24 & 16 \\ -16 & 20 & 0 & 8 & 36 & 24 \\ 8 & -4 & 0 & 0 & -12 & -8 \\ 8 & -4 & 0 & 0 & -12 & -8 \\ -20 & 16 & 0 & 4 & 36 & 24 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -16 & 20 & 0 & 8 & 36 & 24 \\ -12 & 12 & 0 & 4 & 24 & 16 \\ -16 & 20 & 0 & 8 & 36 & 24 \\ 8 & -4 & 0 & 0 & -12 & -8 \\ 8 & -4 & 0 & 0 & -12 & -8 \\ -20 & 16 & 0 & 4 & 36 & 24 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -16z_1 + 20z_2 + 8z_4 + 36z_5 + 24z_6 = 0, \\ -12z_1 + 12z_2 + 4z_4 + 24z_5 + 16z_6 = 0, \\ -16z_1 + 20z_2 + 8z_4 + 36z_5 + 24z_6 = 0, \\ 8\,z_1 - 4\,z_2 - 12z_5 - 8\,z_6 = 0, \\ 8\,z_1 - 4\,z_2 - 12z_5 - 8\,z_6 = 0, \\ -20z_1 + 16z_2 + 4z_4 + 36z_5 + 24z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B-\lambda I_6)^2 &= \{(r_1, r_2, r_3, -r_1 - r_2, r_5, r_1 - r_2/2 - 3\,r_5/2) \\ &\qquad \mid r_1, r_2, r_3, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,-1,0,1), (0,1,0,-1,0,-1/2), \\ &\qquad (0,0,1,0,0,0), (0,0,0,0,1,-3/2)\}. \end{aligned}$$

Kernel of $(\mathbf{B}-\lambda\,\mathbf{I_6})^3$.
We calculate the kernel of $(B-\lambda I_6)^3$.

The matrix $(B-\lambda I_6)^3$ is

$$(B - \lambda I_6)^3 = \begin{pmatrix} -44 & 52 & 0 & 20 & 96 & 64 \\ -24 & 24 & 0 & 8 & 48 & 32 \\ -44 & 52 & 0 & 20 & 96 & 64 \\ 4 & 4 & 0 & 4 & 0 & 0 \\ 4 & 4 & 0 & 4 & 0 & 0 \\ -28 & 20 & 0 & 4 & 48 & 32 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -44 & 52 & 0 & 20 & 96 & 64 \\ -24 & 24 & 0 & 8 & 48 & 32 \\ -44 & 52 & 0 & 20 & 96 & 64 \\ 4 & 4 & 0 & 4 & 0 & 0 \\ 4 & 4 & 0 & 4 & 0 & 0 \\ -28 & 20 & 0 & 4 & 48 & 32 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -44z_1 + 52z_2 + 20z_4 + 96z_5 + 64z_6 = 0, \\ -24z_1 + 24z_2 + 8\,z_4 + 48z_5 + 32z_6 = 0, \\ -44z_1 + 52z_2 + 20z_4 + 96z_5 + 64z_6 = 0, \\ 4\,z_1 + 4\,z_2 + 4\,z_4 = 0, \\ 4\,z_1 + 4\,z_2 + 4\,z_4 = 0, \\ -28z_1 + 20z_2 + 4\,z_4 + 48z_5 + 32z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_6)^3 &= \{(r_1, r_2, r_3, -r_1 - r_2, r_5, r_1 - r_2/2 - 3\,r_5/2) \\ &\qquad \mid r_1, r_2, r_3, r_5 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,-1,0,1), (0,1,0,-1,0,-1/2), \\ &\qquad (0,0,1,0,0,0), (0,0,0,0,1,-3/2)\}. \end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the second power onwards in the following chain of inclusion of sets.

$$\ker(B - \lambda I_6) \subsetneq \ker((B - \lambda I_6)^2) = \ker((B - \lambda I_6)^3) \cdots .$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B-\lambda I_6)$	2	$2 = \dim(\ker(B-\lambda I_6))$
$\ker(B-\lambda I_6)^2$	4	$2 = \dim(\ker(B-\lambda I_6)^2) - \dim(\ker(B-\lambda I_6))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B-\lambda I_6)^i)) - \dim(\ker((B-\lambda I_6)^{i-1}))$. The first number of this last column is $\dim(\ker(B-\lambda I_6))$.

5. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w_1}, \mathbf{w_2}\}$ satisfying

$$\begin{cases} (B-\lambda I_6)\,\mathbf{w_1} = \mathbf{0}, \\ (B-\lambda I_6)\,\mathbf{w_2} = \mathbf{w_1} \end{cases}$$

or

$$\begin{cases} (B-\lambda I_6)^2\,\mathbf{w_2} = \mathbf{0}, \\ (B-\lambda I_6)\,\mathbf{w_2} = \mathbf{w_1} \end{cases}$$

where $\mathbf{w_2}$ is in the vector space $\ker((B-\lambda I_6)^2)$ but not in $\ker(B-\lambda I_6)$.

We look for a generating vector $\mathbf{w_2}$. This vector must be linearly independent of all vectors in $\ker(B-\lambda I_6)$ and must be independent from vectors of height 2 that were already chosen in $\ker(B-\lambda I_6)^2$. We know that

$$\ker((B-\lambda I_6)^2) = \operatorname{span}\{(1,0,0,-1,0,1),(0,1,0,-1,0,-1/2), \\ (0,0,1,0,0,0),(0,0,0,0,1,-3/2)\}.$$

We have at this point not chosen any vector of height 2 of $\ker((B-\lambda I_6)^2)$ in a previous Jordan chain.

We know that a vector in $\ker(B-\lambda I_6)^2$ must be of the form

$$\begin{aligned} a\,(1,0,0,-1,0,1) + b\,(0,1,0,-1,0,-1/2) + c\,(0,0,1,0,0,0) \\ + d\,(0,0,0,0,1,-3/2) \\ = (a,b,c,-a-b,d,a-b/2-3\,d/2). \end{aligned}$$

We remember also

$$\ker(B - \lambda I_6) = \operatorname{span}\{(1,0,-1/2,-1,0,1),(0,1,0,-1,-1,1)\}.$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 0 & -1/2 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ a & b & c & -a-b & d & a-b/2-3\,d/2 \end{pmatrix}.$$

We row reduce this matrix H and if we impose $a + 2\,c \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 & -1 & (b+d)/(a+2\,c) & (2\,a-3\,b+4\,c-3\,d)/(2\,(a+2\,c)) \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 2\,(b+d)/(a+2\,c) & -3\,(b+d)/(a+2\,c) \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + 2\,c \neq 0$. So we can choose $a = 1$, $b = 0$, $c = 0$, $d = 0$.

We have now the generating vector

$$\mathbf{w_2} = (1,0,0,-1,0,1).$$

We start with $\mathbf{w_2} = (1,0,0,-1,0,1)$.

We calculate $\mathbf{w_1}$.

$$\begin{aligned} \mathbf{w_1} &= (B - \lambda I_6)\,\mathbf{w_2} \\ &= \begin{pmatrix} -3 & 5 & 0 & 3 & 8 & 6 \\ -8 & 6 & 2 & -1 & 15 & 8 \\ -6 & 8 & 0 & 3 & 14 & 9 \\ 7 & -3 & -2 & 2 & -11 & -6 \\ 9 & -5 & -2 & 2 & -15 & -8 \\ -13 & 9 & 2 & 0 & 23 & 14 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}. \end{aligned}$$

We have now found a first Jordan chain. It has length 2.

$$\{\mathbf{w_1} = (0,1,0,-1,-1,1), \mathbf{w_2} = (1,0,0,-1,0,1)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B-\lambda I_6)$	2	$\mathbf{w_1}$	1
$\ker(B-\lambda I_6)^2$	4	$\mathbf{w_2}$	1

with

$$\mathbf{w_1} = (0,1,0,-1,-1,1)$$
$$\mathbf{w_2} = (1,0,0,-1,0,1)$$

Calculation of the second Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w_3}, \mathbf{w_4}\}$ satisfying

$$\begin{cases}(B-\lambda I_6)\,\mathbf{w_3} = \mathbf{0},\\ (B-\lambda I_6)\,\mathbf{w_4} = \mathbf{w_3}\end{cases}$$

or

$$\begin{cases}(B-\lambda I_6)^2\,\mathbf{w_4} = \mathbf{0},\\ (B-\lambda I_6)\,\mathbf{w_4} = \mathbf{w_3}\end{cases}$$

where $\mathbf{w_4}$ is in the vector space $\ker((B-\lambda I_6)^2)$ but not in $\ker(B-\lambda I_6)$.

We look for a generating vector $\mathbf{w_4}$. This vector must be linearly independent of all vectors in $\ker(B-\lambda I_6)$ and must be independent from vectors that were already chosen in $\ker(B-\lambda I_6)^2$. We know that

$$\ker((B-\lambda I_6)^2) = \operatorname{span}\{(1,0,0,-1,0,1), (0,1,0,-1,0,-1/2),\\ (0,0,1,0,0,0), (0,0,0,0,1,-3/2)\}.$$

We know that a vector in $\ker(B - \lambda I_6)^2$ must be of the form

$$\begin{aligned}
a\,(1,0,0,-1,0,1) &+ b\,(0,1,0,-1,0,-1/2) \\
&+ c\,(0,0,1,0,0,0) + d\,(0,0,0,0,1,-3/2) \\
&= (a,b,c,-a-b,d,a-b/2-3\,d/2).
\end{aligned}$$

We have at this point chosen in $\ker(B - \lambda I_6)^2$ already the vector $\mathbf{w_2} = (1,0,0,-1,0,1)$ of height 2.

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 0 & -1/2 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & -1 & 0 & 1 \\ a & b & 0 & -a-b & c & a-b/2-3\,c/2 \end{pmatrix}.$$

We row reduce this matrix H and if we impose $b + c \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3/2 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $b + c \neq 0$. So we can take $a = 0$, $b = 2$, $c = 0$ and $d = 0$.

So we have the generating vector

$$\mathbf{w_4} = (0,2,0,-2,0,-1).$$

We start with $\mathbf{w_4} = (0,2,0,-2,0,-1)$.

We calculate $\mathbf{w_3}$.

$$\begin{aligned}
\mathbf{w_3} &= (B - \lambda I_6)\,\mathbf{w_4} \\
&= \begin{pmatrix} -3 & 5 & 0 & 3 & 8 & 6 \\ -8 & 6 & 2 & -1 & 15 & 8 \\ -6 & 8 & 0 & 3 & 14 & 9 \\ 7 & -3 & -2 & 2 & -11 & -6 \\ 9 & -5 & -2 & 2 & -15 & -8 \\ -13 & 9 & 2 & 0 & 23 & 14 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 1 \\ -4 \\ -6 \\ 4 \end{pmatrix}.
\end{aligned}$$

We have now found a second Jordan chain. It has length 2.

$$\{\mathbf{w}_3 = (-2, 6, 1, -4, -6, 4), \mathbf{w}_4 = (0, 2, 0, -2, 0, -1)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	chain 2	remaining dim
$\ker(B - \lambda I_6)$	2	$\mathbf{w}_1$	$\mathbf{w}_3$	0
$\ker(B - \lambda I_6)^2$	4	$\mathbf{w}_2$	$\mathbf{w}_4$	0

with

$\mathbf{w}_1$	$= (0, 1, 0, -1, -1, 1)$
$\mathbf{w}_2$	$= (1, 0, 0, -1, 0, 1)$
$\mathbf{w}_3$	$= (-2, 6, 1, -4, -6, 4)$
$\mathbf{w}_4$	$= (0, 2, 0, -2, 0, -1)$

6. Investigation of the second eigenvalue.

We work now with the eigenvalue $\lambda = 1$.

7. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 8 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 7. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_6.

$$A - \lambda I_6 = \begin{pmatrix} \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

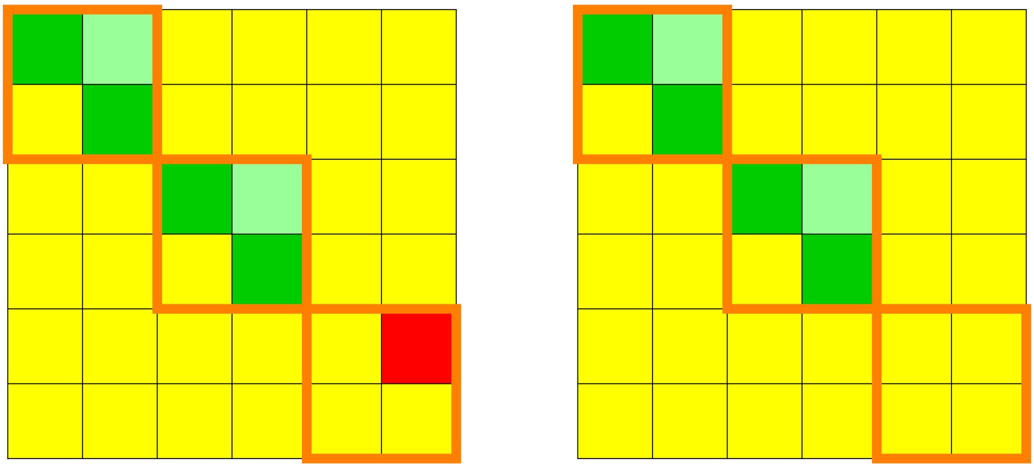
We see that the space span$\{\mathbf{e}_5, \mathbf{e}_6\}$ is invariant relative to $A - \lambda I_6$. If we restrict $A - \lambda I_6$ to this space, we have a nilpotent operator and we can apply the techniques learned in part 2.

We want to investigate the endomorphism $A - \lambda I_6$ restricted to this space. We compute now the powers of $A - \lambda I_6$.

$$A - \lambda I_6 = \begin{pmatrix} \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$(A - \lambda I_6)^2 = \begin{pmatrix} \begin{pmatrix} 4 & -4 \\ 0 & 4 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 4 & -4 \\ 0 & 4 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $(A - \lambda I_6)$. (b) Visualisation of $(A - \lambda I_6)^2$.

Figure 44

We give here some comments about the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. All these green elementary Jordan blocks have the same look however large the exponent of $A - \lambda I_6$. The yellow cells are representing the number zero. The nilpotent part of the matrix is the third elementary Jordan block. It changes by moving the superdiagonal upwards when increasing the exponents of $A - \lambda I_6$.

Investigation of the first Jordan chain.
Let us take a look at the third coloured block. This block is a nilpotent operator on the invariant subspace $\text{span}\{\mathbf{e}_5, \mathbf{e}_6\}$ with respect to the operator $A - \lambda I_6$.

We observe that the original superdiagonal of 1's in the third block of the matrix $A - \lambda I_6$ is going upwards in the powers of the matrix $A - \lambda I_6$ until it finally disappears when taking the second power of $A - \lambda I_6$.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_6) = \text{span}\{\mathbf{e}_5\}; \\ \ker(A - \lambda I_6)^2 = \text{span}\{\mathbf{e}_5, \mathbf{e}_6\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_6)$	1	$1 = \dim(\ker(A - \lambda I_6))$
$\ker(A - \lambda I_6)^2$	2	$1 = \dim(\ker(A - \lambda I_6)^2) - \dim(\ker(A - \lambda I_6))$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A-\lambda I_6)^i) - \dim(\ker(A-\lambda I_6)^{i-1})$. In the first row, we have $\dim(\ker(A - \lambda I_6))$.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A - \lambda I_6) \subsetneq \ker(A - \lambda I_6)^2 = \ker(A - \lambda I_6)^3 = \cdots .$$

We remark by looking at the matrices $(A - \lambda I_6)^i$ that we have the following mappings

$$\begin{cases} (A - \lambda I_6)\,\mathbf{e_6} = \mathbf{e}_5, \\ (A - \lambda I_6)\,\mathbf{e}_5 = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A - \lambda I_6)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases} (A - \lambda I_6)\,\mathbf{e_6} = \mathbf{e}_5, \\ (A - \lambda I_6)^2\,\mathbf{e_6} = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_5 = (A - \lambda I_6)\,\mathbf{e_6}, \mathbf{e_6}\}.$$

After we have found the first Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A - \lambda I_6)$	1	$\mathbf{e}_5$	0
$\ker(A - \lambda I_6)^2$	1	$\mathbf{e}_6$	0

We have now in the last column only 0's and this means that we are done looking for Jordan chains.

8. Kernels of $(\mathbf{B} - \lambda\, \mathbf{I}_6)^{\mathbf{i}}$.

Kernel of $(\mathbf{B} - \lambda \mathbf{I})$.
We calculate the kernel of $B - \lambda I_6$.

The matrix $B - \lambda I_6$ is

$$B - \lambda I_6 = \begin{pmatrix} -5 & 5 & 0 & 3 & 8 & 6 \\ -8 & 4 & 2 & -1 & 15 & 8 \\ -6 & 8 & -2 & 3 & 14 & 9 \\ 7 & -3 & -2 & 0 & -11 & -6 \\ 9 & -5 & -2 & 2 & -17 & -8 \\ -13 & 9 & 2 & 0 & 23 & 12 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -5 & 5 & 0 & 3 & 8 & 6 \\ -8 & 4 & 2 & -1 & 15 & 8 \\ -6 & 8 & -2 & 3 & 14 & 9 \\ 7 & -3 & -2 & 0 & -11 & -6 \\ 9 & -5 & -2 & 2 & -17 & -8 \\ -13 & 9 & 2 & 0 & 23 & 12 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -5z_1 + 5z_2 + 3z_4 + 8z_5 + 6z_6 = 0, \\ -8z_1 + 4z_2 + 2z_3 - z_4 + 15z_5 + 8z_6 = 0, \\ -6z_1 + 8z_2 - 2z_3 + 3z_4 + 14z_5 + 9z_6 = 0, \\ 7z_1 - 3z_2 - 2z_3 - 11z_5 - 6z_6 = 0, \\ 9z_1 - 5z_2 - 2z_3 + 2z_4 - 17z_5 - 8z_6 = 0, \\ -13z_1 + 9z_2 + 2z_3 + 23z_5 + 12z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B-\lambda I_6) &= \{(r_1,0,r_1,r_1,r_1,-r_1) \mid r_1 \in \mathbf{K}\}\\ &= \operatorname{span}\{(1,0,1,1,1,-1)\}.\end{aligned}$$

Kernel of $(\mathbf{B}-\lambda\,\mathbf{I_6})^2$.
We calculate the kernel of $(B-\lambda I_6)^2$. The matrix $(B-\lambda I_6)^2$ is

$$(B-\lambda I_6)^2 = \begin{pmatrix} 0 & 0 & 0 & -4 & 4 & 0\\ 20 & -8 & -8 & 8 & -36 & -16\\ 8 & -12 & 4 & -4 & -20 & -12\\ -20 & 8 & 8 & -4 & 32 & 16\\ -28 & 16 & 8 & -8 & 52 & 24\\ 32 & -20 & -8 & 4 & -56 & -28\end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0 & -4 & 4 & 0\\ 20 & -8 & -8 & 8 & -36 & -16\\ 8 & -12 & 4 & -4 & -20 & -12\\ -20 & 8 & 8 & -4 & 32 & 16\\ -28 & 16 & 8 & -8 & 52 & 24\\ 32 & -20 & -8 & 4 & -56 & -28\end{pmatrix}\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\\ z_5\\ z_6\end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -4z_4+4z_5 = 0,\\ 20z_1-8z_2-8z_3+8z_4-36z_5-16z_6 = 0,\\ 8z_1-12z_2+4z_3-4z_4-20z_5-12z_6 = 0,\\ -20z_1+8z_2+8z_3-4z_4+32z_5+16z_6 = 0,\\ -28z_1+16z_2+8z_3-8z_4+52z_5+24z_6 = 0,\\ 32z_1-20z_2-8z_3+4z_4-56z_5-28z_6 = 0.\end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B-\lambda I_6)^2 &= \{(r_1,r_2,r_1,r_1-2r_2,r_1-2r_2,-r_1+3r_2) \mid r_1,r_2 \in \mathbf{K}\}\\ &= \operatorname{span}\{(1,0,1,1,1,-1),(0,1,0,-2,-2,3)\}.\end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\, \mathbf{I}_6)^3$. We calculate the kernel of $(B - \lambda\, I_6)^3$.

The matrix $(B - \lambda\, I_6)^3$ is

$$(B - \lambda\, I_6)^3 = \begin{pmatrix} 8 & -8 & 0 & 8 & -24 & -8 \\ -48 & 16 & 24 & -28 & 84 & 32 \\ -20 & 28 & -8 & 8 & 48 & 28 \\ 40 & -8 & -24 & 20 & -60 & -24 \\ 64 & -32 & -24 & 28 & -116 & -48 \\ -64 & 32 & 24 & -20 & 108 & 48 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 8 & -8 & 0 & 8 & -24 & -8 \\ -48 & 16 & 24 & -28 & 84 & 32 \\ -20 & 28 & -8 & 8 & 48 & 28 \\ 40 & -8 & -24 & 20 & -60 & -24 \\ 64 & -32 & -24 & 28 & -116 & -48 \\ -64 & 32 & 24 & -20 & 108 & 48 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 8\,z_1 - 8\,z_2 + 8\,z_4 - 24\,z_5 - 8\,z_6 = 0, \\ -48z_1 + 16z_2 + 24z_3 - 28z_4 + 84\,z_5 + 32z_6 = 0, \\ -20z_1 + 28z_2 - 8\,z_3 + 8\,z_4 + 48\,z_5 + 28z_6 = 0, \\ 40z_1 - 8\,z_2 - 24z_3 + 20z_4 - 60\,z_5 - 24z_6 = 0, \\ 64z_1 - 32z_2 - 24z_3 + 28z_4 - 116z_5 - 48z_6 = 0, \\ -64z_1 + 32z_2 + 24z_3 - 20z_4 + 108z_5 + 48z_6 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda\, I_6)^3 &= \{(r_1, r_2, r_1, r_1 - 2\,r_2, r_1 - 2\,r_2, -r_1 + 3\,r_2) \mid r_1, r_2, r_3 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,1,1,1,-1), (0,1,0,-2,-2,3)\}. \end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the second power onwards in the following chain of inclusion of sets.

$$\ker(B - \lambda\, I_6) \subsetneq \ker((B - \lambda\, I_6)^2) = \ker((B - \lambda\, I_6)^3) = \cdots.$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B-\lambda I_6)$	1	$1 = \dim(\ker(B-\lambda I_6))$
$\ker(B-\lambda I_6)^2$	2	$1 = \dim(\ker(B-\lambda I_6)^2) - \dim(\ker(B-\lambda I_6))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B-\lambda I_6)^i)) - \dim(\ker((B-\lambda I_6)^{i-1}))$. The first number of this last column is $\dim(\ker(B-\lambda I_6))$.

9. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w_5}, \mathbf{w_6}\}$ satisfying

$$\begin{cases}(B-\lambda I_6)\,\mathbf{w_5} = \mathbf{0},\\(B-\lambda I_6)\,\mathbf{w_6} = \mathbf{w_5}\end{cases}$$

or

$$\begin{cases}(B-\lambda I_6)^2\,\mathbf{w_6} = \mathbf{0},\\(B-\lambda I_6)\,\mathbf{w_6} = \mathbf{w_5}\end{cases}$$

where $\mathbf{w_6}$ is in the vector space $\ker((B-\lambda I_6)^2)$ but not in $\ker((B-\lambda I_6))$.

We look for a generating vector $\mathbf{w_6}$. This vector must be linearly independent of all vectors in $\ker(B-\lambda I_6)$ and must be independent from vectors that were already chosen in $\ker(B-\lambda I_6)^2$. We know that

$$\ker((B-\lambda I_6)^2) = \operatorname{span}\{(1,0,1,1,1,-1),(0,1,0,-2,-2,3)\}.$$

We have at this point not chosen any vector of height 2 in $\ker((B-\lambda I_6)^2)$.

We know that a vector in $\ker((B-\lambda I_6)^2)$ must be of the form

$$a\,(1,0,1,1,1,-1)+b\,(0,1,0,-2,-2,3) = (a, b, a, a-2\,b, a-2\,b, -a+3\,b).$$

We remember that

$$\ker(B-\lambda I_6) = \operatorname{span}\{(1,0,1,1,1,-1)\}.$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & -1 \\ a & b & a & a-2b & a-2b & 3b-a \end{pmatrix}.$$

We row reduce this matrix H and if we impose $b \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -2 & -2 & 3 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $b \neq 0$. We can choose $a = 0$ and $b = 1$.

We have the generating vector

$$\mathbf{w_6} = (0, 1, 0, -2, -2, 3).$$

We start with $\mathbf{w_6} = (0, 1, 0, -2, -2, 3)$.

We calculate $\mathbf{w_5}$.

$$\begin{aligned} \mathbf{w_5} &= (B - \lambda I_6)\mathbf{w_6} \\ &= \begin{pmatrix} -5 & 5 & 0 & 3 & 8 & 6 \\ -8 & 4 & 2 & -1 & 15 & 8 \\ -6 & 8 & -2 & 3 & 14 & 9 \\ 7 & -3 & -2 & 0 & -11 & -6 \\ 9 & -5 & -2 & 2 & -17 & -8 \\ -13 & 9 & 2 & 0 & 23 & 12 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}. \end{aligned}$$

We have now the Jordan chain

$$\{\mathbf{w_5} = (1, 0, 1, 1, 1, -1), \mathbf{w_6} = (0, 1, 0, -2, -2, 3)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_6)$	1	$\mathbf{w_5}$	0
$\ker(B - \lambda I_6)^2$	2	$\mathbf{w_6}$	0

with

$$\mathbf{w_5} = (1, 0, 1, 1, 1, -1)$$

$$\mathbf{w_6} = (0, 1, 0, -2, -2, 3)$$

10. Result and check of the result.

We construct the matrix of base change P with the coordinates of the vectors $\mathbf{w_i}$ found in the Jordan chains in the columns of P.

$$P = \begin{pmatrix} 0 & 1 & -2 & 0 & 1 & 0 \\ 1 & 0 & 6 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & -4 & -2 & 1 & -2 \\ -1 & 0 & -6 & 0 & 1 & -2 \\ 1 & 1 & 4 & -1 & -1 & 3 \end{pmatrix}.$$

$$P^{-1}\,B\,P = \begin{pmatrix} -1 & 8 & -6 & 5 & 8 & 6 \\ 4 & -6 & 2 & -3 & -9 & -6 \\ 1 & -2 & 1 & -1 & -3 & -2 \\ -1 & 1 & 0 & 0 & 2 & 1 \\ -1 & 2 & 0 & 1 & 3 & 2 \\ -3 & 3 & 0 & 1 & 6 & 4 \end{pmatrix}$$

$$\times \begin{pmatrix} -4 & 5 & 0 & 3 & 8 & 6 \\ -8 & 5 & 2 & -1 & 15 & 8 \\ -6 & 8 & -1 & 3 & 14 & 9 \\ 7 & -3 & -2 & 1 & -11 & -6 \\ 9 & -5 & -2 & 2 & -16 & -8 \\ -13 & 9 & 2 & 0 & 23 & 13 \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & 1 & -2 & 0 & 1 & 0 \\ 1 & 0 & 6 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & -4 & -2 & 1 & -2 \\ -1 & 0 & -6 & 0 & 1 & -2 \\ 1 & 1 & 4 & -1 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

31 exercise. (5×5); $(\mathbf{J}_3(-1), \mathbf{J}_2(2))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} -3 & 0 & -5 & 4 & 1 \\ -4 & 2 & -4 & -1 & -1 \\ 2 & 1 & 3 & -5 & 1 \\ 0 & -1 & 1 & 1 & -2 \\ 3 & -2 & 5 & 0 & -2 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_5| = -(\lambda - 2)^2 (\lambda + 1)^3.$$

We see that the characteristic Cayley-Hamilton polynomial $p_{\text{C-H}}(\lambda)$ can be factorised in linear polynomials over the field $\mathbf{K}$. So we can apply the Jordan normalisation machinery.

The eigenvalue $\lambda = 2$ has algebraic multiplicity 2. The eigenvalue $\lambda = -1$ has algebraic multiplicity 3.

2. Investigation of the first eigenvalue.

We work now with the eigenvalue $\lambda = -1$.

3. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 4 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 3. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_5.

$$A - \lambda I_5 = \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \end{pmatrix}.$$

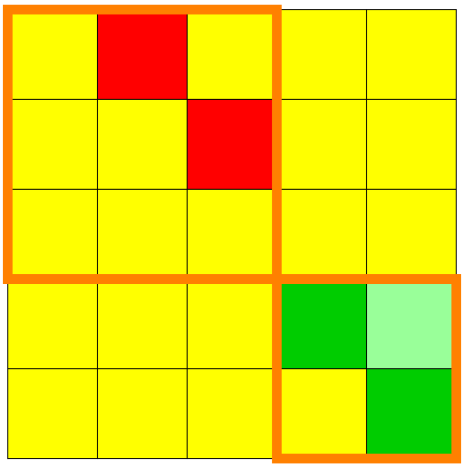
We compute also the powers of $A - \lambda I_5$.

$$A - \lambda I_5 = \begin{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \end{pmatrix},$$

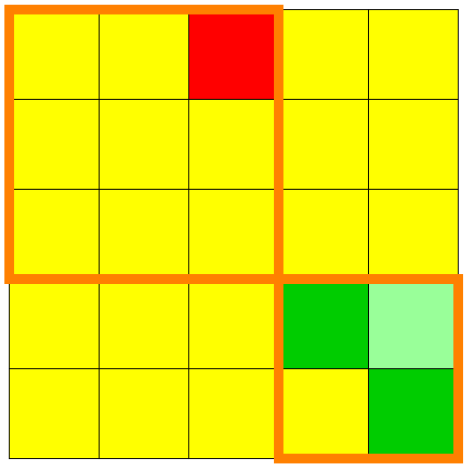
$$(A - \lambda I_5)^2 = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 9 & 6 \\ 0 & 9 \end{pmatrix} \end{pmatrix},$$

$$(A - \lambda I_5)^3 = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 27 & 27 \\ 0 & 27 \end{pmatrix} \end{pmatrix}.$$

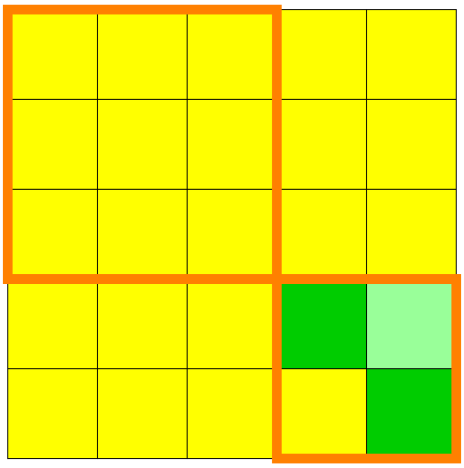
Let us visualise this situation.



(a) Visualisation of $A - \lambda I_5$.



(b) Visualisation of $(A - \lambda I_5)^2$.



(c) Visualisation of $(A - \lambda I_5)^3$.

Figure 45

We give here some comment on the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells represent the number 1. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix.

If we restrict the mapping $A-\lambda I_5$ to the subspace $\text{span}\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ which is an invariant subspace with respect to the operator $A-\lambda I_5$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 3. Let us take a look at the first elementary Jordan block in this matrix.

We observe that the original superdiagonal of 1's in the first block of $A-\lambda I_5$ is going upwards in its elementary Jordan block associated with the nilpotent part of the transformation $A-\lambda I_5$ when increasing the powers of the matrix $A-\lambda I_5$. It finally disappears when taking the third power of $A-\lambda I_5$.

Investigation of the first Jordan chain.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A-\lambda I_5) = \text{span}\{\mathbf{e}_1\}; \\ \ker(A-\lambda I_5)^2 = \text{span}\{\mathbf{e}_1,\mathbf{e}_2\}; \\ \ker(A-\lambda I_5)^3 = \text{span}\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}. \end{cases}$$

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A-\lambda I_5)$	1	$1 = \dim(\ker(A-\lambda I_5))$
$\ker(A-\lambda I_5)^2$	2	$1 = \dim(\ker(A-\lambda I_5)^2) - \dim(\ker(A-\lambda I_5))$
$\ker(A-\lambda I_5)^3$	3	$1 = \dim(\ker(A-\lambda I_5)^3) - \dim(\ker(A-\lambda I_5)^2)$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A-\lambda I_5)^i)-\dim(\ker(A-\lambda I_5)^{i-1})$. In the first row, we have $\dim(\ker(A-\lambda I_5))$.

We see also that there is equality in the inclusion of sets from the third power onwards.

$$\ker(A-\lambda I_5) \subsetneq \ker(A-\lambda I_5)^2 \subsetneq \ker(A-\lambda I_5)^3 = \ker(A-\lambda I_5)^4 = \cdots .$$

We remark by looking at the matrices $(A - \lambda I_5)^i$ that we have the following mappings

$$\begin{cases} (A - \lambda I_5)\,\mathbf{e_3} = \mathbf{e_2}, \\ (A - \lambda I_5)\,\mathbf{e_2} = \mathbf{e_1}, \\ (A - \lambda I_5)\,\mathbf{e_1} = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A - \lambda I_5)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases} (A - \lambda I_5)\,\mathbf{e_3} = \mathbf{e_2}, \\ (A - \lambda I_5)^2\,\mathbf{e_3} = \mathbf{e_1}, \\ (A - \lambda I_5)^3\,\mathbf{e_3} = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e_1} = (A - \lambda I_5)^2\,\mathbf{e_3}, \mathbf{e_2} = (A - \lambda I_5)\,\mathbf{e_3}, \mathbf{e_3}\}.$$

After we have found the first Jordan chain of length 3, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A - \lambda I_5)$	1	$\mathbf{e_1}$	0
$\ker(A - \lambda I_5)^2$	2	$\mathbf{e_2}$	0
$\ker(A - \lambda I_5)^3$	3	$\mathbf{e_3}$	0

We have now in the last column only 0's and this means that we are done looking for Jordan chains.

4. Kernels of $(\mathbf{B} - \lambda\,\mathbf{I_5})^{\mathbf{i}}$.

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I_5})$.
We calculate the kernel of $B - \lambda I_5$.

The matrix $B - \lambda I_5$ is

$$B - \lambda I_5 = \begin{pmatrix} -2 & 0 & -5 & 4 & 1 \\ -4 & 3 & -4 & -1 & -1 \\ 2 & 1 & 4 & -5 & 1 \\ 0 & -1 & 1 & 2 & -2 \\ 3 & -2 & 5 & 0 & -1 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -2 & 0 & -5 & 4 & 1 \\ -4 & 3 & -4 & -1 & -1 \\ 2 & 1 & 4 & -5 & 1 \\ 0 & -1 & 1 & 2 & -2 \\ 3 & -2 & 5 & 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -2z_1 - 5z_3 + 4z_4 + z_5 = 0, \\ -4z_1 + 3z_2 - 4z_3 - z_4 - z_5 = 0, \\ 2z_1 + z_2 + 4z_3 - 5z_4 + z_5 = 0, \\ -z_2 + z_3 + 2z_4 - 2z_5 = 0, \\ 3z_1 - 2z_2 + 5z_3 - z_5 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_5) &= \{(r_1, r_1/2, -r_1/2, 0, -r_1/2) \mid r_1 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, 1/2, -1/2, 0, -1/2)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda \mathbf{I}_5)^2$.
We calculate the kernel of $(B - \lambda I_5)^2$.

The matrix $(B - \lambda I_5)^2$ is

$$(B - \lambda I_5)^2 = \begin{pmatrix} -3 & -11 & -1 & 25 & -16 \\ -15 & 8 & -14 & -1 & -8 \\ 3 & 10 & 2 & -23 & 14 \\ 0 & 0 & 0 & 0 & 0 \\ 9 & 1 & 8 & -11 & 11 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -3 & -11 & -1 & 25 & -16 \\ -15 & 8 & -14 & -1 & -8 \\ 3 & 10 & 2 & -23 & 14 \\ 0 & 0 & 0 & 0 & 0 \\ 9 & 1 & 8 & -11 & 11 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -3z_1 - 11z_2 - z_3 + 25z_4 - 16z_5 = 0, \\ -15z_1 + 8z_2 - 14z_3 - z_4 - 8z_5 = 0, \\ 3z_1 + 10z_2 + 2z_3 - 23z_4 + 14z_5 = 0, \\ 9z_1 + z_2 + 8z_3 - 11z_4 + 11z_5 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_5)^2 &= \{(r_1, r_2, -2r_1/3 + r_2/3, -r_1/3 + 2r_2/3, -2r_1/3 + r_2/3) \\ &\qquad \mid r_1, r_2 \in \mathbf{K}\} \\ &= \text{span}\{(1, 0, -2/3, -1/3, -2/3), (0, 1, 1/3, 2/3, 1/3)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda \mathbf{I_5})^3$.
We calculate the kernel of $(B - \lambda I_5)^3$.

The matrix $(B - \lambda I_5)^3$ is

$$(B - \lambda I_5)^3 = \begin{pmatrix} 0 & -27 & 0 & 54 & -27 \\ -54 & 27 & -54 & 0 & -27 \\ 0 & 27 & 0 & -54 & 27 \\ 0 & 0 & 0 & 0 & 0 \\ 27 & 0 & 27 & -27 & 27 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & -27 & 0 & 54 & -27 \\ -54 & 27 & -54 & 0 & -27 \\ 0 & 27 & 0 & -54 & 27 \\ 0 & 0 & 0 & 0 & 0 \\ 27 & 0 & 27 & -27 & 27 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} \quad\quad - 27z_2 \quad\quad + 54z_4 - 27z_5 = 0, \\ -54z_1 + 27z_2 - 54z_3 \quad\quad - 27z_5 = 0, \\ \quad\quad + 27z_2 \quad\quad - 54z_4 + 27z_5 = 0, \\ 27z_1 \quad\quad + 27z_3 - 27z_4 + 27z_5 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_5)^3 &= \{(r_1, r_2, r_3, -r_1 + r_2 - r_3, -2r_1 + r_2 - 2r_3) \\ &\qquad\qquad \mid r_1, r_2, r_3 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,0,-1,-2), (0,1,0,1,1), (0,0,1,-1,-2)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda \mathbf{I_5})^4$.
We calculate the kernel of $(B - \lambda I_5)^4$.

The matrix $(B - \lambda I_5)^4$ is

$$(B - \lambda I_5)^4 = \begin{pmatrix} 27 & -81 & 27 & 135 & -54 \\ -189 & 81 & -189 & 27 & -108 \\ -27 & 81 & -27 & -135 & 54 \\ 0 & 0 & 0 & 0 & 0 \\ 81 & 0 & 81 & -81 & 81 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 27 & -81 & 27 & 135 & -54 \\ -189 & 81 & -189 & 27 & -108 \\ -27 & 81 & -27 & -135 & 54 \\ 0 & 0 & 0 & 0 & 0 \\ 81 & 0 & 81 & -81 & 81 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 27z_1 - 81z_2 + 27z_3 + 135z_4 - 54z_5 = 0, \\ -189z_1 + 81z_2 - 189z_3 + 27z_4 - 108z_5 = 0, \\ -27z_1 + 81z_2 - 27z_3 - 135z_4 + 54z_5 = 0, \\ 81z_1 \quad\quad + 81z_3 - 81z_4 + 81z_5 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B-\lambda I_5)^4 &= \{(r_1,r_2,r_3,-r_1+r_2-r_3,-2\,r_1+r_2-2\,r_3) \mid r_1,r_2,r_3\in\mathbf{K}\}\\ &= \text{span}\{(1,0,0,-1,-2),(0,1,0,1,1),(0,0,1,-1,-2)\}.\end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the third power onwards in the following chain of inclusion of sets.

$$\ker(B-\lambda I_5) \subsetneq \ker((B-\lambda I_5)^2) \subsetneq \ker((B-\lambda I_5)^3) = \ker((B-\lambda I_5)^4) = \cdots.$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B-\lambda I_5)$	1	$1 = \dim(\ker(B-\lambda I_5))$
$\ker(B-\lambda I_5)^2$	2	$1 = \dim(\ker(B-\lambda I_5)^2) - \dim(\ker(B-\lambda I_5))$
$\ker(B-\lambda I_5)^3$	3	$1 = \dim(\ker(B-\lambda I_5)^3) - \dim(\ker(B-\lambda I_5)^2)$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B-\lambda I_5)^i)) - \dim(\ker((B-\lambda I_5)^{i-1}))$. The first number of this last column is $\dim(\ker(B-\lambda I_5))$.

5. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w_1},\mathbf{w_2},\mathbf{w_3}\}$ satisfying

$$\begin{cases}(B-\lambda I_5)\,\mathbf{w_1} = \mathbf{0},\\ (B-\lambda I_5)\,\mathbf{w_2} = \mathbf{w_1},\\ (B-\lambda I_5)\,\mathbf{w_3} = \mathbf{w_2}\end{cases}$$

or

$$\begin{cases}(B-\lambda I_5)^3\,\mathbf{w_3} = \mathbf{0},\\ (B-\lambda I_5)^2\,\mathbf{w_3} = \mathbf{w_1},\\ (B-\lambda I_5)\,\mathbf{w_3} = \mathbf{w_2}\end{cases}$$

where $\mathbf{w}_3$ is in the vector space $\ker((B - \lambda I_5)^3)$ but not in $\ker((B - \lambda I_5)^2)$.

We look for a generating vector $\mathbf{w}_3$. This vector must be linearly independent of all vectors in $\ker(B - \lambda I_5)^2$ and must be independent from vectors of height 3 that were already chosen in $\ker(B - \lambda I_5)^3$. We know that

$$\ker((B - \lambda I_5)^3) = \text{span}\{(1, 0, 0, -1, -2), (0, 1, 0, 1, 1), (0, 0, 1, -1, -2)\}.$$

We have at this point not chosen any vector of height 3 in $\ker((B-\lambda I_5)^3)$. We know that a vector in $\ker((B - \lambda I_5)^3)$ must be of the form

$$\begin{aligned} &a\,(1, 0, 0, -1, -2) + b\,(0, 1, 0, 1, 1) + c\,(0, 0, 1, -1, -2) \\ &\quad = (a, b, c, -a + b - c, -2\,a + b - 2\,c). \end{aligned}$$

We remember that

$$\ker(B - \lambda I_5)^2 = \text{span}\{(1, 0, -2/3, -1/3, -2/3), (0, 1, 1/3, 2/3, 1/3)\}.$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 0 & -2/3 & -1/3 & -2/3 \\ 0 & 1 & 1/3 & 2/3 & 1/3 \\ a & b & c & -a + b - c & -2\,a + b - 2\,c \end{pmatrix}.$$

We row reduce this matrix H and if we impose $2\,a - b + 3\,c \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -2 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $2\,a - b + 3\,c \neq 0$. We can choose $a = 1$, $b = 0$ and $c = 0$.

We have the generating vector

$$\mathbf{w}_3 = (1, 0, 0, -1, -2).$$

We start with $\mathbf{w}_3 = (1, 0, 0, -1, -2)$.

We calculate $\mathbf{w_2}$.

$$\begin{aligned}\mathbf{w_2} &= (B - \lambda I_5)\mathbf{w_3} \\ &= \begin{pmatrix} -2 & 0 & -5 & 4 & 1 \\ -4 & 3 & -4 & -1 & -1 \\ 2 & 1 & 4 & -5 & 1 \\ 0 & -1 & 1 & 2 & -2 \\ 3 & -2 & 5 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -8 \\ -1 \\ 5 \\ 2 \\ 5 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathbf{w_2} &= (B - \lambda I_5)^2\mathbf{w_4} \\ &= \begin{pmatrix} -3 & -11 & -1 & 25 & -16 \\ -15 & 8 & -14 & -1 & -8 \\ 3 & 10 & 2 & -23 & 14 \\ 0 & 0 & 0 & 0 & 0 \\ 9 & 1 & 8 & -11 & 11 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -2 \\ 0 \\ -2 \end{pmatrix}.\end{aligned}$$

So we have the Jordan chain

$$\{\mathbf{w_1} = (4, 2, -2, 0, -2), \mathbf{w_2} = (-8, -1, 5, 2, 5), \mathbf{w_3} = (1, 0, 0, -1, -2)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_5)$	1	$\mathbf{w_1}$	0
$\ker((B - \lambda I_5)^2)$	2	$\mathbf{w_2}$	0
$\ker((B - \lambda I_5)^3)$	3	$\mathbf{w_3}$	0

$$\mathbf{w_1} = (4, 2, -2, 0, -2)$$
$$\mathbf{w_2} = (-8, -1, 5, 2, 5)$$
$$\mathbf{w_3} = (1, 0, 0, -1, -2)$$

6. Investigation of the second eigenvalue.

We work now with the eigenvalue $\lambda = 2$.

7. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 8 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 7. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_5.

$$A - \lambda I_5 = \begin{pmatrix} \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

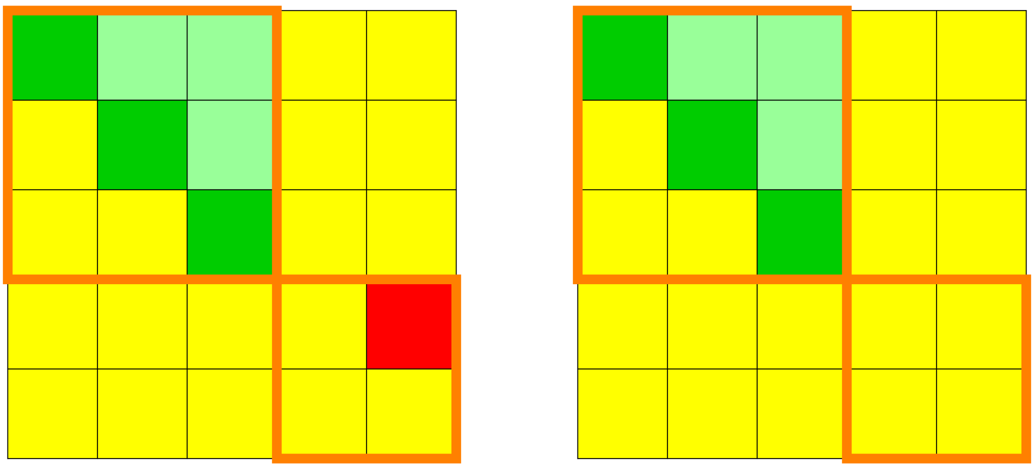
We see that the space span$\{\mathbf{e}_4, \mathbf{e}_5\}$ is invariant relative to $A - \lambda I_5$. If we restrict $A - \lambda I_5$ to this space, we have a nilpotent operator and we can apply the techniques learned in part 2.

We want to investigate the endomorphism $A - \lambda I_5$ restricted to this space. We compute now the powers of $A - \lambda I_5$.

$$A - \lambda I_5 = \begin{pmatrix} -3 & 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(A - \lambda I_5)^2 = \begin{pmatrix} 9 & -6 & 1 & 0 & 0 \\ 0 & 9 & -6 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $(A - \lambda I_5)$. (b) Visualisation of $(A - \lambda I_5)^2$.

Figure 46

We give here some comment on the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells represent the number 0. The red cells represent the number 1. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix.

If we restrict the mapping $A - \lambda I_5$ to the subspace $\text{span}\{\mathbf{e}_4, \mathbf{e}_5\}$ which is an invariant subspace with respect to the operator $A - \lambda I_5$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 2. Let us take a look at the second elementary Jordan block in this matrix. We observe that the original superdiagonal of 1's in the second block of $A - \lambda I_5$ are going upwards in their respective elementary Jordan blocks associated with the nilpotent part of the transformation $A - \lambda I_5$ when increasing the powers of the matrix $A - \lambda I_5$. It finally disappears when taking the second power of $A - \lambda I_5$.

All these green elementary Jordan blocks have the same look however large the exponent of $A - \lambda I_5$. The yellow cells are representing the number zero. The nilpotent part of the matrix is the second elementary Jordan block. It changes by moving the superdiagonal upwards when increasing the exponents of $A - \lambda I_5$.

Investigation of the first Jordan chain.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_5) = \text{span}\{\mathbf{e}_4\}; \\ \ker(A - \lambda I_5)^2 = \text{span}\{\mathbf{e}_4, \mathbf{e}_5\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_5)$	1	$1 = \dim(\ker(A - \lambda I_5))$
$\ker(A - \lambda I_5)^2$	2	$1 = \dim(\ker(A - \lambda I_5)^2) - \dim(\ker(A - \lambda I_5))$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A-\lambda I_5)^i) - \dim(\ker(A-\lambda I_5)^{i-1})$. In the first row, we have $\dim(\ker(A-\lambda I_5))$.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A-\lambda I_5) \subsetneq \ker(A-\lambda I_5)^2 = \ker(A-\lambda I_5)^3 = \cdots .$$

We remark by looking at the matrices $(A-\lambda I_5)^i$ that we have the following mappings

$$\begin{cases}(A-\lambda I_5)\,\mathbf{e}_5 = \mathbf{e}_4,\\ (A-\lambda I_5)\,\mathbf{e}_4 = \mathbf{0}.\end{cases}$$

We remark by looking at the matrices $(A-\lambda I_5)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases}(A-\lambda I_5)\,\mathbf{e}_5 = \mathbf{e}_4,\\ (A-\lambda I_5)^2\,\mathbf{e}_5 = \mathbf{0}.\end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_4 = (A-\lambda I_5)\,\mathbf{e}_5, \mathbf{e}_5\}.$$

After we have found the Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A-\lambda I_5)$	1	$\mathbf{e}_4$	0
$\ker(A-\lambda I_5)^2$	2	$\mathbf{e}_5$	0

We have now in the last column only 0's and this means that we are done looking for Jordan chains.

8. Kernels of $(\mathbf{B} - \lambda\,\mathbf{I_5})^{\mathbf{i}}$.

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I_5})$.
We calculate the kernel of $B - \lambda\, I_5$.

The matrix $B - \lambda\, I_5$ is

$$B - \lambda\, I_5 = \begin{pmatrix} -2 & 0 & -5 & 4 & 1 \\ -4 & 3 & -4 & -1 & -1 \\ 2 & 1 & 4 & -5 & 1 \\ 0 & -1 & 1 & 2 & -2 \\ 3 & -2 & 5 & 0 & -1 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -2 & 0 & -5 & 4 & 1 \\ -4 & 3 & -4 & -1 & -1 \\ 2 & 1 & 4 & -5 & 1 \\ 0 & -1 & 1 & 2 & -2 \\ 3 & -2 & 5 & 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -2z_1 - 5z_3 + 4z_4 + z_5 = 0, \\ -4z_1 + 3z_2 - 4z_3 - z_4 - z_5 = 0, \\ 2z_1 + z_2 + 4z_3 - 5z_4 + z_5 = 0, \\ - z_2 + z_3 + 2z_4 - 2z_5 = 0, \\ 3z_1 - 2z_2 + 5z_3 - z_5 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda\, I_5) &= \{(r_1, r_1/2, -r_1/2, 0, -r_1/2) \mid r_1 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, 1/2, -1/2, 0, -1/2)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I_5})^2$.
We calculate the kernel of $(B - \lambda\, I_5)^2$.

The matrix $(B - \lambda\, I_5)^2$ is

$$(B-\lambda I_5)^2 = \begin{pmatrix} -3 & -11 & -1 & 25 & -16 \\ -15 & 8 & -14 & -1 & -8 \\ 3 & 10 & 2 & -23 & 14 \\ 0 & 0 & 0 & 0 & 0 \\ 9 & 1 & 8 & -11 & 11 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -3 & -11 & -1 & 25 & -16 \\ -15 & 8 & -14 & -1 & -8 \\ 3 & 10 & 2 & -23 & 14 \\ 0 & 0 & 0 & 0 & 0 \\ 9 & 1 & 8 & -11 & 11 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -3z_1 - 11z_2 - z_3 + 25z_4 - 16z_5 = 0, \\ -15z_1 + 8z_2 - 14z_3 - z_4 - 8z_5 = 0, \\ 3z_1 + 10z_2 + 2z_3 - 23z_4 + 14z_5 = 0, \\ 9z_1 + z_2 + 8z_3 - 11z_4 + 11z_5 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B-\lambda I_5)^2 &= \{(r_1, r_2, -2r_1/3 + r_2/3, -r_1/3 + 2r_2/3, -2r_1/3 + r_2/3) \\ &\qquad \mid r_1, r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,-2/3,-1/3,-2/3), (0,1,1/3,2/3,1/3)\}. \end{aligned}$$

Kernel of $(\mathbf{B}-\lambda\,\mathbf{I}_5)^3$.
We calculate the kernel of $(B-\lambda I_5)^3$.

The matrix $(B-\lambda I_5)^3$ is

$$(B-\lambda I_5)^3 = \begin{pmatrix} -54 & 72 & -126 & -63 & 144 \\ -27 & 9 & -36 & -18 & 18 \\ 27 & -36 & 63 & 18 & -72 \\ 0 & -27 & 27 & 27 & -54 \\ 27 & -63 & 90 & 72 & -126 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -54 & 72 & -126 & -63 & 144 \\ -27 & 9 & -36 & -18 & 18 \\ 27 & -36 & 63 & 18 & -72 \\ 0 & -27 & 27 & 27 & -54 \\ 27 & -63 & 90 & 72 & -126 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -54z_1 + 72z_2 - 126z_3 - 63z_4 + 144z_5 = 0, \\ -27z_1 + 9z_2 - 36z_3 - 18z_4 + 18z_5 = 0, \\ 27z_1 - 36z_2 + 63z_3 + 18z_4 - 72z_5 = 0, \\ -27z_2 + 27z_3 + 27z_4 - 54z_5 = 0, \\ 27z_1 - 63z_2 + 90z_3 + 72z_4 - 126z_5 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_5)^3 &= \{(r_1, r_2, -r_1, 0, -r_1/2 - r_2/2) \mid r_1, r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, 0, -1, 0, -1/2), (0, 1, 0, 0, -1/2)\}. \end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the second power onwards in the following chain of inclusion of sets.

$$\ker(B - \lambda I_5) \subsetneq \ker((B - \lambda I_5)^2) = \ker((B - \lambda I_5)^3) \cdots .$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B - \lambda I_5)$	1	$1 = \dim(\ker(B - \lambda I_5))$
$\ker(B - \lambda I_5)^2$	2	$1 = \dim(\ker(B - \lambda I_5)^2) - \dim(\ker(B - \lambda I_5))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B - \lambda I_5)^i)) - \dim(\ker((B - \lambda I_5)^{i-1}))$. The first number of this last column is $\dim(\ker(B - \lambda I_5))$.

9. Calculation of Jordan chains.

Calculation of the Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w}_4, \mathbf{w}_5\}$ satisfying

$$\begin{cases}(B - \lambda I_5)\,\mathbf{w}_4 = \mathbf{0},\\ (B - \lambda I_5)\,\mathbf{w}_5 = \mathbf{w}_4,\end{cases}$$

or

$$\begin{cases}(B - \lambda I_5)^2\,\mathbf{w}_5 = \mathbf{0},\\ (B - \lambda I_5)\,\mathbf{w}_5 = \mathbf{w}_4\end{cases}$$

where $\mathbf{w}_5$ is in the vector space $\ker((B - \lambda I_5)^2)$ but not in $\ker(B - \lambda I_5)$.

We look for a generating vector $\mathbf{w}_5$. This vector must be linearly independent of all vectors in $\ker(B - \lambda I_5)$ and must be independent from vectors that were already chosen in $\ker(B - \lambda I_5)^2$. We know that

$$\ker((B - \lambda I_5)^2) = \operatorname{span}\{(1, 0, -1, 0, -1/2), (0, 1, 0, 0, -1/2)\}.$$

We have at this point not chosen any vector of height 2 in $\ker((B - \lambda I_5)^2)$.

We know that a vector in $\ker((B - \lambda I_5)^2)$ must be of the form

$$a\,(1, 0, -1, 0, -1/2) + b\,(0, 1, 0, 0, -1/2) = (a, b, -a, 0, -a/2).$$

We remember also

$$\ker(B - \lambda I_5) = \operatorname{span}\{(1, 1/2, -1/2, 0, -1/2)\}.$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 1/2 & -1/2 & 0 & -1/2 \\ a & b & -a & 0 & -a/2 - b/2 \end{pmatrix}.$$

We row reduce this matrix H and if we impose $a - 2\,b \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & \frac{b-a}{a-2\,b} & 0 & \frac{a-b}{2\,(2\,b-a)} \\ 0 & 1 & \frac{a}{a-2\,b} & 0 & -\frac{b}{2\,b-a} \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a - 2\,b \neq 0$. We can choose $a = 2$ and $b = 0$.

We have the generating vector

$$\mathbf{w_5} = (2, 0, -2, 0, -1).$$

We start with $\mathbf{w_5} = (2, 0, -2, 0, -1)$.

We calculate $\mathbf{w_4}$.

$$\begin{aligned}\mathbf{w_4} &= (B - \lambda I_5)\mathbf{w_5} \\ &= \begin{pmatrix} -5 & 0 & -5 & 4 & 1 \\ -4 & 0 & -4 & -1 & -1 \\ 2 & 1 & 1 & -5 & 1 \\ 0 & -1 & 1 & -1 & -2 \\ 3 & -2 & 5 & 0 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.\end{aligned}$$

So we have the Jordan chain

$$\{\mathbf{w_4} = (-1, 1, 1, 0, 0), \mathbf{w_5} = (2, 0, -2, 0, -1)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_5)$	1	$\mathbf{w_4}$	0
$\ker(B - \lambda I_5)^2$	2	$\mathbf{w_5}$	0

with

$\mathbf{w_4} = (-1, 1, 1, 0, 0)$
$\mathbf{w_5} = (2, 0, -2, 0, -1)$

10. Result and check of the result.

We construct the matrix of base change P with the coordinates of the vectors $\mathbf{w_i}$ found in the Jordan chains in the columns of P.

$$P = \begin{pmatrix} 4 & -8 & 1 & -1 & 2 \\ 2 & -1 & 0 & 1 & 0 \\ -2 & 5 & 0 & 1 & -2 \\ 0 & 2 & -1 & 0 & 0 \\ -2 & 5 & -2 & 0 & -1 \end{pmatrix}.$$

$$\begin{aligned} P^{-1} B P &= \begin{pmatrix} 1/2 & -1/8 & 5/8 & 1 & -1/4 \\ 0 & -1/4 & 1/4 & 1 & -1/2 \\ 0 & -1/2 & 1/2 & 1 & -1 \\ -1 & 1 & -1 & -1 & 0 \\ -1 & 0 & -1 & 1 & -1 \end{pmatrix} \\ &\times \begin{pmatrix} -3 & 0 & -5 & 4 & 1 \\ -4 & 2 & -4 & -1 & -1 \\ 2 & 1 & 3 & -5 & 1 \\ 0 & -1 & 1 & 1 & -2 \\ 3 & -2 & 5 & 0 & -2 \end{pmatrix} \\ &\times \begin{pmatrix} 4 & -8 & 1 & -1 & 2 \\ 2 & -1 & 0 & 1 & 0 \\ -2 & 5 & 0 & 1 & -2 \\ 0 & 2 & -1 & 0 & 0 \\ -2 & 5 & -2 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}. \end{aligned}$$

32 exercise. (4×4); $(\mathbf{J}_2(-2), \mathbf{J}_2(2))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 10 & 5 & 3 & 4 \\ -12 & -7 & -3 & -4 \\ -8 & -5 & -1 & -4 \\ -5 & -1 & -4 & -2 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_4| = (\lambda - 2)^2 (\lambda + 2)^2.$$

We see that the characteristic Cayley-Hamilton polynomial $p_{\text{C-H}}(\lambda)$ can be factorised in linear polynomials over the field $\mathbf{K}$. So we can apply the Jordan normalisation machinery.

The eigenvalue $\lambda = -2$ has algebraic multiplicity 2, The eigenvalue $\lambda = 2$ has algebraic multiplicity 2.

2. Investigation of the first eigenvalue.

We work now with the eigenvalue $\lambda = -2$.

3. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 4 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 3. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_4.

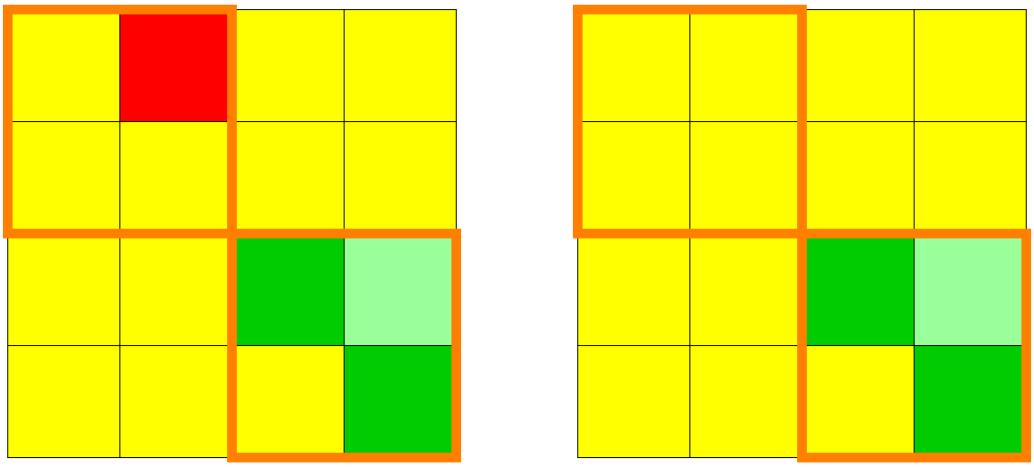
$$A - \lambda I_4 = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} \end{pmatrix}.$$

We compute also the powers of $A - \lambda I_4$.

$$A - \lambda I_4 = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} \end{pmatrix},$$

$$(A - \lambda I_4)^2 = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 16 & 8 \\ 0 & 16 \end{pmatrix} \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $A - \lambda I_4$. (b) Visualisation of $(A - \lambda I_4)^2$.

Figure 47

We give some comment on the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells represent the number 1. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix.

If we restrict the mapping $A - \lambda I_4$ to the subspace span$\{\mathbf{e}_1, \mathbf{e}_2\}$ which is an invariant subspace with respect to the operator $A - \lambda I_4$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 2. Let us take a look at the first elementary Jordan block in this matrix. We observe that the original superdiagonals of 1's in the first block of $A - \lambda I_4$ is going upwards in its elementary Jordan block associated with the nilpotent part of the transformation $A - \lambda I_4$ when increasing the powers of the matrix $A - \lambda I_4$. It finally disappears when taking the second power of $A - \lambda I_4$.

Investigation of the first Jordan chain.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_4) = \operatorname{span}\{\mathbf{e}_1\}; \\ \ker(A - \lambda I_4)^2 = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_4)$	1	$1 = \dim(\ker(A - \lambda I_4))$
$\ker(A - \lambda I_4)^2$	2	$1 = \dim(\ker(A - \lambda I_4)^2) - \dim(\ker(A - \lambda I_4))$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A-\lambda I_4)^i) - \dim(\ker(A-\lambda I_4)^{i-1})$. In the first row, we have $\dim(\ker(A - \lambda I_4))$.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A - \lambda I_4) \subsetneq \ker(A - \lambda I_4)^2 = \ker(A - \lambda I_4)^3 = \cdots .$$

We remark by looking at the matrices $(A - \lambda I_4)^i$ that we have the following mappings

$$\begin{cases} (A - \lambda I_4)\,\mathbf{e}_2 = \mathbf{e}_1, \\ (A - \lambda I_4)\,\mathbf{e}_1 = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A - \lambda I_4)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases} (A - \lambda I_4)\,\mathbf{e}_2 = \mathbf{e}_1, \\ (A - \lambda I_4)^2\,\mathbf{e}_2 = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_1 = (A - \lambda I_4)\,\mathbf{e}_2, \mathbf{e}_2\}.$$

After we have found the first Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A - \lambda I_4)$	1	$\mathbf{e_1}$	0
$\ker(A - \lambda I_4)^2$	1	$\mathbf{e_2}$	0

We have now in the last column only 0's and this means that we are done looking for Jordan chains.

4. Kernels of $(\mathbf{B} - \lambda \mathbf{I_4})^{\mathbf{i}}$.

Kernel of $(\mathbf{B} - \lambda \mathbf{I_4})$.
We calculate the kernel of $B - \lambda I_4$.

The matrix $B - \lambda I_4$ is

$$B - \lambda I_4 = \begin{pmatrix} 12 & 5 & 3 & 4 \\ -12 & -5 & -3 & -4 \\ -8 & -5 & 1 & -4 \\ -5 & -1 & -4 & 0 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 12 & 5 & 3 & 4 \\ -12 & -5 & -3 & -4 \\ -8 & -5 & 1 & -4 \\ -5 & -1 & -4 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 12z_1 + 5z_2 + 3z_3 + 4z_4 = 0, \\ -12z_1 - 5z_2 - 3z_3 - 4z_4 = 0, \\ -8z_1 - 5z_2 + z_3 - 4z_4 = 0, \\ -5z_1 - z_2 - 4z_3 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B - \lambda I_4) &= \{(r_1, -r_1, -r_1, -r_1) \mid r_1 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, -1, -1, -1)\}.\end{aligned}$$

Kernel of $(\mathbf{B} - \lambda \mathbf{I_4})^2$.
We calculate the kernel of $(B - \lambda I_4)^2$.

The matrix $(B - \lambda I_4)^2$ is

$$(B - \lambda I_4)^2 = \begin{pmatrix} 40 & 16 & 8 & 16 \\ -40 & -16 & -8 & -16 \\ -24 & -16 & 8 & -16 \\ -16 & 0 & -16 & 0 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 40 & 16 & 8 & 16 \\ -40 & -16 & -8 & -16 \\ -24 & -16 & 8 & -16 \\ -16 & 0 & -16 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 40z_1 + 16z_2 + 8z_3 + 16z_4 = 0, \\ -40z_1 - 16z_2 - 8z_3 - 16z_4 = 0, \\ -24z_1 - 16z_2 + 8z_3 - 16z_4 = 0, \\ -16z_1 - 16z_3 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B - \lambda I_4)^2 &= \{(r_1, r_2, -r_1, -2r_1 - r_2) \mid r_1, r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, 0, -1, -2), (0, 1, 0, -1)\}.\end{aligned}$$

Kernel of $(\mathbf{B} - \lambda \mathbf{I_4})^3$.
We calculate the kernel of $(B - \lambda I_4)^3$.

The matrix $(B - \lambda I_4)^3$ is

$$(B-\lambda I_4)^3 = \begin{pmatrix} 144 & 64 & 16 & 64 \\ -144 & -64 & -16 & -64 \\ -80 & -64 & 48 & -64 \\ -64 & 0 & -64 & 0 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 144 & 64 & 16 & 64 \\ -144 & -64 & -16 & -64 \\ -80 & -64 & 48 & -64 \\ -64 & 0 & -64 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 144z_1 + 64z_2 + 16z_3 + 64z_4 = 0, \\ -144z_1 - 64z_2 - 16z_3 - 64z_4 = 0, \\ -80\, z_1 - 64z_2 + 48z_3 - 64z_4 = 0, \\ -64\, z_1 \qquad - 64z_3 \qquad = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B-\lambda I_4)^3 &= \{(r_1, r_2, -r_1, -2\,r_1 - r_2) \mid r_1, r_2 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,0,-1,-2),(0,1,0,-1)\}. \end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the second power onwards in the following chain of inclusion of sets.

$$\ker(B-\lambda I_4) \subsetneq \ker((B-\lambda I_4)^2) = \ker((B-\lambda I_4)^3) \cdots .$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B-\lambda I_4)$	1	$1 = \dim(\ker(B-\lambda I_4))$
$\ker(B-\lambda I_4)^2$	2	$1 = \dim(\ker(B-\lambda I_4)^2) - \dim(\ker(B-\lambda I_4))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B-\lambda I_4)^i)) - \dim(\ker((B-\lambda I_4)^{i-1}))$. The first number of this last column is $\dim(\ker(B-\lambda I_4))$.

5. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w_1}, \mathbf{w_2}\}$ satisfying

$$\begin{cases} (B-\lambda I_4)\,\mathbf{w_1} = \mathbf{0}, \\ (B-\lambda I_4)\,\mathbf{w_2} = \mathbf{w_1} \end{cases}$$

or

$$\begin{cases} (B-\lambda I_4)^2\,\mathbf{w_2} = \mathbf{0}, \\ (B-\lambda I_4)\,\mathbf{w_2} = \mathbf{w_1} \end{cases}$$

where $\mathbf{w_2}$ is in the vector space $\ker((B-\lambda I_4)^2)$ but not in $\ker((B-\lambda I_4))$.

We look for a generating vector $\mathbf{w_2}$. This vector must be linearly independent of all vectors in $\ker(B-\lambda I_4)$ and must be independent from vectors of height 2 that were already chosen in $\ker(B-\lambda I_4)^2$. We know that

$$\ker((B-\lambda I_4)^2) = \operatorname{span}\{(1,0,-1,-2),(0,1,0,-1)\}.$$

We have at this point not chosen any vector of height 2 in $\ker((B-\lambda I_4))^2$.

We know that a vector in $\ker((B-\lambda I_4)^2)$ must be of the form

$$a\,(1,0,-1,-2) + b\,(0,1,0,-1) = (a,b,-a,-2\,a-b).$$

We remember that

$$\ker(B-\lambda I_4) = \operatorname{span}\{(1,-1,-1,-1)\}.$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & -1 & -1 & -1 \\ a & b & -a & -2\,a-b \end{pmatrix}.$$

We row reduce this matrix H and if we impose $a+b \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + b \neq 0$. We can choose $a = 1$ and $b = 0$.

We have the generating vector

$$\mathbf{w_2} = (1, 0, -1, -2).$$

We start with $\mathbf{w_2} = (1, 0, -1, -2)$.

We calculate

$$\begin{aligned}\mathbf{w_1} &= (B - \lambda I_4)\,\mathbf{w_2} \\ &= \begin{pmatrix} 12 & 5 & 3 & 4 \\ -12 & -5 & -3 & -4 \\ -8 & -5 & 1 & -4 \\ -5 & -1 & -4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}.\end{aligned}$$

We have the Jordan chain

$$\{\mathbf{w_1} = (1, -1, -1, -1), \mathbf{w_2} = (1, 0, -1, -2)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_4)$	1	$\mathbf{w_1}$	0
$\ker((B - \lambda I_4)^2)$	2	$\mathbf{w_2}$	0

with

$\mathbf{w_1} = (1, -1, -1, -1)$
$\mathbf{w_2} = (1, 0, -1, -2)$

6. Investigation of the second eigenvalue.

We work now with the eigenvalue $\lambda = 2$.

7. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 8 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 7. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_4.

$$A - \lambda I_4 = \begin{pmatrix} \begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

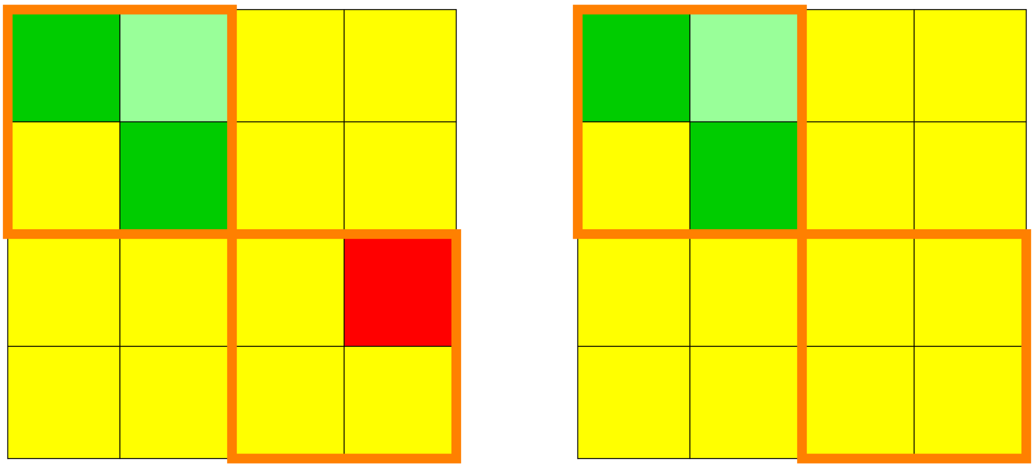
We see that the space $\operatorname{span}\{\mathbf{e}_3, \mathbf{e}_4\}$ is invariant relative to $A - \lambda I_4$. If we restrict $A - \lambda I_4$ to this space, we have a nilpotent operator and we can apply the techniques learned in part 2.

We want to investigate the endomorphism $A - \lambda I_4$ restricted to this space. We compute now the powers of $A - \lambda I_4$.

$$A - \lambda I_4 = \begin{pmatrix} \begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$(A - \lambda I_4)^2 = \begin{pmatrix} \begin{pmatrix} 16 & -8 \\ 0 & 16 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $(A - \lambda I_4)$. (b) Visualisation of $(A - \lambda I_4)^2$.

Figure 48

We give here some comment on the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells represent the number 1. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix.

If we restrict the mapping $A - \lambda I_4$ to the subspace $\text{span}\{\mathbf{e}_3, \mathbf{e}_4\}$ which is an invariant subspace with respect to the operator $A - \lambda I_4$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 2. Let us take a look at the second elementary Jordan block in this matrix. We observe that the original superdiagonal of 1's in the second block of $A - \lambda I_4$ is going upwards in its elementary Jordan block associated with the nilpotent part of the transformation $A - \lambda I_4$ when increasing the powers of the matrix $A - \lambda I_4$. It finally disappears when taking the second power of $A - \lambda I_4$.

Investigation of the first Jordan chain.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_4) = \operatorname{span}\{\mathbf{e}_3\}; \\ \ker(A - \lambda I_4)^2 = \operatorname{span}\{\mathbf{e}_3, \mathbf{e}_4\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_4)$	1	$1 = \dim(\ker(A - \lambda I_4))$
$\ker(A - \lambda I_4)^2$	2	$1 = \dim(\ker(A - \lambda I_4)^2) - \dim(\ker(A - \lambda I_4))$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A-\lambda I_4)^i) - \dim(\ker(A-\lambda I_4)^{i-1})$. In the first row, we have $\dim(\ker(A - \lambda I_4))$.

We see that the kernels are starting to stabilise. We mean by this that we are having equality from the second power onwards in the following chain of inclusion of sets.

$$\ker(A - \lambda I_4) \subsetneq \ker(A - \lambda I_4)^2 = \ker(A - \lambda I_4)^3 = \cdots .$$

We remark by looking at the matrices $(A - \lambda I_4)^i$ that we have the following mappings

$$\begin{cases}(A-\lambda\, I_4)\,\mathbf{e}_4 = \mathbf{e}_3,\\ (A-\lambda\, I_4)\,\mathbf{e}_3 = \mathbf{0}.\end{cases}$$

We remark by looking at the matrices $(A-\lambda\, I_4)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases}(A-\lambda\, I_4)\,\mathbf{e}_4 = \mathbf{e}_3,\\ (A-\lambda\, I_4)^2\,\mathbf{e}_4 = \mathbf{0}.\end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e}_3 = (A-\lambda\, I_4)\,\mathbf{e}_4, \mathbf{e}_4\}.$$

After we have found the first Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A-\lambda\, I_4)$	1	$\mathbf{e}_3$	0
$\ker(A-\lambda\, I_4)^2$	1	$\mathbf{e}_4$	0

We have now in the last column only 0's and this means that we are done looking for Jordan chains.

8. Kernels of $(\mathbf{B}-\lambda\,\mathbf{I}_4)^{\mathbf{i}}$.

Kernel of $(\mathbf{B}-\lambda\,\mathbf{I}_4)$.
We calculate the kernel of $B-\lambda\, I_4$.

The matrix $B-\lambda\, I_4$ is

$$B-\lambda\, I_4 = \begin{pmatrix} 12 & 5 & 3 & 4\\ -12 & -5 & -3 & -4\\ -8 & -5 & 1 & -4\\ -5 & -1 & -4 & 0\end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 12 & 5 & 3 & 4 \\ -12 & -5 & -3 & -4 \\ -8 & -5 & 1 & -4 \\ -5 & -1 & -4 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 12z_1 + 5z_2 + 3z_3 + 4z_4 = 0, \\ -12z_1 - 5z_2 - 3z_3 - 4z_4 = 0, \\ -8z_1 - 5z_2 + z_3 - 4z_4 = 0, \\ -5z_1 - z_2 - 4z_3 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_4) &= \{(r_1, -r_1, -r_1, -r_1) \mid r_1 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, -1, -1, -1)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\, \mathbf{I})^2$.
We calculate the kernel of $(B - \lambda I_4)^2$.

The matrix $(B - \lambda I_4)^2$ is

$$(B - \lambda I_4)^2 = \begin{pmatrix} -40 & -24 & -16 & -16 \\ 56 & 40 & 16 & 16 \\ 40 & 24 & 16 & 16 \\ 24 & 8 & 16 & 16 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -40 & -24 & -16 & -16 \\ 56 & 40 & 16 & 16 \\ 40 & 24 & 16 & 16 \\ 24 & 8 & 16 & 16 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -40z_1 - 24z_2 - 16z_3 - 16z_4 = 0, \\ 56z_1 + 40z_2 + 16z_3 + 16z_4 = 0, \\ 40z_1 + 24z_2 + 16z_3 + 16z_4 = 0, \\ 24z_1 + 8z_2 + 16z_3 + 16z_4 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B-\lambda I_4)^2 &= \{(r_1,-r_1,r_3,-r_1-r_3) \mid r_1,r_3 \in \mathbf{K}\}\\ &= \operatorname{span}\{(1,-1,0,-1),(0,0,1,-1)\}.\end{aligned}$$

Kernel of $(\mathbf{B}-\lambda\,\mathbf{I})^3$.
We calculate the kernel of $(B-\lambda I_4)^3$.

The matrix $(B-\lambda I_4)^3$ is

$$(B-\lambda I_4)^3 = \begin{pmatrix} 176 & 112 & 64 & 64\\ -240 & -176 & -64 & -64\\ -176 & -112 & -64 & -64\\ -112 & -48 & -64 & -64 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 176 & 112 & 64 & 64\\ -240 & -176 & -64 & -64\\ -176 & -112 & -64 & -64\\ -112 & -48 & -64 & -64 \end{pmatrix}\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 176z_1 + 112z_2 + 64z_3 + 64z_4 = 0,\\ -240z_1 - 176z_2 - 64z_3 - 64z_4 = 0,\\ -176z_1 - 112z_2 - 64z_3 - 64z_4 = 0,\\ -112z_1 - 48\,z_2 - 64z_3 - 64z_4 = 0.\end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B-\lambda I_4)^3 &= \{(r_1,-r_1,r_3,-r_1-r_3) \mid r_1,r_3 \in \mathbf{K}\}\\ &= \operatorname{span}\{(1,-1,0,-1),(0,0,1,-1)\}.\end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the second power onwards in the following chain of inclusion of sets.

$$\ker(B-\lambda I_4) \subsetneq \ker((B-\lambda I_4)^2) = \ker((B-\lambda I_4)^3) = \cdots .$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B-\lambda I_4)$	1	$1 = \dim(\ker(B-\lambda I_4))$
$\ker(B-\lambda I_4)^2$	2	$1 = \dim(\ker(B-\lambda I_4)^2) - \dim(\ker(B-\lambda I_4))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B-\lambda I_4)^i)) - \dim(\ker((B-\lambda I_4)^{i-1}))$. The first number of this last column is $\dim(\ker(B-\lambda I_4))$.

9. Calculation of Jordan chains.

Calculation of the Jordan chain.
We look for a linearly independent set of vectors $\{\mathbf{w_3}, \mathbf{w_4}\}$.

$$\begin{cases} (B-\lambda I_4)\,\mathbf{w_3} = \mathbf{0}, \\ (B-\lambda I_4)\,\mathbf{w_4} = \mathbf{w_3}. \end{cases}$$

where $\mathbf{w_4}$ is in the vector space $\ker((B-\lambda I_4)^2)$ but not in $\ker(B-\lambda I_4)$.

We look for a generating vector $\mathbf{w_4}$. This vector must be linearly independent of all vectors in $\ker(B-\lambda I_4)$ and must be independent from vectors of height 2 that were already chosen in $\ker(B-\lambda I_4)^2$. We know that

$$\ker((B-\lambda I_4)^2) = \operatorname{span}\{(1,-1,0,-1),(0,0,1,-1)\}.$$

We have at this point not chosen in $\ker((B-\lambda I_4)^2)$ any vector of height 2.

We know that a vector in $\ker((B-\lambda I_4)^2)$ must be of the form

$$a\,(1,-1,0,-1) + b\,(0,0,1,-1) = (a,-a,b,-a-b).$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & -1 & -1 & 0 \\ a & -a & b & -a-b \end{pmatrix}.$$

We row reduce this matrix H and if we impose $a + b \neq 0$, we find

$$\begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + b \neq 0$. We can choose $a = 1$ and $b = 0$.

We have the generating vector

$$\mathbf{w_4} = (1, -1, 0, -1).$$

We start with $\mathbf{w_4} = (1, -1, 0, -1)$.

We calculate $\mathbf{w_3}$.

$$\begin{aligned} \mathbf{w_3} &= (B - \lambda I_4)\mathbf{w_4} \\ &= \begin{pmatrix} 8 & 5 & 3 & 4 \\ -12 & -9 & -3 & -4 \\ -8 & -5 & -3 & -4 \\ -5 & -1 & -4 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}. \end{aligned}$$

We have the Jordan chain

$$\{\mathbf{w_3} = (-1, 1, 1, 0), \mathbf{w_4} = (1, -1, 0, -1)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_4)$	1	$\mathbf{w_3}$	0
$\ker(B - \lambda I_4)^2$	2	$\mathbf{w_4}$	0

with

$$\mathbf{w_3} = (-1, 1, 1, 0)$$
$$\mathbf{w_4} = (1, -1, 0, -1)$$

10. Result and check of the result.

We construct the matrix of base change P with the coordinates of the vectors $\mathbf{w_i}$ found in the Jordan chains in the columns of P.

$$P = \begin{pmatrix} 1 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 1 & 0 \\ -1 & -2 & 0 & -1 \end{pmatrix}.$$

$$\begin{aligned} P^{-1}\,B\,P &= \begin{pmatrix} -3 & -2 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ -2 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \\ &\times \begin{pmatrix} 10 & 5 & 3 & 4 \\ -12 & -7 & -3 & -4 \\ -8 & -5 & -1 & -4 \\ -5 & -1 & -4 & -2 \end{pmatrix} \\ &\times \begin{pmatrix} 1 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 1 & 0 \\ -1 & -2 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \end{aligned}$$

33 exercise. (3×3); $(J_3(-2))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} -4 & 0 & 1 \\ -3 & -3 & 2 \\ -5 & -1 & 1 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_3| = -(\lambda + 2)^3.$$

We see that the characteristic Cayley-Hamilton polynomial $p_{\text{C-H}}(\lambda)$ can be factorised in linear polynomials over the field $\mathbf{K}$. So we can apply the Jordan normalisation machinery.

The eigenvalue $\lambda = -2$ has algebraic multiplicity 3.

2. investigating the eigenvalue

We work now with the eigenvalue $\lambda = -2$.

3. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely

this subsection and start with subsection 4 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 3. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \left(\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}\right).$$

We subtract from this matrix A the matrix λI_3.

$$A - \lambda I_3 = \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right).$$

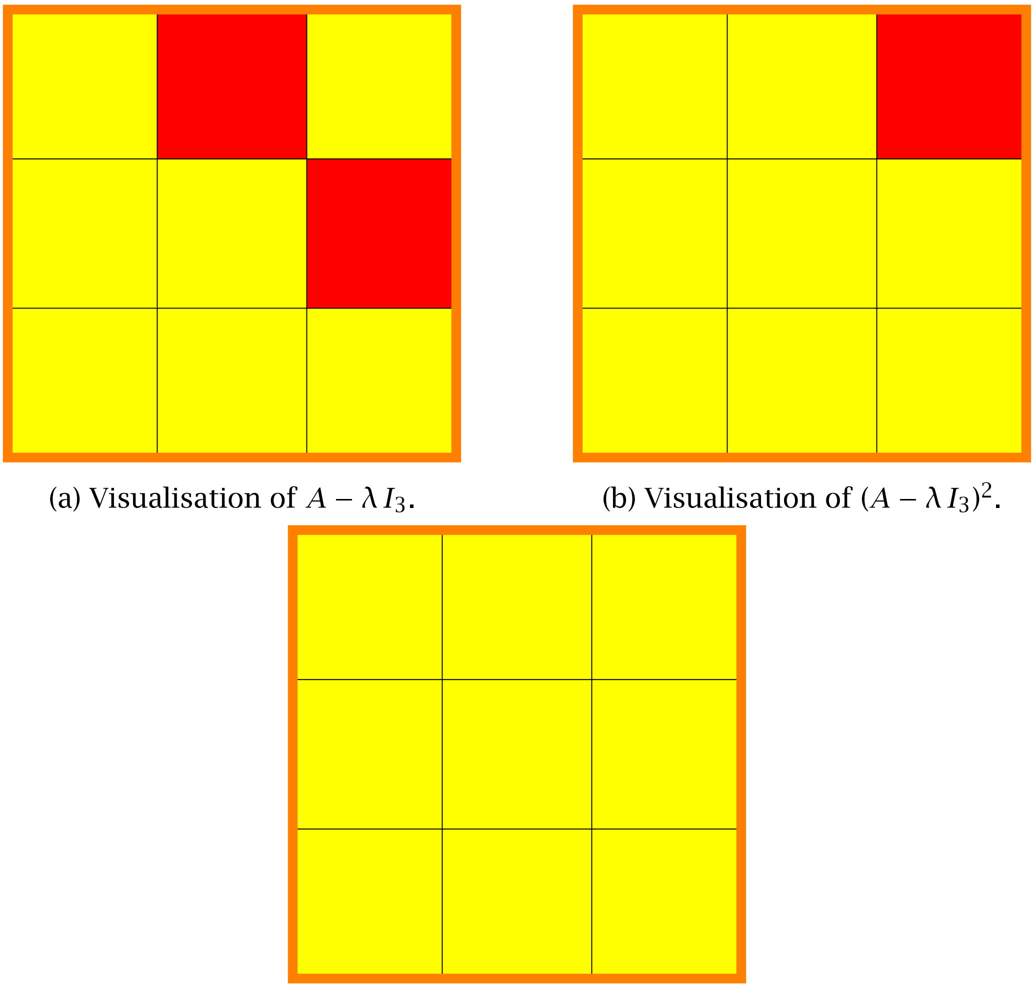
We compute also the powers of $A - \lambda I_3$.

$$A - \lambda I_3 = \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right),$$

$$(A - \lambda I_3)^2 = \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right),$$

$$(A - \lambda I_3)^3 = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right).$$

Let us visualise this situation.



(a) Visualisation of $A - \lambda I_3$.

(b) Visualisation of $(A - \lambda I_3)^2$.

(c) Visualisation of $(A - \lambda I_3)^3$.

Figure 49

We give some comment on the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells represent the number 1. All the blocks with green cells in it will be keeping the same look however large we take the powers of this matrix.

If we restrict the mapping $A - \lambda I_3$ to the subspace span$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ which is an invariant subspace with respect to the operator $A - \lambda I_3$, then we have the classic case of a nilpotent operator on a finite dimensional space. Remark that the word *restriction* is not necessary here because we have only one elementary Jordan block. The nilpotent operator has degree of nilpotency 3. Let us take a look at the first and only elementary Jordan block in this matrix. We observe that the original superdiagonal of 1's in the first and only block of $A - \lambda I_3$ is going upwards in its elementary Jordan block associated with the nilpotent part of the transformation $A - \lambda I_3$ when increasing the powers of the matrix $A - \lambda I_3$. It finally disappears when taking the third power of $A - \lambda I_3$.

Investigation of the Jordan chain.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_3) = \operatorname{span}\{\mathbf{e_1}\}; \\ \ker(A - \lambda I_3)^2 = \operatorname{span}\{\mathbf{e_1}, \mathbf{e_2}\}; \\ \ker(A - \lambda I_3)^3 = \operatorname{span}\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_3)$	1	$1 = \dim(\ker(A - \lambda I_3))$
$\ker(A - \lambda I_3)^2$	2	$1 = \dim(\ker(A - \lambda I_3)^2) - \dim(\ker(A - \lambda I_3))$
$\ker(A - \lambda I_3)^3$	3	$1 = \dim(\ker(A - \lambda I_3)^3) - \dim(\ker(A - \lambda I_3)^2)$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A-\lambda I_3)^i)-\dim(\ker(A-\lambda I_3)^{i-1})$. In the first row, we have $\dim(\ker(A - \lambda I_3))$.

We see also that there is equality in the inclusion of sets from the third power onwards.

$$\ker(A - \lambda I_3) \subsetneq \ker(A - \lambda I_3)^2 \subsetneq \ker(A - \lambda I_3)^3 = \ker(A - \lambda I_3)^4 = \cdots.$$

We remark by looking at the matrices $(A - \lambda I_3)^i$ that we have the following mappings

$$\begin{cases} (A - \lambda I_3)\, \mathbf{e_3} = \mathbf{e_2}, \\ (A - \lambda I_3)\, \mathbf{e_2} = \mathbf{e_1}, \\ (A - \lambda I_3)\, \mathbf{e_1} = \mathbf{0}. \end{cases}$$

We remark by looking at the matrices $(A - \lambda I_3)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases} (A - \lambda I_3)\, \mathbf{e_3} = \mathbf{e_2}, \\ (A - \lambda I_3)^2\, \mathbf{e_3} = \mathbf{e_1}, \\ (A - \lambda I_3)^3\, \mathbf{e_3} = \mathbf{0}. \end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e_1} = (A - \lambda I_3)^2\, \mathbf{e_3}, \mathbf{e_2} = (A - \lambda I_3)\, \mathbf{e_3}, \mathbf{e_3}\}.$$

After we have found the first Jordan chain of length 3, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A - \lambda I_3)$	1	$\mathbf{e_1}$	0
$\ker(A - \lambda I_3)^2$	2	$\mathbf{e_2}$	0
$\ker(A - \lambda I_3)^3$	3	$\mathbf{e_3}$	0

We have now in the last column only 0's and this means that we are done looking for Jordan chains.

4. Kernels of $(\mathbf{B} - \lambda \mathbf{I_3})^{\mathbf{i}}$.

Kernel of $(\mathbf{B} - \lambda \mathbf{I_3})$.
We calculate the kernel of $B - \lambda I_3$.

The matrix $B - \lambda I_3$ is

$$B - \lambda I_3 = \begin{pmatrix} -2 & 0 & 1 \\ -3 & -1 & 2 \\ -5 & -1 & 3 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -2 & 0 & 1 \\ -3 & -1 & 2 \\ -5 & -1 & 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -2z_1 \phantom{{}-z_2} + z_3 = 0, \\ -3z_1 - z_2 + 2z_3 = 0, \\ -5z_1 - z_2 + 3z_3 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_3) &= \{(r_1, r_1, 2\,r_1) \mid r_1 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1,1,2)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I_3})^2$.
We calculate the kernel of $(B - \lambda I_3)^2$.

The matrix $(B - \lambda I_3)^2$ is

$$(B - \lambda I_3)^2 = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ -2 & -2 & 2 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -z_1 - z_2 + z_3 = 0, \\ -z_1 - z_2 + z_3 = 0, \\ -2z_1 - 2z_2 + 2z_3 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B-\lambda I_3)^2 &= \{(r_1, r_2, r_1+r_2) \mid r_1, r_2 \in \mathbf{K}\}\\ &= \operatorname{span}\{(1,0,1),(0,1,1)\}.\end{aligned}$$

Kernel of $(\mathbf{B}-\lambda\,\mathbf{I_3})^3$.
We calculate the kernel of $(B-\lambda I_3)^3$.

The matrix $(B-\lambda I_3)^3$ is

$$(B-\lambda I_3)^3 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} z_1\\ z_2\\ z_3\end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0\end{pmatrix}.$$

We solve this system and this gives us the solutions set which is a subspace

$$\ker(B-\lambda I_3)^3 = \mathbf{K}^3.$$

Kernel of $(\mathbf{B}-\lambda\,\mathbf{I_3})^4$.
We calculate the kernel of $(B-\lambda I_3)^4$. The matrix $(B-\lambda I_3)^4$ is

$$(B-\lambda I_3)^4 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

We have again that $\ker(B-\lambda I_3)^4 = \mathbf{K}^3$.

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the third power onwards in the following chain of inclusion of sets.

$$\ker(B-\lambda I_3) \subsetneq \ker((B-\lambda I_3)^2) \subsetneq \ker((B-\lambda I_3)^3) = \ker((B-\lambda I_3)^4) = \cdots.$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B-\lambda I_3)$	1	$1 = \dim(\ker(B-\lambda I_3))$
$\ker(B-\lambda I_3)^2$	2	$1 = \dim(\ker((B-\lambda I_3)^2)) - \dim(\ker(B-\lambda I_3))$
$\ker(B-\lambda I_3)^3$	3	$1 = \dim(\ker((B-\lambda I_3)^3)) - \dim(\ker((B-\lambda I_3)^2))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B-\lambda I_3)^i)) - \dim(\ker((B-\lambda I_3)^{i-1}))$. The first number of this last column is $\dim(\ker(B-\lambda I_3))$.

5. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ satisfying

$$\begin{cases} (B-\lambda I_3)\,\mathbf{w_1} = \mathbf{0}, \\ (B-\lambda I_3)\,\mathbf{w_2} = \mathbf{w_1}, \\ (B-\lambda I_3)\,\mathbf{w_3} = \mathbf{w_2}. \end{cases}$$

where $\mathbf{w_3}$ is in the vector space $\ker((B-\lambda I_3)^3)$ but not in $\ker((B-\lambda I_3)^2)$.

We look for a generating vector $\mathbf{w_3}$. This vector must be linearly independent of all vectors in $\ker(B-\lambda I_3)^2$ and must be independent from vectors of height 3 that were already chosen in $\ker(B-\lambda I_3)^3$. We know that

$$\ker((B-\lambda I_3)^3) = \operatorname{span}\{(1,0,0),(0,1,0),(0,0,1)\}.$$

We know that a vector in $\ker((B-\lambda I_3)^3)$ must be of the form

$$a\,(1,0,0) + b\,(0,1,0) + c\,(0,0,1) = (a,b,c).$$

We remember also that

$$\ker(B-\lambda I_3)^2 = \operatorname{span}\{(1,0,1),(0,1,1)\}.$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ a & b & c \end{pmatrix}.$$

We row reduce this matrix H and if we impose $a + b - c \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $a + b - c \neq 0$. We can choose $a = 1$, $b = 0$ and $c = 0$.

We have the generating vector

$$\mathbf{w_3} = (1, 0, 0).$$

We start with $\mathbf{w_3} = (1, 0, 0)$.

We calculate $\mathbf{w_2}$.

$$\mathbf{w_2} = (B - \lambda I_3)\,\mathbf{w_3} = \begin{pmatrix} -2 & 0 & 1 \\ -3 & -1 & 2 \\ -5 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ -5 \end{pmatrix}$$

and

$$\mathbf{w_1} = (B - \lambda I_3)^2\,\mathbf{w_4} = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}.$$

We have the Jordan chain

$$\{\mathbf{w_1} = (-1, -1, -2), \mathbf{w_2} = (-2, -3, -5), \mathbf{w_3} = (1, 0, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_3)$	1	$\mathbf{w_1}$	0
$\ker(B - \lambda I_3)^2$	2	$\mathbf{w_2}$	0
$\ker(B - \lambda I_3)^3$	3	$\mathbf{w_3}$	0

with

$$\mathbf{w_1} = (-1, -1, -2)$$

$$\mathbf{w_2} = (-2, -3, -5)$$

$$\mathbf{w_3} = (1, 0, 0)$$

6. Result and check of the result.

We construct the matrix of base change P with the coordinates of the vectors $\mathbf{w_i}$ found in the Jordan chains in the columns of P.

$$P = \begin{pmatrix} -1 & -2 & 1 \\ -1 & -3 & 0 \\ -2 & -5 & 0 \end{pmatrix}.$$

$$\begin{aligned} P^{-1} B P &= \begin{pmatrix} 0 & 5 & -3 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -4 & 0 & 1 \\ -3 & -3 & 2 \\ -5 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -2 & 1 \\ -1 & -3 & 0 \\ -2 & -5 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix} \\ &= \left(\begin{pmatrix} \mathbf{-2} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-2} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{-2} \end{pmatrix}\right). \end{aligned}$$

34 exercise. (3×3); $(\mathbf{J}_2(3), \mathbf{J}_1(-2))$.

Use the relevant vector spaces $\ker(B - \lambda I)^j$ to investigate and predict the structure of the Jordan normal form of the matrix B. Find if possible a matrix A similar to B such that A is in Jordan normal form. Find explicitly a matrix P that is invertible and represents the base change: $A = P^{-1} B P$. Find explicitly Jordan chains that are necessary to construct the matrix P.

$$B = \begin{pmatrix} 10 & 1 & 5 \\ -14 & 1 & -10 \\ -12 & -1 & -7 \end{pmatrix}.$$

Solution.

1. Eigenvalues and the characteristic polynomial.

We check first the eigenvalues by computing the Cayley-Hamilton or characteristic polynomial.

$$p_{\text{C-H}}(\lambda) = |B - \lambda I_3| = -(\lambda - 3)^2 (\lambda + 2).$$

We see that the characteristic Cayley-Hamilton polynomial $p_{\text{C-H}}(\lambda)$ can be factorised in linear polynomials over the field $\mathbf{K}$. So we can apply the Jordan normalisation machinery.

The eigenvalue $\lambda = 3$ has algebraic multiplicity 2. The eigenvalue $\lambda = -2$ has algebraic multiplicity 1.

2. Investigation of the first eigenvalue.

We work now with the eigenvalue $\lambda = 3$.

3. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely

this subsection and start with subsection 4 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 3. The solution will be completely independent from this section.

We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & (-2) \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_3.

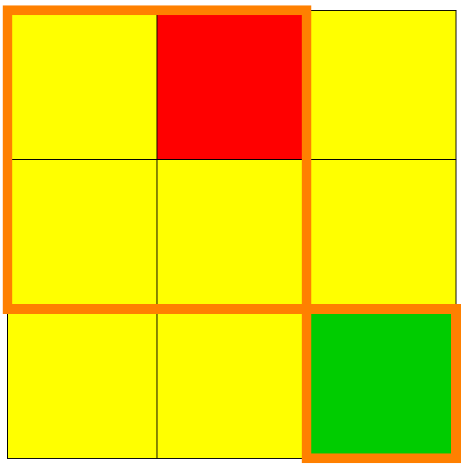
$$A - \lambda I_3 = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & (-5) \end{pmatrix}.$$

We compute also the powers of $A - \lambda I_3$.

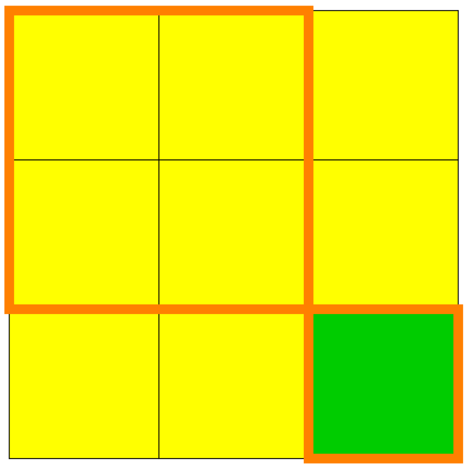
$$A - \lambda I_3 = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & (-5) \end{pmatrix},$$

$$(A - \lambda I_3)^2 = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & (25) \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $A - \lambda I_3$.



(b) Visualisation of $(A - \lambda I_3)^2$.

Figure 50

We give here some comment on the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The yellow cells are representing the number zero. The red cells represent the number 1.

All the block with green cells in it will be keeping the same look however large we take the powers of this matrix. If we restrict the mapping $A - \lambda I_3$ to the subspace $\operatorname{span}\{\mathbf{e_1}, \mathbf{e_2}\}$ which is an invariant subspace with respect to the operator $A - \lambda I_3$, then we have the classic case of a nilpotent operator on a finite dimensional space. The nilpotent operator has degree of nilpotency 2. Let us take a look at the first elementary Jordan block in this matrix. We observe that the original superdiagonal of 1's in the first block of $A - \lambda I_3$ is going upwards in its elementary Jordan block associated with the nilpotent part of the transformation $A - \lambda I_3$ when increasing the powers of the matrix $A - \lambda I_3$. They finally disappear when taking the second power of $A - \lambda I_3$.

Investigation of the first Jordan chain.

It is interesting to observe how the kernels change. We can see almost without calculation that

$$\begin{cases} \ker(A - \lambda I_3) = \operatorname{span}\{\mathbf{e_1}\}; \\ \ker(A - \lambda I_3)^2 = \operatorname{span}\{\mathbf{e_1}, \mathbf{e_2}\}. \end{cases}$$

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_3)$	1	$1 = \dim(\ker(A - \lambda I_3))$
$\ker(A - \lambda I_3)^2$	2	$1 = \dim(\ker(A - \lambda I_3)^2) - \dim(\ker(A - \lambda I_3))$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A - \lambda I_3)^i) - \dim(\ker(A - \lambda I_3)^{i-1})$. In the first row, we have $\dim(\ker(A - \lambda I_3))$.

We see also that there is equality in the inclusion of sets from the second power onwards.

$$\ker(A - \lambda I_3) \subsetneq \ker(A - \lambda I_3)^2 = \ker(A - \lambda I_3)^3 = \cdots .$$

We remark by looking at the matrices $(A - \lambda I_3)^i$ that we have the following mappings

$$\begin{cases}(A - \lambda I_3)\,\mathbf{e_2} = \mathbf{e_1},\\ (A - \lambda I_3)\,\mathbf{e_1} = \mathbf{0}.\end{cases}$$

We remark by looking at the matrices $(A - \lambda I_3)^i$ or as a consequence of the previous mappings that we have also the following mappings

$$\begin{cases}(A - \lambda I_3)\,\mathbf{e_2} = \mathbf{e_1},\\ (A - \lambda I_3)^2\,\mathbf{e_2} = \mathbf{0}.\end{cases}$$

One sees that we have a Jordan chain of linearly independent vectors. We write a Jordan chain in reverse order.

$$\{\mathbf{e_1} = (A - \lambda I_3)\,\mathbf{e_2}, \mathbf{e_2}\}.$$

After we have found the first Jordan chain of length 2, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A - \lambda I_3)$	1	$\mathbf{e_1}$	0
$\ker(A - \lambda I_3)^2$	2	$\mathbf{e_2}$	0

We have now in the last column only 0's and this means that we are done looking for Jordan chains.

4. Kernels of $(\mathbf{B} - \lambda\,\mathbf{I_3})^{\mathbf{i}}$.

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I_3})$.

We calculate the kernel of $B - \lambda I_3$.

The matrix $B - \lambda I_3$ is

$$B - \lambda I_3 = \begin{pmatrix} 7 & 1 & 5 \\ -14 & -2 & -10 \\ -12 & -1 & -10 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 7 & 1 & 5 \\ -14 & -2 & -10 \\ -12 & -1 & -10 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 7\,z_1 + z_2 + 5\,z_3 = 0, \\ -14z_1 - 2z_2 - 10z_3 = 0, \\ -12z_1 - z_2 - 10z_3 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_3) &= \{(r_1, -2\,r_1, -r_1) \mid r_1 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, -2, -1)\}. \end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I_3})^2$.
We calculate the kernel of $(B - \lambda I_3)^2$.

The matrix $(B - \lambda I_3)^2$ is

$$(B - \lambda I_3)^2 = \begin{pmatrix} -25 & 0 & -25 \\ 50 & 0 & 50 \\ 50 & 0 & 50 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} -25 & 0 & -25 \\ 50 & 0 & 50 \\ 50 & 0 & 50 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} -25z_1 - 25z_3 = 0, \\ 50z_1 + 50z_3 = 0, \\ 50z_1 + 50z_3 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B - \lambda I_3)^2 &= \{(r_1, r_2, -r_1) \mid r_1, r_2 \in \mathbf{K}\}\\ &= \operatorname{span}\{(1, 0, -1), (0, 1, 0)\}.\end{aligned}$$

Kernel of $(\mathbf{B} - \lambda\,\mathbf{I_3})^3$.
We calculate the kernel of $(B - \lambda I_3)^3$.

The matrix $(B - \lambda I_3)^3$ is

$$(B - \lambda I_3)^3 = \begin{pmatrix} 125 & 0 & 125 \\ -250 & 0 & -250 \\ -250 & 0 & -250 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 125 & 0 & 125 \\ -250 & 0 & -250 \\ -250 & 0 & -250 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 125z_1 + 125z_3 = 0, \\ -250z_1 - 250z_3 = 0, \\ -250z_1 - 250z_3 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned}\ker(B - \lambda I_3)^3 &= \{(r_1, r_2, -r_1) \mid r_1, r_2 \in \mathbf{K}\}\\ &= \operatorname{span}\{(1, 0, -1), (0, 1, 0)\}.\end{aligned}$$

Stabilisation of the kernels.
We see that the kernels are starting to stabilise. We mean by this that we are having equality from the second power onwards in the following chain of inclusion of sets.

$$\ker(B - \lambda I_3) \subsetneq \ker((B - \lambda I_3)^2) = \ker((B - \lambda I_3)^3) = \cdots.$$

We assemble all this information in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B - \lambda I_3)$	1	$1 = \dim(\ker(B - \lambda I_3))$
$\ker(B - \lambda I_3)^2$	2	$1 = \dim(\ker(B - \lambda I_3)^2) - \dim(\ker(B - \lambda I_3))$

The last column in this table is the column of the consecutive differences of the first column, $\dim(\ker((B - \lambda I_3)^i)) - \dim(\ker((B - \lambda I_3)^{i-1}))$. The first number of this last column is $\dim(\ker(B - \lambda I_3))$.

5. Calculation of Jordan chains.

We look for a linearly independent set of vectors $\{\mathbf{w_1}, \mathbf{w_2}\}$ satisfying

$$\begin{cases} (B - \lambda I_3)\,\mathbf{w_1} = \mathbf{0}, \\ (B - \lambda I_3)\,\mathbf{w_2} = \mathbf{w_1} \end{cases}$$

or

$$\begin{cases} (B - \lambda I_3)^2\,\mathbf{w_2} = \mathbf{0}, \\ (B - \lambda I_3)\,\mathbf{w_2} = \mathbf{w_1} \end{cases}$$

where $\mathbf{w_2}$ is in the vector space $\ker((B - \lambda I_3)^2)$ but not in $\ker((B - \lambda I_3))$.

We look for a generating vector $\mathbf{w_2}$. This vector must be linearly independent of all vectors in $\ker(B - \lambda I_3)$ and must be independent from vectors that were already chosen in $\ker(B - \lambda I_3)$. We know that

$$\ker((B - \lambda I_3)^2) = \operatorname{span}\{(1, 0, -1), (0, 1, 0)\}.$$

We have at this point not chosen any vector of height 2 in a previous Jordan chain of $\ker((B - \lambda I_3)^2)$.

We know that a vector in $\ker((B - \lambda I_3)^2)$ must be of the form

$$a\,(1, 0, -1) + b\,(0, 1, 0) = (a, b, -a).$$

We remember that

$$\ker(B - \lambda I_3) = \operatorname{span}\{(1, -2, -1)\}.$$

So we have that the vectors just mentioned must be linearly independent and we place them in the rows of a matrix H.

$$H = \begin{pmatrix} 1 & -2 & -1 \\ a & b & -a \end{pmatrix}.$$

We row reduce this matrix H and if we impose $2a + b \neq 0$, we find

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We see that these vectors are independent if we impose the condition $2a + b \neq 0$. We can choose $a = 0$ and $b = 1$.

We have the generating vector

$$\mathbf{w_2} = (0, 1, 0).$$

We start with $\mathbf{w_2} = (0, 1, 0)$.

So we calculate

$$\mathbf{w_1} = (B - \lambda I_3)\mathbf{w_2} = \begin{pmatrix} 7 & 1 & 5 \\ -14 & -2 & -10 \\ -12 & -1 & -10 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

So we have the Jordan chain

$$\{\mathbf{w_1} = (1, -2, -1), \mathbf{w_2} = (0, 1, 0)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_3)$	1	$\mathbf{w_1}$	0
$\ker(B - \lambda I_3)^2$	2	$\mathbf{w_2}$	0

with

$\mathbf{w_1} = (1, -2, -1)$
$\mathbf{w_2} = (0, 1, 0)$

6. Investigation of the second eigenvalue.

We work now with the eigenvalue $\lambda = -2$.

7. Digression about a related matrix.

We take a look in this subsection at a related matrix A that is already in Jordan normal form. It will later turn out that this is exactly the matrix A that we are looking for. A good analysis of this matrix A can be a way to better understand what we will do later on in the solution of this exercise starting from the next subsection. Almost every action or calculation we will make from there on has its counterpart here in this subsection in the analysis of A. There will be much less surprises left after this analysis. A more advanced reader can of course skip entirely this subsection and start with subsection 8 of the solution. The solution from that subsection onwards will make no reference at all to this subsection 7. The solution will be completely independent from this section.

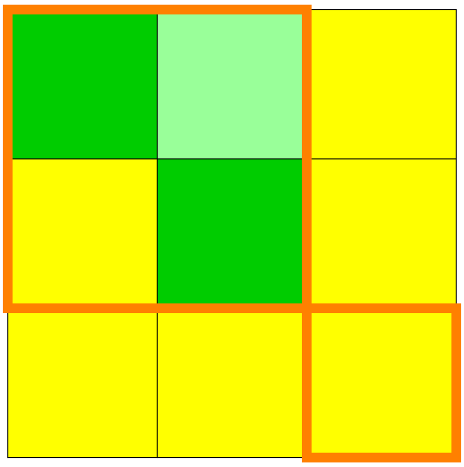
We start from the matrix A.

$$A = \begin{pmatrix} \begin{pmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & (\mathbf{-2}) \end{pmatrix}.$$

We subtract from this matrix A the matrix λI_3.

$$A - \lambda I_3 = \begin{pmatrix} \begin{pmatrix} \mathbf{5} & \mathbf{1} \\ \mathbf{0} & \mathbf{5} \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & (\mathbf{0}) \end{pmatrix}.$$

Let us visualise this situation.



(a) Visualisation of $(A - \lambda I_3)$.

Figure 51

We give here some comment on the preceding figure.

We visualise this type of matrix by colouring the diagonal elements which are not zero by dark green and the colour light green is representing any number. The green elementary Jordan blocks have the same look however large the exponent of $A - \lambda I_3$. The yellow cells are representing the number zero. The nilpotent part of the matrix is the second elementary Jordan block. The superdiagonal does not exist in this case. We need only the first power of $A - \lambda I_3$ to make the non existing superdiagonal disappear. We observe that it is enough to find one nonzero vector in the kernel of the matrix $A - \lambda I_3$. This is an eigenvector of A.

Investigation of the first Jordan chain.

It is interesting to observe how the kernels change. We can see almost without calculation that $\ker(A - \lambda I_3) = \text{span}\{\mathbf{e}_3\}$.

After having investigated the kernels, we can look at the data we have found in the following table.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A - \lambda I_3)$	1	$1 = \dim(\ker(A - \lambda I_3))$

We find in the first column the dimensions of the kernels. In the second column, we have in every row i but the first the consecutive differences of the kernel dimensions numbers $\dim(\ker(A-\lambda I_3)^i)-\dim(\ker(A-\lambda I_3)^{i-1})$. In the first row, we have $\dim(\ker(A-\lambda I_3))$.

We remark by looking at the matrices $(A-\lambda I_3)$ that we have the following mappings

$$(A-\lambda I_3)\,\mathbf{e_3} = \mathbf{0}.$$

One sees that we have a Jordan chain of one linearly independent vector.

$$\{\mathbf{e_3}\}.$$

After we have found the first Jordan chain of length 1, we have then the following table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(A-\lambda I_3)$	1	$\mathbf{e_3}$	0

We have now in the last column only 0's and this means that we are done looking for Jordan chains.

8. Kernels of $(\mathbf{B}-\lambda\,\mathbf{I_3})^{\mathbf{i}}$.

Kernels of $\mathbf{B}-\lambda\,\mathbf{I_3}$.
We calculate the kernel of $(B-\lambda I_3)$.

The matrix $B-\lambda I_3$ is

$$B-\lambda I_3 = \begin{pmatrix} 12 & 1 & 5 \\ -14 & 3 & -10 \\ -12 & -1 & -5 \end{pmatrix}.$$

We have to solve the matrix equation

$$\begin{pmatrix} 12 & 1 & 5 \\ -14 & 3 & -10 \\ -12 & -1 & -5 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This results in having to solve the following system of linear equations.

$$\begin{cases} 12z_1 + z_2 + 5z_3 = 0, \\ -14z_1 + 3z_2 - 10z_3 = 0, \\ -12z_1 - z_2 - 5z_3 = 0. \end{cases}$$

We solve this system and this gives us the solutions set which is a subspace

$$\begin{aligned} \ker(B - \lambda I_3) &= \{(r_1, -2r_1, -2r_1) \mid r_1 \in \mathbf{K}\} \\ &= \operatorname{span}\{(1, -2, -2)\}. \end{aligned}$$

9. Calculation of Jordan chains.

We have the Jordan chain

$$\{\mathbf{w_3} = (1, -2, -2)\}.$$

Let us take a look at our current information table.

Keeping track of chains and dimensions.

	dim	chain 1	remaining dim
$\ker(B - \lambda I_3)$	1	$\mathbf{w_3}$	0

with

$$\mathbf{w_3} = (1, -2, -2)$$

10. Result and check of the result.

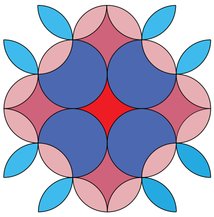
We construct the matrix of base change P with the coordinates of the vectors $\mathbf{w_i}$ found in the Jordan chains in the columns of P.

$$P = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & -2 \\ -1 & 0 & -2 \end{pmatrix}.$$

$$P^{-1}BP = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 10 & 1 & 5 \\ -14 & 1 & -10 \\ -12 & -1 & -7 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & -2 \\ -1 & 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & (\mathbf{-2}) \end{pmatrix}.$$



Part 4
Appendix

1. Calculation of Jordan chains. Choosing a generating vector in a single Jordan chain.

We have used a strategy for choosing a generating vector of an elementary Jordan chain in the text. We will try to illustrate the reasoning behind it without giving a complete proof.

Suppose we have a nilpotent operator B. Suppose that a Jordan chain of length e.g. 4 exists. If one has to look for an elementary Jordan chain, say e.g. of length 4, then we must look for a vector in $\ker(B^4)$ that is not in $\ker(B^3)$. We set out explaining why this produces a chain of linearly independent vectors.

Vectors of a Jordan chain are necessarily linearly independent.

We prove here that vectors in one Jordan chain are necessarily linearly independent.

Suppose that we have a vector $\mathbf{w_4}$ that is in $\ker(B^4)$ but not in $\ker(B^3)$. Suppose that such a vector indeed exists and that we have thorougly checked the existence of such a vector what we tried to do in our solutions. Then we prove that the chain

$$\{B^3\,\mathbf{w_4}, B^2\,\mathbf{w_4}, B\,\mathbf{w_4}, \mathbf{w_4}\}$$

is a chain of linearly independent vectors.

Suppose given the following equation

$$\lambda_1\,B^3\,\mathbf{w_4} + \lambda_2\,B^2\,\mathbf{w_4} + \lambda_3\,B\mathbf{w_4} + \lambda_4\,\mathbf{w_4} = \mathbf{0}.$$

Then we multiply the left hand side and the right hand side of this equation with B^3 which is the same as applying the operator B^3 on the vector on the left hand side and on the vector on the right hand side.
We have then the equation

$$\lambda_1\,B^6\mathbf{w_4} + \lambda_2\,B^5\,\mathbf{w_4} + \lambda_3\,B^4\,\mathbf{w_4} + \lambda_4\,B^3\,\mathbf{w_4} = 0.$$

But $B^6\,\mathbf{w_4} = 0$, because $\mathbf{w_4} \in \ker(B^4) \subseteq \ker(B^3)$, and we can reason analogously for $B^5\,\mathbf{w_4} = 0$, $B^4\,\mathbf{w_4} = 0$.

So there remains from the equation only $\lambda_4\,B^3\,\mathbf{w_4} = 0$. Now we know by the very choice of $\mathbf{w_4}$ that $\mathbf{w_4} \notin \ker(B^3)$, so $B^3\,\mathbf{w_4} \neq \mathbf{0}$ and we have $\lambda_4 = 0$. The equation is reduced to

$$\lambda_1 B^3 \mathbf{w_4} + \lambda_2 B^2 \mathbf{w_4} + \lambda_3 B\mathbf{w_4} = 0.$$

We multiply then the left-hand side and the right-hand side by B^2 and find similarly that $\lambda_3 = 0$.

So we have by reasoning in a similar way that vectors in one Jordan chain are necessarily linearly independent and that by the choice of $\mathbf{w_4}$ the length is exactly 4 because $\mathbf{w_4} \in \ker(B^4)$ but not in $\ker(B^3)$.

Choosing a generating vector when building a second Jordan chain.

Suppose once again that we have a nilpotent operator B. Suppose that we have already found a first Jordan chain as described above of length 4.

Suppose now that we have to choose a generating vector for a *second* Jordan chain of length 2. We stated in the text that it is enough to choose *first* a generating vector $\mathbf{w_6}$ for that chain so that $\mathbf{w_6}$ is in $\ker(B^2)$ but not in $\ker(B)$. This ensures that $\mathbf{w_6}$ is generating a chain of length two and that the vectors in this chain are an independent set.

So we take care that $\mathbf{w_6}$ is linearly independent from all basis vectors in $\ker(B)$. We *secondly* also demand that it is at the same time linearly independent from the already chosen vector in the previous Jordan chain $B^2\mathbf{w_4}$ of exactly height 2 which is also in $\ker(B^2)$ but not in $\ker(B)$. To be crystal clear, we have now that the set

$$\{\mathbf{w_6}, \mathbf{w_2} = B^2\mathbf{w_4}, \text{a basis of } \ker(B)\}$$

is a set consisting of linearly independent vectors. This is the procedure that we always followed in our solutions. We have to prove now that the set

$$\{\mathbf{w_1} = B^3\mathbf{w_4}, \mathbf{w_2} = B^2\mathbf{w_4}, \mathbf{w_3} = B\mathbf{w_4}, \mathbf{w_4}, \mathbf{w_5} = B\mathbf{w_6}, \mathbf{w_6}\}$$

is a linearly independent set of vectors. We stress that we know already that the separate sets $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}\}$ and $\{\mathbf{w_5}, \mathbf{w_6}\}$ consist of linearly independent vectors by what we proved for separate Jordan chains.

Let us consider the following equation

$$\mu_1\mathbf{w_1} + \mu_2\mathbf{w_2} + \mu_3\mathbf{w_3} + \mu_4\mathbf{w_4} + \mu_5\mathbf{w_5} + \mu_6\mathbf{w_6} = \mathbf{0}$$

or

$$\mu_1 B^3 \mathbf{w_4} + \mu_2 B^2 \mathbf{w_4} + \mu_3 B \mathbf{w_4} + \mu_4 \mathbf{w_4} + \mu_5 B \mathbf{w_6} + \mu_6 \mathbf{w_6} = \mathbf{0}.$$

We have to prove for linearly independence that it follows that $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = 0$. We multiply left hand side and right hand side of the equation with B^3. Then

$$\mu_1 B^6 \mathbf{w_4} + \mu_2 B^5 \mathbf{w_4} + \mu_3 B^4 \mathbf{w_4} + \mu_4 B^3 \mathbf{w_4} + \mu_5 B^4 \mathbf{w_6} + \mu_6 B^3 \mathbf{w_6} = \mathbf{0}.$$

All terms $B^i \mathbf{w_j}$ but $B^3 \mathbf{w_4}$ are $\mathbf{0}$. The reason for this is that $B^i \mathbf{w_j}$ are in the $\ker(B^3)$ for $j = 4$ because $\mathbf{w_4}$ is in $\ker(B^4)$ and are in $\ker(B^2)$ for $j = 6$ because $\mathbf{w_6}$ is in $\ker(B^2)$.

There remains $\mu_4 B^3 \mathbf{w_4} = \mathbf{0}$ and we know that $B^3 \mathbf{w_4} \neq \mathbf{0}$. So we have $\mu_4 = 0$.

There remains from the equation that

$$\mu_1 \mathbf{w_1} + \mu_2 \mathbf{w_2} + \mu_3 \mathbf{w_3} + \mu_5 \mathbf{w_5} + \mu_6 \mathbf{w_6} = \mathbf{0}$$

or

$$\mu_1 B^3 \mathbf{w_4} + \mu_2 B^2 \mathbf{w_4} + \mu_3 B \mathbf{w_4} + \mu_5 B \mathbf{w_6} + \mu_6 \mathbf{w_6} = \mathbf{0}.$$

We multiply left hand side and right hand side of the equation with B^2 and we have now

$$\mu_1 B^5 \mathbf{w_4} + \mu_2 B^4 \mathbf{w_4} + \mu_3 B^3 \mathbf{w_4} + \mu_5 B^3 \mathbf{w_6} + \mu_6 B^2 \mathbf{w_6} = \mathbf{0}.$$

As before we have $B^5 \mathbf{w_4} = \mathbf{0}$, $B^4 \mathbf{w_4} = \mathbf{0}$, $B^3 \mathbf{w_6} = \mathbf{0}$, $B^2 \mathbf{w_6} = \mathbf{0}$.

There remains from the equation that

$$\mu_3 B^3 \mathbf{w_4} = \mathbf{0}$$

and we have now $\mu_3 = 0$ because $B^3 \mathbf{w_4} \neq \mathbf{0}$.

There remains from the equation

$$\mu_1 \mathbf{w_1} + \mu_2 \mathbf{w_2} + \mu_5 \mathbf{w_5} + \mu_6 \mathbf{w_6} = \mathbf{0}.$$

or

$$\mu_1 B^3 \mathbf{w_4} + \mu_2 B^2 \mathbf{w_4} + \mu_5 B \mathbf{w_6} + \mu_6 \mathbf{w_6} = \mathbf{0}.$$

We multiply left hand side and right hand side of the equation with B and we have

$$\mu_1 B^4 \mathbf{w_4} + \mu_2 B^3 \mathbf{w_4} + \mu_5 B^2 \mathbf{w_6} + \mu_6 B \mathbf{w_6} = \mathbf{0}.$$

We are left with

$$\mu_2 B^3 \mathbf{w_4} + \mu_6 B \mathbf{w_6} = \mathbf{0}.$$

We can rewrite this as

$$B \left(\mu_2 B^2 \mathbf{w_4} + \mu_6 \mathbf{w_6} \right) = \mathbf{0}.$$

We conclude that $\mu_2 B^2 \mathbf{w_4} + \mu_6 \mathbf{w_6} \in \ker(B)$. So $\mu_2 B^2 \mathbf{w_4} + \mu_6 \mathbf{w_6} = \mathbf{b}$ where $\mathbf{b} \in \ker(B)$. But $\mathbf{w_6}$ was explicitly chosen to be independent from $B^2 \mathbf{w_4}$ and any vector in $\ker(B)$. This means that the set $\{\mathbf{w_6}, B^2 \mathbf{w_4}, \mathbf{b}\}$ is a set of linearly independent vectors. We have thus that $\mathbf{b} = \mathbf{0}$ and therefore that $\mu_2 = 0$ and $\mu_6 = 0$.

There remains from the equation that

$$\mu_1 \mathbf{w_1} + \mu_5 \mathbf{w_5} = \mathbf{0}$$

or

$$\mu_1 B^3 \mathbf{w_4} + \mu_5 B \mathbf{w_6} = \mathbf{0}.$$

We can rewrite this equation as

$$B \left(\mu_1 B^2 \mathbf{w_4} + \mu_5 \mathbf{w_6} \right) = \mathbf{0}.$$

We conclude that $\mu_1 B^2 \mathbf{w_4} + \mu_5 \mathbf{w_6} \in \ker(B)$ and reason as before that $\mu_1 = 0$ and $\mu_5 = 0$.

Remarks.

1. It is wise to do independency checks while making Jordan exercises. But it is only necessary to do them while choosing generating vectors for a Jordan chain. We have indicated this in the text with the words “we choose”.

2. There is an opinion that says that it is not worthwhile to do any independency check. The justification is then that if one has a linearly independent set of vectors in V that is not a basis in V, then making a random choice of a supplementary vector is enough to

guarantee that this vector can considered to be statistically linearly independent from the previous set. This is undeniably true. There is a big “but” though in daily practice. Because one tends to choose vectors with many zero’s to ensure easy calculations afterwards and one almost automatically chooses digits that are already used many times in the same exercise and last but not least one chooses almost invariably coefficients in a *very* bounded set in $\mathbf{N}$, one is way far-off from *true randomness*. The probability of not being linearly dependent is far larger then 0 in daily practice when one chooses a so called “random” vector. If one chooses this way of proceeding, then it is even more important to check the end result. The matrix P can often not be found because in that case it is possible to choose generating vectors having the wrong height and then columns to build P are missing or in other words a basis of Jordan chains cannot be found. It can also be the case that one sees problems first by *zero divisions*. A matrix P of the basis change is *miraculously* not invertible. So there is a caveat and some caution is advised.

3. One subtle mistake can be made. In the notation of the calculations we have done in our comments on choosing the generating vectors, we have used as criterion that the set $\{B^2\mathbf{w}_4, \mathbf{w}_6, \mathbf{b}\}$, $\mathbf{b}$ is a set of basis vectors of $\ker(B)$, must consist of linearly independent vectors. It is not enough to check that $\mathbf{w}_6$ is independent from all basis vectors $\mathbf{b} \in \ker(B)$ separately, and afterwards checking that $B^2\mathbf{w}_4$ and $\mathbf{w}_6$ are linearly independent. This does not guarantee that the set $\{B^2\mathbf{w}_4, \mathbf{w}_6, \mathbf{b}\}$, $\mathbf{b}$ is a set of basis vectors of $\ker(B)$, consists of linearly independent vectors.

How to check linear independence of vectors in a Jordan chain.
Suppose we have already found a Jordan chain $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Suppose we have to choose a generating vector of height 2.

1. Choose a generic vector $\mathbf{w}$. This vector is a generic linear combination of generators of $\ker(B^2)$.

2. Take the vectors of height exactly 2 already found in previously chosen Jordan chains.

3. Choose a basis for $\ker(B)$.

4. Take care that all the vectors previously mentioned are a linearly independent set.

2. Some remarks about the tables used in the text.

The geometric dimension.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(A-\lambda I_7)$	2	$2 = \dim(\ker(A-\lambda I_7))$
$\ker(A-\lambda I_7)^2$	4	$2 = \dim(\ker(A-\lambda I_7)^2) - \dim(\ker(A-\lambda I_7))$

Figure 52. The first number in the column with label “remaining dimensions” in the **initial table** for the eigenvalue λ is the geometric dimension for that eigenvalue. It indicates how many elementary Jordan chains can be build for this eigenvalue.

The last non zero number in the column “remaining dimensions”.

Keeping track of chains and dimensions.

	dim	remaining dim
$\ker(B-\lambda I_7)$	2	$2 = \dim(\ker(B-\lambda I_7))$
$\ker(B-\lambda I_7)^2$	4	$2 = \dim(\ker(B-\lambda I_7)^2) - \dim(\ker(B-\lambda I_7))$

Figure 53. The last nonzero number in the column with label “remaining dimensions” in **any table** for the eigenvalue λ is the number of remaining Jordan chains of length exactly equal to the row number in which this number is situated. These remaining Jordan chains can be chosen with linearly independent vectors. In this example the last non zero number is in row number 2. This means that two Jordan chains can still be found with length 2 (the row number). These chains can be chosen with linearly independent vectors.

3. Geometric interpretation of a nilpotent Jordan matrix.

It can be interesting to see what a matrix does by looking at the map it can represent. Let us take a nilpotent Jordan matrix and see what it does in the affine space A^3 in $\mathbf{R}$.

The nilpotent Jordan matrix that we will study is

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let A^3 be the real affine space. Let us look at the mapping

$$g : A^3 \longrightarrow A^3 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow P \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ 0 \end{pmatrix}.$$

We remark immediately that consecutive applications of this map give

$$P^2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix}$$

and

$$P^3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

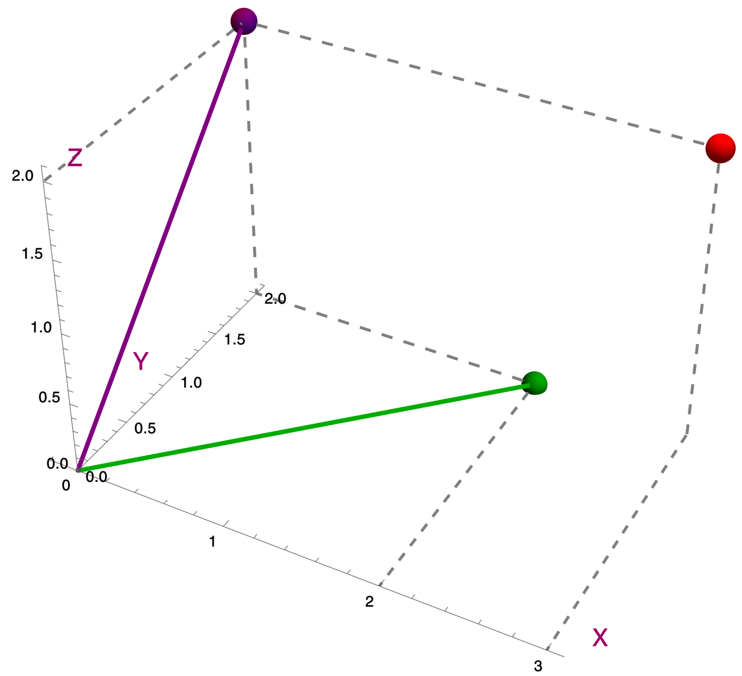


Figure 54. We can look upon P as a transformation build in two steps. We start with the red point. That red point is projected along the X-direction in an intermediate step following the transformation $(x, y, z) \mapsto (0, y, z)$. This is a projection on the Y-Z plane along the X direction. This mapped point is the purple point. We see that the red point loses its X-component. This is essentially due to the fact that the X axis is essentially the kernel. This purple point is then copied to the X-Y plane with the same X-Y coordinates as the coordinates of the purple point in the Y-Z plane. The resulting point is green coloured. This is the transformation $(0, y, z) \mapsto (y, z, 0)$. We will apply the map P once again on our resulting point in green.

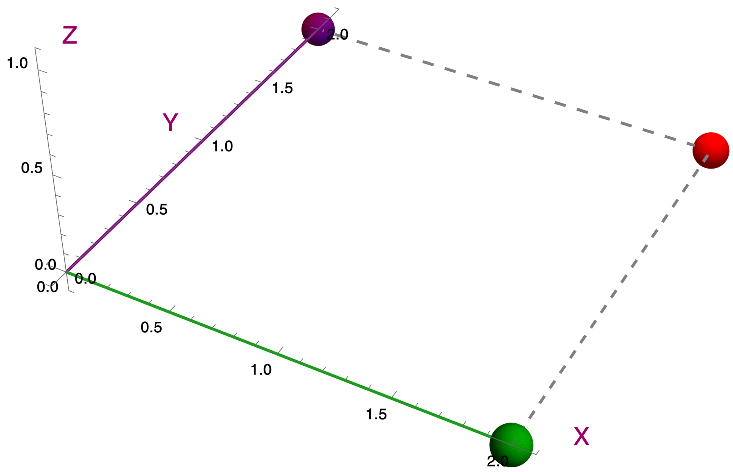


Figure 55. The green point in the previous figure is now coloured in red. We will apply the map defined by P once again. We can look upon P again as a transformation build in two steps. We start with the red point. That red point is projected along the X direction in an intermediate step following the transformation $(x, y, z) \mapsto (0, y, z)$. This is a projection on the Y-Z plane along the X direction. This mapped point is the purple point. We see that the red point loses its X-component. This is essentially due to the fact that the X axis is essentially the kernel. This purple point is then copied to the X-Y plane with the same X-Y coordinates as the coordinates of the purple point in the Y-Z plane. This is the transformation $(0, y, z) \mapsto (y, z, 0)$.

It is clear that if we map this green point again that it will be the zero vector when projecting along the X direction. So the mapped vector will be the zero point **0**.

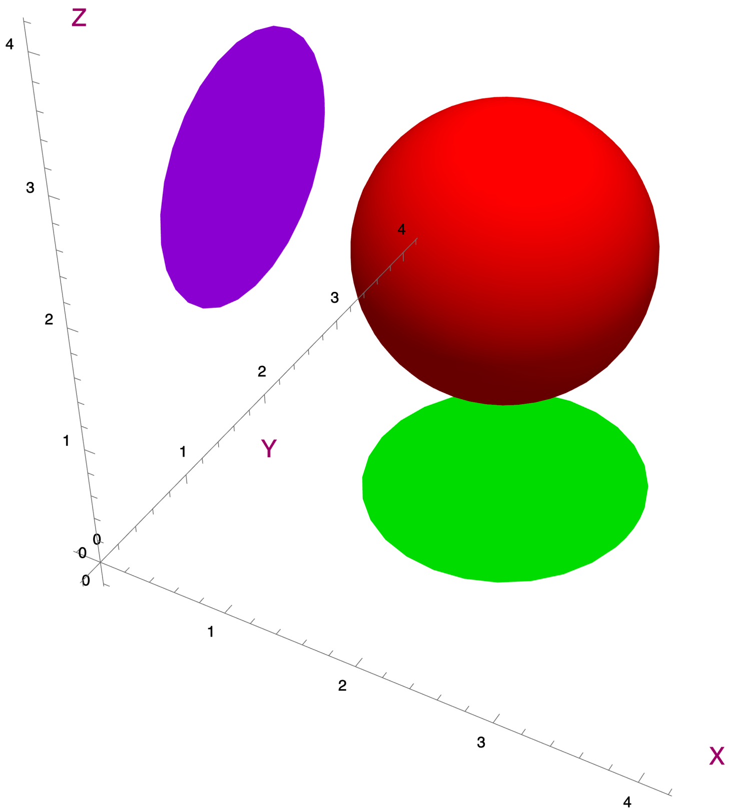


Figure 56. We have here a sphere coloured in red. We use the word sphere for the set of points satisfying the equation $(x-2)^2+(y-2)^2+(z-2)^2=1$. We will apply the map defined by P once again. We can look upon P again as a transformation build in two steps. We start with the red sphere. That red sphere is projected along the X-direction in an intermediary step following the transformation $(x,y,z)\mapsto(0,y,z)$. This is a projection on the Y-Z-plane along the X-direction. This mapped sphere is then the purple disk. We see that the red sphere loses its X component. This is essentially due to the fact that the X-axis is essentially the kernel. This purple disk is then copied to the X-Y-plane with the same X-Y-position as the position of the purple disk in the Y-Z-plane. This is the transformation $(0,y,z)\mapsto(y,z,0)$. The by P transformed sphere is now the green disk. We will map the green disk in the following figure once again following P.

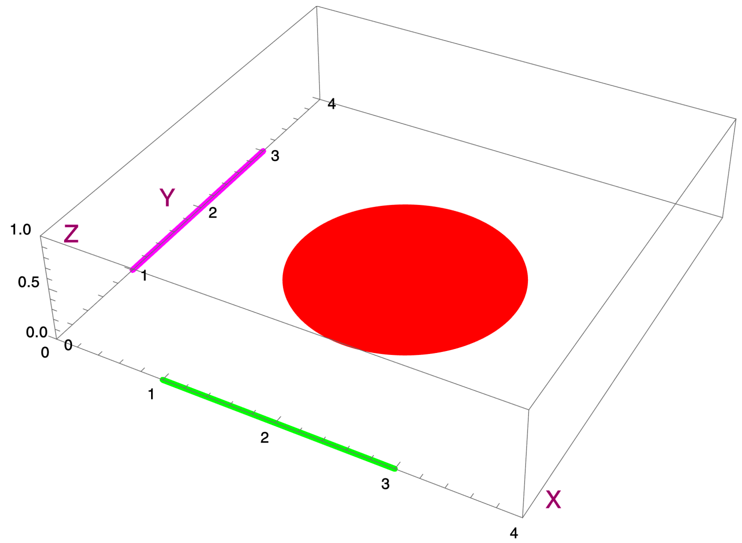


Figure 57. The green disk of the preceding figure is now coloured in red. We will apply the map defined by P on the red disk once again. We can look upon P again as a transformation build in two steps. We start with the red disk. That red disk is projected along the X-direction in an intermediate step following the transformation $(x, y, z) \mapsto (0, y, z)$. This is a projection on the Y-Z-plane along the X-direction. This mapped disk is then the purple interval. We see that the red disk loses its X-component. This is essentially due to the fact that the X-axis is essentially the kernel. This purple interval is then copied to the X-Y-plane with the same X-Y-position as the position of the purple interval in the Y-Z-plane. This is the transformation $(0, y, z) \mapsto (y, z, 0)$. The by P transformed disk is now the green interval.

It is clear that if we map this green interval again that it will be the zero vector when projecting along the X-direction. So the mapped interval will be the zero point $\mathbf{0}$ by another application of P.

4. The minimal polynomial of B.

Suppose that the characteristic polynomial is

$$(\lambda - \lambda_1)^{e_1}(\lambda - \lambda_2)^{e_2} \cdots (\lambda - \lambda_n)^{e_n}$$

where the λ_i are the eigenvalues of B.
The minimal polynomial $m(\lambda)$ is defined as the monic generator of the ideal in $\mathbf{K}[\lambda]$ that annihilates B. It is a divisor of the characteristic

polymial because the characteristic polynomial is also in the annihilating ideal. So the general form of the minimal polynomial is

$$m(\lambda) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_n)^{m_n}$$

where for all i we have that $m_i \le e_i$.

We remark that we can write down this polynomial at the end of every exercise we have made in these notes. We have made enough calculations to construct this polynomial. For every i we have investigated the space $V(\lambda_i) = \bigcup_{k\ge 1} \ker(B - \lambda_i)^k$. The number m_i is now equal to the following numbers which are all equal

1. either the height of nilpotency of the operator $B - \lambda_i I$ restricted on the subspace $V(\lambda_i)$
2. or the first exponent where the ascending chain $\ker(B-\lambda_i) \subseteq \ker(B-\lambda_i)^2 \subseteq \ker(B - \lambda_i)^3 \cdots$ starts to stabilise
3. or the size of the largest elementary Jordan block associated to the eigenvalue λ_i.

Part 5
Copy & Paste

The following matrices are written in code with no typographic instructions. So they are ideal for copy and paste to the editor of a calculator. Brackets and delimiters can be changed in that editor with its search and replace functionality.

Exercise 1.
Jordan structure: $(\mathbf{4} \times \mathbf{4})$; $(\mathbf{J}_4(\mathbf{0}))$.

```
B =
(
( 1  , -2 , 3 , 1 ),
( 4  , -5 , 7 , 2 ),
( 3  , -3 , 4 , 1 ),
( -1 , 0  , 0 , 0 )
)
```

Exercise 2.
$(\mathbf{4} \times \mathbf{4})$; $(\mathbf{J}_3(\mathbf{0}), \mathbf{J}_1(\mathbf{0}))$.

```
B =
(
( 1 , 1 , -2 , 0 ),
( 1 , 0 , -1 , 0 ),
( 1 , 0 , -1 , 0 ),
( 0 , 1 , -1 , 0 )
)
```

Exercise 3.
$(\mathbf{4} \times \mathbf{4})$; $(\mathbf{J}_2(\mathbf{0}), \mathbf{J}_2(\mathbf{0}))$.

```
B =
(
( 1  , 0 , 1  , 0     ),
( 0  , 0 , 1  , -1    ),
( -1 , 0 , -1 , 0     ),
( -1 , 0 , -1 , 0     )
)
```

Exercise 4.
(7×7); $(\mathbf{J}_3(\mathbf{0}), \mathbf{J}_3(\mathbf{0}), \mathbf{J}_1(\mathbf{0}))$.

```
B =
(
( -2 , 1  , 0  , -2 , 1  , 3  , 2  ),
( 1  , -1 , 0  , 1  , -1 , -2 , -1 ),
( 1  , -1 , 0  , 1  , -1 , -2 , -1 ),
( 0  , -1 , 0  , 2  , 2  , 1  , -2 ),
( 0  , 1  , 1  , 0  , 2  , 2  , 0  ),
( -1 , 0  , -1 , -1 , -1 , 0  , 1  ),
( -1 , 1  , 1  , 1  , 5  , 5  , -1 )
)
```

Exercise 5.
(7×7); $(\mathbf{J}_3(\mathbf{0}), \mathbf{J}_2(\mathbf{0}), \mathbf{J}_2(\mathbf{0}))$.

```
B =
(
( 3  , 3  , -4 , 2  , 1  , 1  , -1 ),
( -3 , -3 , 3  , -2 , 0  , 0  , 1  ),
( -1 , -1 , 1  , -1 , 0  , 0  , 1  ),
( -1 , -1 , 3  , -1 , -2 , -2 , 1  ),
( -5 , -5 , 6  , -4 , 0  , 0  , 3  ),
( 4  , 4  , -5 , 3  , 0  , 0  , -2 ),
( 0  , 0  , 1  , 0  , -1 , -1 , 0  )
)
```

Exercise 6.
(7×7); $(\mathbf{J}_2(\mathbf{0}), \mathbf{J}_2(\mathbf{0}), \mathbf{J}_2(\mathbf{0}), \mathbf{J}_1(\mathbf{0}))$.

```
B =
(
( 0  , -1 , 0  , -1 , 1  , 0  , 0  ),
( 0  , -1 , -1 , -4 , 3  , -1 , -1 ),
( 0  , -2 , -1 , -5 , 4  , -1 , -1 ),
( 1  , -1 , -2 , -5 , 4  , -1 , -1 ),
( 1  , -2 , -3 , -9 , 7  , -2 , -2 ),
( -1 , -1 , 2  , 3  , -2 , 1  , 1  ),
( 0  , 2  , -1 , -1 , 0  , -1 , -1 )
)
```

Exercise 7.
(6×6); $(\mathbf{J}_5(\mathbf{0}), \mathbf{J}_1(\mathbf{0}))$.

```
B =
(
( 1  , 2  , -1 , 0 , 2  , 1  ),
( -1 , -1 , 1  , 0 , -2 , -1 ),
( 0  , 1  , 0  , 0 , 0  , 0  ),
( 1  , 1  , -1 , 0 , 2  , 1  ),
( 1  , 1  , -1 , 0 , 0  , 0  ),
( -1 , -1 , 1  , 0 , 1  , 0  )
)
```

Exercise 8.
(6×6); $(\mathbf{J}_4(\mathbf{0}), \mathbf{J}_2(\mathbf{0}))$.

```
B =
(
( 2  , -2 , -1 , 2  , 1 , 1 ),
( 4  , -5 , -3 , 5  , 1 , 2 ),
( -1 , 2  , 1  , -3 , 1 , 0 ),
( 1  , -1 , -1 , 1  , 1 , 1 ),
( -1 , 1  , 0  , 0  , 0 , 0 ),
( 3  , -4 , -2 , 2  , 0 , 1 )
)
```

Exercise 9.
(6×6); $(\mathbf{J}_3(\mathbf{0}), \mathbf{J}_3(\mathbf{0}))$.

```
B =
(
( 3   , 3   , -1 , 2  , 2  , 1  ),
( -5  , -5  , 1  , -3 , -3 , -1 ),
( -12 , -10 , 3  , -8 , -5 , -3 ),
( -1  , -1  , 1  , -1 , -1 , -1 ),
(  0  , 1   , 0  , 0  , 1  , 0  ),
( -4  , -4  , 1  , -3 , -2 , -1 )
)
```

Exercise 10.
(5×5); $(\mathbf{J}_4(\mathbf{0}), \mathbf{J}_1(\mathbf{0}))$.

```
B =
(
( 0 , 2  , -1 , -4 , -2 ),
( 0 , 1  , 0  , -2 , -1 ),
( 0 , -1 , 0  , 1  , 1  ),
( 0 , 1  , 0  , -1 , -1 ),
( 0 , 0  , -1 , -2 , 0  )
)
```

Exercise 11.
(5×5); $(\mathbf{J}_3(\mathbf{0}), \mathbf{J}_2(\mathbf{0}))$.

```
B =
(
( 0  , 0  , 1  , 1  , 0  ),
( 1  , 0  , 1  , 2  , 1  ),
( 3  , -1 , 1  , 3  , 2  ),
( -2 , 1  , -1 , -2 , -1 ),
( 2  , -1 , 0  , 1  , 1  )
)
```

Exercise 12.
(5×5); $(\mathbf{J}_2(\mathbf{0}), \mathbf{J}_2(\mathbf{0}), \mathbf{J}_1(\mathbf{0}))$.

```
B =
(
( 1  , 0  , 1  , -1 , -2 ),
( -1 , 0  , -1 , 1  , 2  ),
( 1  , -1 , 0  , 0  , -2 ),
( 0  , -1 , -1 , 1  , 0  ),
( 1  , 0  , 1  , -1 , -2 )
)
```

Exercise 13.
(3×3); $(J_2(0), J_1(0))$.

```
B =
(
( -1 , -1 , 1  ),
( 1  , 1  , -1 ),
( 0  , 0  , 0  )
)
```

Exercise 14.
(3×3); $(J_3(0))$.

```
B =
(
( -2 , 1  , -2 ),
( -1 , 0  , -1 ),
( 2  , -1 , 2  )
)
```

Exercise 15.
(2×2); $(J_2(0))$.

```
B =
(
( -1 , 1 ),
( -1 , 1 )
)
```

Exercise 16.
(8×8); $(J_4(0), J_2(0), J_2(0))$.

```
B =
(
( 10  , -1 , 9   , -11  , 6   , 3  , -5  , -9  ),
( 20  , 0  , 19  , -20  , 16  , 4  , -15 , -19 ),
( -5  , 0  , -5  , 5    , -4  , -1 , 4   , 5   ),
( 6   , -1 , 5   , -7   , 3   , 2  , -2  , -5  ),
( -3  , 0  , -3  , 3    , -2  , -1 , 2   , 3   ),
( -23 , 1  , -21 , 24   , -15 , -6 , 13  , 20  ),
( 21  , -1 , 19  , -22  , 15  , 5  , -13 , -19 ),
( -25 , 1  , -23 , 26   , -18 , -6 , 16  , 23  )
)
```

Exercise 17.
$(\mathbf{3} \times \mathbf{3})$; $(\mathbf{J}_3(\mathbf{0}))$.

```
B =
(
( 3  , 2  , -1 ),
( -3 , -2 , 1  ),
( 2  , 1  , -1 )
)
```

Exercise 18.
$(\mathbf{5} \times \mathbf{5})$; $(\mathbf{J}_3(\mathbf{0}), \mathbf{J}_2(\mathbf{0}))$.

```
B =
(
( 2  , 1  , 1  , -2 , -1 ),
( -2 , -1 , 0  , 1  , 2  ),
( -3 , -1 , -1 , 2  , 2  ),
( -1 , 0  , 0  , 0  , 1  ),
( 1  , 0  , 1  , -1 , 0  )
)
```

Exercise 19.
$(\mathbf{4} \times \mathbf{4})$; $(\mathbf{J}_4(\mathbf{0}))$.

```
B =
(
( -1 , 1  , -1 , 0 ),
( 7  , -4 , 9  , 3 ),
( 3  , -2 , 4  , 1 ),
( 3  , -2 , 3  , 1 )
)
```

Exercise 23.
(7×7); $(\mathbf{J}_2(\mathbf{1}), \mathbf{J}_2(\mathbf{1}), \mathbf{J}_3(-\mathbf{1}))$.

```
B =
(
( 0  , 2  , 0  , -2 , -1 , 0  , 0  ),
( 0  , 1  , 0  , 0  , 0  , 0  , 0  ),
( 1  , -1 , 1  , 2  , 1  , 0  , 0  ),
( 0  , -1 , 0  , 4  , 3  , -1 , 0  ),
( 1  , 0  , 0  , -4 , -4 , 2  , 0  ),
( -1 , 1  , 1  , -3 , -2 , 0  , 1  ),
( -1 , 1  , -2 , -2 , -1 , 0  , -1 )
)
```

Exercise 24.
(7×7); $(\mathbf{J}_3(\mathbf{1}), \mathbf{J}_2(\mathbf{1}), \mathbf{J}_2(\mathbf{2}))$.

```
B =
(
( 2  , 1  , 0  , 0 , 0  , 1  , 0  ),
( -2 , 0  , 1  , 1 , 1  , -1 , -2 ),
( -2 , -2 , 3  , 1 , -1 , -3 , -1 ),
( 1  , 1  , -1 , 1 , 1  , 2  , 1  ),
( -1 , 0  , 1  , 0 , 1  , -1 , -1 ),
( 0  , -1 , 0  , 0 , -1 , 0  , 1  ),
( 0  , 0  , 0  , 0 , 0  , 0  , 2  )
)
```

Exercise 25.
(7×7); $(\mathbf{J}_2(\mathbf{2}), \mathbf{J}_2(\mathbf{2}), \mathbf{J}_1(\mathbf{2}), \mathbf{J}_2(\mathbf{1}))$.

```
B =
(
( 1   , 0  , 0  , 0  , 0  , 0  , 0  ),
( -2  , 1  , 0  , -2 , 1  , 2  , 1  ),
( -3  , -1 , 2  , -1 , 1  , 2  , 1  ),
( 0   , 0  , 0  , 2  , 0  , 0  , 0  ),
( 3   , 1  , 0  , 1  , 1  , -2 , -1 ),
( 4   , 1  , -1 , 1  , -1 , -1 , -1 ),
( -15 , -4 , 2  , -5 , 4  , 10 , 6  )
)
```

Exercise 26.
(7×7); $(\mathbf{J_4(1)}, \mathbf{J_1(1)}, \mathbf{J_2(-1)})$.

```
B =
(
( -2 , 1  , -2 , -1 , 0 , 0  , -2 ),
( 1  , 1  , 2  , 0  , 0 , -1 , 1  ),
( 2  , 0  , 2  , 1  , 0 , -1 , 1  ),
( 1  , -1 , 0  , 2  , 0 , 2  , 0  ),
( 1  , 0  , 2  , 0  , 1 , -1 , 1  ),
( 1  , 0  , 1  , 0  , 0 , -2 , 1  ),
( 0  , -1 , 0  , 0  , 0 , 0  , 1  )
)
```

Exercise 27.
(6×6); $(\mathbf{J_3(-1)}, \mathbf{J_1(-1)}, \mathbf{J_2(1)})$.

```
B =
(
( -1 , -2 , 3  , 2  , -4  , 2  ),
( 0  , -1 , -4 , -3 , -2  , -3 ),
( 0  , 0  , 2  , 2  , 1   , 2  ),
( 0  , 4  , -2 , -4 , 4   , -3 ),
( 0  , 0  , 3  , 2  , 0   , 2  ),
( 0  , -8 , 0  , 3  , -10 , 2  )
)
```

Exercise 28.
(6×6); $(\mathbf{J_2(-1)}, \mathbf{J_2(-1)}, \mathbf{J_2(1)})$.

```
B =
(
( -4  , 5  , 0  , 3  , 8   , 6  ),
( -8  , 5  , 2  , -1 , 15  , 8  ),
( -6  , 8  , -1 , 3  , 14  , 9  ),
( 7   , -3 , -2 , 1  , -11 , -6 ),
( 9   , -5 , -2 , 2  , -16 , -8 ),
( -13 , 9  , 2  , 0  , 23  , 13 )
)
```

Exercise 29.
$(\mathbf{6} \times \mathbf{6}); (\mathbf{J}_3(-\mathbf{1}), \mathbf{J}_1(-\mathbf{1}), \mathbf{J}_2(\mathbf{1}))$.

```
B =
(
( -7 , 7  , 1  , 2  , 13  , 8  ),
( -8 , 5  , 2  , -1 , 15  , 8  ),
( -5 , 7  , -1 , 3  , 12  , 8  ),
( 10 , -5 , -3 , 2  , -16 , -8 ),
( 9  , -5 , -2 , 2  , -16 , -8 ),
( -16 , 11 , 3 , -1 , 28  , 15 )
)
```

Exercise 30.
$(\mathbf{6} \times \mathbf{6}); (\mathbf{J}_2(-\mathbf{1}), \mathbf{J}_2(-\mathbf{1}), \mathbf{J}_2(\mathbf{1}))$.

```
B =
(
( -4  , 5  , 0  , 3  , 8   , 6  ),
( -8  , 5  , 2  , -1 , 15  , 8  ),
( -6  , 8  , -1 , 3  , 14  , 9  ),
( 7   , -3 , -2 , 1  , -11 , -6 ),
( 9   , -5 , -2 , 2  , -16 , -8 ),
( -13 , 9  , 2  , 0  , 23  , 13 )
)
```

Exercise 31.
$(\mathbf{5} \times \mathbf{5}); (\mathbf{J}_3(-\mathbf{1}), \mathbf{J}_2(\mathbf{2}))$.

```
B =
(
( -3 , 0  , -5 , 4  , 1  ),
( -4 , 2  , -4 , -1 , -1 ),
( 2  , 1  , 3  , -5 , 1  ),
( 0  , -1 , 1  , 1  , -2 ),
( 3  , -2 , 5  , 0  , -2 )
)
```

Exercise 32.
(4×4); $(J_2(-2), J_2(2))$.

```
B =
(
( 10  , 5  , 3  , 4  ),
( -12 , -7 , -3 , -4 ),
( -8  , -5 , -1 , -4 ),
( -5  , -1 , -4 , -2 )
)
```

Exercise 33.
(3×3); $(J_3(-2))$.

```
B =
(
( -4 , 0  , 1 ),
( -3 , -3 , 2 ),
( -5 , -1 , 1 )
)
```

Exercise 34.
(3×3); $(J_2(3), J_1(-2))$.

```
B =
(
( 10  , 1  , 5   ),
( -14 , 1  , -10 ),
( -12 , -1 , -7  )
)
```

Contact the author:
Dirk.Bollaerts@protonmail.com